MATHEMATICAL PAPERS

BY

W. K. Clifford.

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BY

WILLIAM KINGDON CLIFFORD.

EDITED BY ROBERT TUCKER,

WITH AN INTRODUCTION BY H. J. STEPHEN SMITH.

"If he had lived we might have known something."

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My dear Mrs Clifford,

When towards the end of April 1879 I called upon you in answer to an urgent request to do so, it was with much surprise that I heard from you that it was one of your lamented Husband's last requests in regard to his work that I should be one of two persons to be asked to see the remaining Manuscript of his admirable book on *Dynamic* through the press. Though much gratified by his kind approval of what I had written upon the portion already in print, I could hardly look upon myself as qualified to undertake such a task—who indeed could hope to succeed in fitly editing fragments which had not received his last energizing touches? Whatever diffidence may have at first deterred me, yielded on further reflection to your pressing solicitation.

At this same interview you told me that it had been suggested, I believe by Mr Spottiswoode and Mr Macmillan, that a collected edition of Clifford's Mathematical Works, including the Memoirs already printed as well as such posthumous papers as might

be found in a state fit for publication, should be brought out under the auspices of a double editorship, one editor to be a resident at Cambridge and the other to be myself. If I hesitated to undertake the editing of the Manuscript of the *Dynamic*, you may believe I was even more afraid to undertake so great a responsibility as the editing of the Papers. Finding however at an interview I had with Dr Jack that it was desired that the papers should be brought out with all speed, and learning from Mr Spottiswoode that the plan of a double editorship had fallen through, I consented to do the best I could, feeling the less hesitation in making the attempt as I was able to obtain the assistance of Mr Spottiswoode himself and of Professors Cayley, Henrici and Smith.

Thus much of explanation I feel to be due from me to show that I did not engage in so great an undertaking of my own mere motion, but that I entered upon my task with no slight idea of its magnitude, although the actuality has far surpassed my expectation.

Your own assistance and kind interest in the progress of the work have been most valuable and most encouraging to me; and I can now only wish that the result of my efforts may deserve your approval, and may be not unworthy of the illustrious mathematician, your much loved Husband.

"If I have done well, it is that which I desired: but if slenderly and meanly, it is that which I could attain unto."

I need not enter into any account of the causes which have operated to bring about so tardy a completion of my labours. Sufficient it is for me to know that you have understood and allowed for the many hindrances which have arisen from circumstances connected with the fragmentary condition of some of the papers, and from the pressing claims upon my own time and upon that of the eminent mathematicians, whose advice and assistance have so ungrudgingly been extended to me.

Thanking you, my dear Mrs Clifford, for the honour you have conferred upon me in entrusting to me the work of raising this monument to the memory of one you love so well,

I remain,

Yours faithfully,

R. Tucker.

December 30th, 1881.

DIRECTIONS TO BINDER.

Papers now published for the first time are distinguished by an * prefixed to the number indicating their order in this volume.

CORRIGENDA.

On p. 22, *for* xv. *read* xvii.

,, 114, 6 up, *read* convertical.

,, 134, in connexion with the formulæ, refer to *Quarterly Journal of Mathematics*, No. 25. For this reference I am indebted to Mr J. J. Walker, who also suggests the following corrections "on p. 135, 4 up, *for* κ *read* $\frac{1}{\kappa}$; p. 136, *omit* $\left[\frac{1}{\kappa}\right]$, l. 10, -2ι, l. 11."

,, 205, r is v in Clifford's MS. The paper is printed as in the *London Math. Society's Proceedings*.

,, 621, l. 12, *read* vol. xxxv. p. 21.

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II. BIBLIOGRAPHICAL.

BY

ROBERT TUCKER, M.A.,

HON. SEC. TO LONDON MATHEMATICAL SOCIETY,

SOMETIME SCHOLAR OF ST JOHN'S COLLEGE, CAMBRIDGE.

I

WILLIAM KINGDON CLIFFORD,

Born at Exeter, May 4th, 1845. Died at Madeira, March 3rd, 1879.

After the touching Biography prefixed to the *Lectures and Essays*, written with consummate literary skill by one who knew and loved Clifford well—"as his own soul,"—I feel it is unnecessary for me to say more than a few words as a preface to the present collection of his Mathematical Papers, and in the few following lines I shall confine myself to the simple statement of facts bearing upon his life and work as a Mathematician.

Clifford received his early education at Mr Templeton's school in Exeter, where in 1858 and 1859 he gained numerous distinctions in a very extended range of subjects at the University Local Examinations, both of Oxford and Cambridge. From this school he proceeded, in 1860, to King's College, London, where in like manner his efforts were crowned with success in more than one branch of study. Having in 1863 obtained a minor scholarship, he entered at Trinity College, Cambridge; at the end of his course in 1867 he graduated Second Wrangler and gained the Second Smith's Prize. In 1868, he was elected a Fellow of his College. He proceeded to M.A. in 1870; joined the Eclipse Expedition to Sicily of the same year, and was in 1871 elected Professor of Applied Mathematics and

Mechanics in University College, London, a post he occupied until the time of his death.

In June, 1874, Clifford was elected a Fellow of the Royal Society. He became a Member of the London Mathematical Society, June 18th, 1866, and served on its Council for every session from 1868—9 to 1876—7. Though he then ceased to be a member of the Council, he continued to take a warm interest in the proceedings of the Society, and on many a night he was present at the meetings when he was far from well and ought to have remained at home. I go back in imagination to the first meeting he attended—his was ever a welcome face in our room at University College and subsequently at Burlington House and Albemarle Street—it was on the evening of January 23rd, 1868, when Dr Hirst, our Treasurer, occupied the chair in the absence of our then President, Prof. Sylvester. Of the little band then present four have since been removed by death, viz., Clifford, Clerk Maxwell, Archibald Smith, and Thomas Cotterill. Clerk Maxwell made two communications: i. "on the construction of stereograms of surfaces," ii. "on the doctrine of Reciprocal Diagrams of Forces with the extension of Airy's Function of Stress from two dimensions to three." The former was made interesting by the fact that the author brought with him a real Image Stereoscope constructed after his own directions, and which invested the stereograms with a marvellous resemblance to the solids of which they were the plane presentments[1]. It is possible that Clifford spoke upon both these papers, but I have no record of his having done so. Mr J. J. Walker read a third paper in connection with which Clerk Maxwell asked him if he could point out a method of determining in what cases all the possible parts of the impossible roots of an equation are negative. In studying the motion of governors for regulating machines, he had found that the stability of the motion depended on this

[1] These diagrams may be seen in Vol. ix of the *Quarterly Journal of Pure and Applied Mathematics*. The paper is entitled "On the Cyclide."

condition, which is easily obtained for a cubic but is more difficult to find in the case of equations of higher degrees. Clifford at once said that we obtain the condition required by forming an equation whose roots are the sums of the roots of the original equation taken in pairs and determining the condition that the real roots of this equation should be negative. This may serve to give an idea of his readiness and helpfulness at our meetings. He rarely rose to speak of his own accord, but when a direct appeal was made to him he was ever ready to contribute a few pertinent remarks of an eminently suggestive kind, often showing in the course of the discussion that he had carefully considered the subject for himself. In this way he often threw out hints which enabled others to advance still further the researches which they had hitherto almost looked upon as their own. His readiness is exemplified in the following extract from a letter I have received from his private tutor, the Rev. Percival Frost[1]: "We were capital friends, yet I was so much engaged with a large number of pupils that I did not see very much of him except in a professional way. Even when he came to see me out of his working hours we used to get upon some mathematical curiosity, and both being fond of mathematics for their own sakes, we have often pursued our amusement into the small hours—once between 2 and 3—for which his tutor called him to account, good-naturedly excusing him when he heard of how he had been occupied. He often used to amuse me by solving in his head difficult problems, when some conversation like the following would take place. *Fr.* The men in the next room tell me this problem won't come out: there must be a mistake: just read it over and tell me where the setter has blundered. *Cl.* (reads it over and thinks a few minutes) I see how it is, he has, &c., &c."

Few anecdotes of Clifford's young days which show any foreshadowing of his future mathematical power have reached

[1] See also *Nature* (March 13th, 1879).

me. The following illustration, though trifling, may find a place here. His aunt (Mrs McLeod) writes: "In the year of the first Exhibition, 1851, his parents came to town, and I had the charge of him with my own child. When putting him to bed one night, I saw dear Willie looking very thoughtful. When I asked him what it was all about, he looked up in his smiling, loving way, saying, 'Aunt Annie, I don't think you would know.' But on my asking again, he said he was calculating 'how many edges (*sharp*) of a penknife it would take to go round the wheel of a coach.'

"We had a little talk about it, for it seemed impossible to me he could arrive at any conclusion. He then gave the figures and I daresay the size of the wheel: but all that I have long forgotten." The question and answer were submitted to an uncle, the late Mr Frank Kingdon, who was no mean mathematician, and he said the result was correct within a few figures.

A passage in Mr Pollock's Introduction to the *Lectures and Essays* (p. 41) would lead one to infer that Clifford turned his attention to kites at a somewhat late period of his life: it is however clear from a letter he himself wrote to Mr Miller that he had practised the flying of kites when a boy, for in 1863 he says "I have had in my mind, almost from the time I began to fly kites (I have not yet left off), the problem of finding the form of a kite-string under the action of the wind. On a rough trial, the other day, the intrinsic equation seemed not very difficult to obtain; if I get any result, I will send it to you hereafter[1]."

One more anecdote which will give an idea of the quick-

[1] The question figures as 6009 in the *Educational Times*, July 1879, May 1880 (Reprint, vol. xxxiii. p. 59) In the same letter he adds, "I have been trying to construct a second interpretation of mechanical equations similar to that of tangential co-ordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong." Cf. *Lectures and Essays*, p. 41, "his thoughts often ran upon mechanical inventions."

ness and clearness of his perception of complicated relations in space, must close this brief sketch. Mr Frost, in his previously cited letter, goes on to say, "My brother A. H. Frost, who was in England for a short holiday from his missionary work in India, brought with him a complicated puzzle which was to be taken to pieces. It was not a two-dimension one, like many, but solid: about as big as a good snowball. My brother said, 'I have heard you talk of the wonderful capacity of Clifford, prove it to me by asking him to tea, and I will believe you if he can take my puzzle to pieces.' I accordingly asked him, and on my brother's giving him the thing, he, without fingering it but simply looking it over for a few minutes, put his head in his hands for some ten minutes and then took hold of the puzzle, and at once, to my brother's astonishment, dislocated it, and my brother believed in him ever afterwards."

These few personal details of one who stood in the front rank of the mathematicians of our time, though trivial, will not, I think, be deemed out of place in such a volume as this. Ever kindly and unselfish, Clifford maintained in his relations with his brother mathematicians the same amiable bearing which endeared him to a large circle of private friends.

II.

If it were possible to ascertain the chronological order of creation of Clifford's papers, that order would have been followed in this work: but lacking the guidance of the 'vanished hand,' I soon found that I could not attempt such an arrangement and reconciled myself, unwillingly I must say to the publication of them in almost any order. In adopting this plan I received the approval of all the mathematicians whom I consulted upon the matter. In the following Bibliographical sketch, I have endeavoured to overcome the defects of this arrangement by assigning in all cases, to the best of my ability, the lower limit of publication or, in the case of posthumous papers, of composition. It is possible that some readers may be able to furnish data for correcting my statements: such information will be gratefully accepted.

When the number of a "Reprint" problem is given in black type, this shows that the *solution* is Clifford's, but not the *problem**. The titles of unpublished papers are printed in italics.

The MSS even the roughest, are all very neatly written and by the style of penmanship indicate four or five different epochs: it is by attention to this fact that I have been led to assign the respective dates to the posthumous papers.

* "Reprint" stands for "Mathematical Questions, with their Solutions, from the *Educational Times*," edited by W. J. C. Miller; "L. and E." stands for "Lectures and Essays," edited by Leslie Stephen and F. Pollock.

A lithographed specimen of the smallest handwriting faces the title-page*.

1863.

"On Jacobians and Polar Opposites," VI. pp. 23—33.

"Analogues of Pascal's Theorem," Nov. 20th, X. pp. 72—79.

Reprint, 1362, **1373**, 1378, **1379**, 1387, **1389**, **1393**, 1399, 1409, 4097 (1415)†, **1418**, 1423, 1448, 4143 (1459).

1864.

"Analytical Metrics" (see footnote, p. 80) XI. pp. 80—109.

"Geometrical Theorem," XLVII. pp. 410, 411.

Reprint: **1319**, **1394**, **1416**, **1421**, **1442**, **1443**, 1468, 1479, 1486, **1505**, 1507, **1514**, **1517**, **1519**, 2108 (1526), 1585, 1605.

1865.

"On Triangular Symmetry," XLVIII. pp. 412—414.

Reprint: 1497, 1638, 1652, 1675, **1679**, **1680**, 1691, 1724, **1732**, **1733**, 1748, 1750, 1775, 1795, 1823.

1866.

"On some extensions of the fundamental proposition in M. Chasles's Theory of Characteristics," March, XLIX. pp. 415—418.

"On the General Theory of Anharmonics," Nov. 22. XII. pp. 110—114.

"Bitangent Circles of a Conic," pp. 543—545, and

"Of Power-coordinates in General," pp. 546—555, may belong to this year.

Reprint: 1878, **1888**, 4199 (1907), 1918, 1929, 1962, 1996, 2220, 2229, 2253, 4696 (2281), 2301.

* Professor Henrici tells me that Clifford proposed to him that they should write in conjunction a series of books on Mathematics, beginning at the very commencement and carrying the subject in each case rapidly to the most advanced stages. The question was often discussed during the years in which Clifford was in full health, but this sheet is all that remains. Professor Henrici's occupation as Examiner to the London University prevented his actively commencing his part of the work. The short abstract xi. p. 650, is suggestive as to the place which some of the papers published in this volume were intended to take in a more extended scheme.

† The number in the bracket is that belonging to the first proposal of the question, the other number is that of the re-proposed question to which the solution was given.

On Question 2229, M. Chasles (Rapport sur les Progrès de la Géométrie, p. 353), writes .

"Ces surfaces anallagmatiques du quatrième ordre n'ont point tardé à fixer l'attention des géomètres. M. W. K. Clifford, notamment, en a fait connaître plusieurs propriétés (voir *Ed. Times*, Sept. 1866, p. 134)."

1867.

"On the Principal axes of a Rigid body," VII. pp. 34—37.
Reprint: 2343, 2383, 2446, 2510, 2522.

1868.

"On some of the conditions of Mental Development," March 6. L. and E., Vol. I., pp. 75—108.
"On some Porismatic Problems," Nov. 9, III. pp. 17—19.
"On a general investigation of the theory of Polars," Nov. 26. XIII. pp. 115—118.
"On the Powers of Spheres" (?) *XXXIV. pp. 332—336. I assign the origin of this paper to this year, see my note, p. 332, though I am disposed to think that the paper was not actually written out long before 1876.
Reprint: 3885 (2674), 2732, 2748, 2776, 2793.

1869.

"On the Theory of Distances," XVI. pp. 130—164.
"On Syzygetic relations among the powers of Linear Quantics," Nov. 25. XIV. pp. 119—122.
"On Syzygetic relations connecting the powers of Linear Quantics." *XV. pp. 123—129.
"*On the Umbilici of Anallagmatic surfaces.*" Title only given in the British Association Report for this year.
"*Lectures on Geometry,*" given to a Class of Ladies at S. Kensington. Syllabus given, pp. 628—637.
"On Boundaries in General," printed at end of "Seeing and Thinking," see infra; also published in Macmillan's Magazine, August, 1879.
"Analysis of Cremona's Transformations" (?) pp. 538—542.
Reprint: 4236 (2817), 2858, 2923, 2924, 2932, 2942, 2960, 2979, 5626 (3000), 3021.

1870.

"On a case of Evaporation in the order of a Resultant," Feb. 10, XVII. pp. 165—167.

"On Theories of the Physical Forces," Feb. 18. L. and E., Vol. I. pp. 100—123.

"Proof that every rational Equation has a root," Feb. 21. IV. p. 20.

"On the Space-theory of Matter," Feb. 21. V. pp. 21, 2.

"Synthetic Proof of Miquel's Theorem," March. VIII. pp. 38—54.

"*On an unexplained contradiction in Geometry.*" Title only in British Association Report of this year.

"Lecture Notes," pp. 524—530.

Reprint: 3197, 3255, 3282.

1871.

"*The History of the Sun: being an explanation of the nebular hypothesis and of recent controversies in regard to the time which can be allowed for the evolution of life.*" April 16. L. and E. Vol. I. p. 68.

"On a Canonical form of Spherical Harmonics." August. XXV. pp. 234, 5.*

"*Note on the Secular Cooling and the Figure of the Earth.*" Title only in British Association Report for this year.

Reprint, 4034 (3308).

1872.

"Atoms." Jan. 7. L. and E., Vol. I. pp. 158—190.

"*Ether; the evidence for its Existence and the Phenomena it explains.*" April 14. L. and E., Vol. I. p. 68.

"Remarks on a Theory of the Exponential Function derived from the equation $\frac{du}{dt}=pu$." May 9. XLV. p. 406.

"*On Babbage's Calculating Machines.*" May 24. L. and E., Vol. I. p. 69.

"On the Aims and Instruments of Scientific Thought." August. L. and E., Vol. I. pp. 124—157.

* In *Nature*, Sept. 7, there is little more than the title of this paper given.

"*On the Contact of Surfaces of the Second Order with other Surfaces.*" Title only in British Association Report of this year.

An *Athenæum* correspondent says (Aug. 24), "Prof. Clifford explained that the radial polarization settles the fact of there being floating clouds of solid or liquid matter in the corona." No. 2339.

"On a Theorem relating to Polyhedra, analogous to Mr Cotterill's Theorem on Plane Polygons." Nov. 14. XVIII. pp. 168—176.

"*The Dawn of the Sciences in Europe.*" Nov. 17. L. and E., Vol. I. p. 68.

"Geometry on an Ellipsoid." Dec. 12. XIX pp. 177—180.

1873.

"The Philosophy of the Pure Sciences." March 1, 8, 15. L. and E, Vol. I. pp. 254—340.

"On the Hypotheses which lie at the bases of Geometry." May 1, 8. IX. pp. 55—71.

"*The Relations between Science and some Modern Poetry.*" May 4. Recast as "Cosmic Emotion." L. and E., Vol. I. p. 68.

"Preliminary Sketch of Biquaternions." June 12. XX. pp. 181—200.

"On Mr Spottiswoode's Contact Problems." June 19. XXXII. pp. 287—304.

"*On some Curves of the Fifth Class,*" and "*On a Surface of Zero Curvature and Finite Extent.*" Titles only in British Association Report for this year.

"Review of De Morgan's 'Budget of Paradoxes.'" August 15. pp. 559—561.

"Graphic representation of the Harmonic Components of a Periodic Motion." Dec. 11. XXI. pp. 201—204.

"Syllabus of Lectures on Motion." (?) pp. 516—523.

Reprint: **3876**, 3961, 3980, 4010, 4069.

1874.

"Review of Vol. I. of G. H. Lewes' 'Problems of Life and Mind.'" Feb. 7. L. and E., Vol. I. p. 68.

"The First and Last Catastrophe." April 12. L. and E., Vol. I. pp. 191—227.

"*On the Education of the People.*" May 22. L. and E., Vol. I. p. 68.

"*On a Message from Prof. Sylvester,*" and "*On the General Equations of Chemical Decomposition.*" Titles only given in the British Association Report for this year.

"Prof. Clifford exhibited a jointed frame in illustration of some beautiful discoveries recently made in connexion with what machinists call parallel motion." *Athenæum*, Sept. 5. No. 2445 (alluding to the discoveries of Peaucellier, Hart, and Sylvester).

"This paper (*Chemical Decomposition*) was read before Section A. The Author thinks that Chemical Equations may be brought under a general formula. Thus, $H_2+Cl_2=2HCl$. If we assume that there is a structure common to the hydrogen and the chlorine atoms, also a structure confined to the hydrogen, and likewise a structure confined to the chlorine atoms, we may represent the equation thus: $XYY+XZZ=2\ XYZ$, when X represents the common structure, and Y and Z the structures which are confined to hydrogen and chlorine respectively. So $2H_2+O_2=2H_2O$ may be represented thus: $2\,XY+XXZZ=2\,XXYZ$. These equations involve no hypotheses, because the fundamental facts of the molecular theory are now firmly established. Reasoning from these and similar equations, the author deduces the result that the ordinary equations of chemistry, such as those just stated, are expressive of facts, and that the hydrogen molecule really consists of two equal atoms." *Nature*, Sept. 24, No. 256.

"Body and Mind." Nov. 1. L. and E., Vol. II. pp. 31—70.

"On the Nature of Things in themselves." L. and E, Vol. II. pp. 71—88.

"Seeing and Thinking." December. Published in 1879 (*Nature* Series).

"Motion of a Solid in Elliptic Space." *XLI. pp. 378—384. (See note, p. 378, but I do not think the paper was *written* until 1876).

1875.

"*The general features of the History of Science.*" Feb. 27, March 6, 13, 20. L. and E., Vol. I. p. 69.

"*Ultramontanism.*" April 28. L. and E., Vol. I. p. 68.

"The Unseen Universe." June. L. and E., Vol. I. pp. 228—253.

"On the Scientific Basis of Morals." L. and E., Vol. II. pp. 106—123.

"*On the Theory of Linear Transformations:* (i) *the Graphical representation of Invariants;* (ii) *the Expansion of Unsymmetrical*

Functions in Symmetrical Functions and Determinants; (iii) *the Notation of Matrices.*" Title only in British Association Report of this year.

"Prof Clifford astonished the section by some remarkable applications of Grassmann's "polar multiplication" defined by the law that *ba* is minus *ab*. He applied it to the graphical representation of invariants, to the expansion of unsymmetrical functions, and to the notation of matrices, illustrating his remarks by drawings representing atoms hung together in various ways."—*Athenæum*. Sept. 11th. No. 2498.

"A Fragment on Matrices," *XXXV. pp. 337—341, perhaps belongs to this period.

"Right and Wrong: the scientific ground of their distinction." Nov. 7th. L. and E., Vol. II. pp. 124—176.

"On the Transformation of Elliptic Functions." Dec. 9th. XXII. pp. 205—217.

Reprint: 4641, 4819, 4843.

1876.

"On the Free Motion under no forces of a Rigid System in an n-fold homaloid." (Provisional Notice.) Jan. 13th. XXVI. pp. 236—240.

"*Sight and what it tells us.*" Feb. 24. L. and E., Vol. I. p. 69.

"Instruments used in Measurement." L. pp. 419—423; L. and E., Vol. II. p. 3—8.

"Instruments illustrating Kinematics, Statics and Dynamics." LI. pp. 424—440; L. and E., Vol. II. pp. 9—30.

"The Ethics of Belief." L. and E., Vol. II. pp. 177—211.

"Notes on the communication entitled 'On the Transformation of Elliptic Functions.'" XXII. pp. 218—228.

"On the Classification of Geometric Algebras," *XLIII. pp. 397—401.

"On In-and-circumscribed Polyhedra" (?) *XXIV. pp. 229—233.

"On the Theory of Screws in a space of constant positive curvature" (?) *XLIV. pp. 402—405.

"On Tricircular Sextics" (?) *XXXVI. pp. 342—345.

"On the Double Theta-functions" (?) *XL. pp. 368—377.

"Further Note on Biquaternions" (?) *XLII. pp. 385—396.

Reprint: 4871, 4897, 4925, 4950, 4972, 4996.

1877.

"On the Types of Compound Statement involving four classes." Jan. 9th: I. pp. 1—13. L. and E., Vol. II. pp. 89—106.

"The Ethics of Religion." March 4. L. and E, Vol. II. pp. 212—243.

"The Influence upon Morality of a Decline in Religious Belief." April: L. and E., Vol. II. pp. 244—252.

Review of Dr Booth's "New Geometrical Methods." June: pp. 562—564.

"On the Canonical Form and Dissection of a Riemann's Surface." June 14th: XXVII. pp. 241—254.

"Cosmic Emotion." October: L. and E., Vol. II. pp. 253—285.

"Notes on Vortex-Motion, on the Triple-generation of Three-bar Curves, and on the Mass-centre of an Octahedron." November 8th: XLVI. pp. 407—409.

"Enumeration of the Types of Compound Statements." (?) * II. pp. 14—16.

In the Michaelmas Term of this year (cf. L. and E., Vol. I. p. 27) Prof. Clifford delivered a course of ten lectures on Quaternions with a view to their physical applications, for students of Physics, "who are not able or willing to read very high Mathematics."

For an admirable collection of Notes of these Lectures (see *infra*, pp. 478—515), I am indebted to the kindness of Miss Ellen Watson*, who also placed at my disposal other Lecture Notes which I have not made use of in the present work. Mr G. Griffith, of Harrow, who also attended the Course, lent me his skeleton Notes which enabled me to clear up one or two minor points.

Two other courses were announced. In the Lent Term of 1878, "On Elliptic Functions and some of their Physical Applications treated on a basis of Elementary Algebra, but assuming a Knowledge of the Elements of the Differential Calculus†."

* Miss Watson was the first woman to enter the Classes of Mathematics at University College, London. Of her, Clifford said, "that her proficiency would have been very rare in a man," and that "he was totally unprepared to find it in a woman."—She was obliged to leave England on account of failing health, and died in December 1880, at Grahamstown, South Africa.

† One or two Lectures of this course were delivered, but the few Notes of them in Miss Watson's MSS. are too fragmentary to be of service.

These were to be followed by a third Course in the Midsummer Term, "On Spherical Harmonics and other functions rising out of the Theory of the Potential and Allied Theories treated by means of Partial Differential Equations and series with special attention to problems of Electricity."

"Algebraic Introduction to Elliptic Functions" (? commenced and added to at different times), pp. 443—477.

"On Groups of Periodic Functions" (?) * XXXVIII. pp. 350—355. Reprint: 5304, 5457.

1878.

"Remarks on the Chemico-Algebraical Theory." XXVIII. pp. 255—257.

"Virchow on the Teaching of Science." April: L. and E., Vol. II. pp. 286—321.

"On the Classification of Loci." April 8: XXXIII. pp. 305—329.

"Childhood and Ignorance." May: L. and E., Vol. I. p. 70.

"Applications of Grassmann's Extensive Algebra." XXX. pp. 266—276.

"Elements of Dynamic: an Introduction to the Study of Motion and Rest in Solid and Fluid Bodies." Part I. Kinematic.

It may not be out of place to give here an analysis of what Clifford intended to give in Vol. II. of his book, the manuscript of which is now in my possession.

BOOK IV. FORCES.

Cap. I. The Laws of Motion.

Cap. II. The Conditions of Equilibrium of a Rigid Body.

Cap. III. The Composition of Forces.

§ 1. The Link-polygon, Reciprocal diagrams, &c.

§ 2. Centres of Inertia, Second Moment, Cores.

§ 3. Attractions, Potential and Level Surfaces, Sources and Sinks. Theorems of Stokes and Chasles. Electric Images. Centrobaric Bodies.

Cap. IV. Motion of a Rigid Body.

BOOK V. STRESSES.

Cap. I. Solids.

"Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general." (?) *XXIX. pp. 258—265.

"Binary Forms of Alternate Variables." (?) *XXXI. pp. 277—286.

"On Bessel's Functions." *XXXVII. pp. 346—349.

"Theory of Marks of Multiple Theta-functions." (?) *XXXIX. pp. 356—367.

The manuscript of "The common sense of the Exact Sciences" is in Prof. Rowe's hands, and is almost ready for publication.

All introduced matter is, with one or two exceptions, included in [ ]: this has necessitated the adoption of another form of *bracket* when [ ] occurs in Clifford's papers. I am in the main responsible for these additions, though I have in all cases submitted the suggestions to one or more of the mathematicians mentioned in my dedicatory letter: their own additions are suitably initialled.

INTRODUCTION

BY

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INTRODUCTION.

It will be generally admitted that the publication in a collected form of the works of the eminent men, who have moulded the mathematical sciences into their present form, has become little less than a necessity to those who desire to follow in their footsteps, and to advance, if possible, beyond the limits attained by them. And mathematicians will gratefully acknowledge that no inconsiderable progress has already been made towards satisfying this requirement. To the Academy of Sciences at Paris we are indebted for magnificent editions of the complete works of Laplace and of Lagrange; the Government of Norway has given to the world the collected memoirs of Abel; the Academy of Goettingen has fulfilled the same duty toward the great names of Gauss and Riemann; and the Academy of Berlin has followed the example by undertaking editions of the works of Steiner and Jacobi. In our own country we have the collected works of Green, of Mac Cullagh, of Gregory, of Leslie Ellis, and of Macquorn Rankine; not to mention the volumes of reprinted memoirs which we owe to living writers; for example, to Sir William Thomson and to Professor Stokes. Such collections, we may hope, will form an increasing portion of every scientific library. At the present time the results of mathematical research almost always appear in the Transactions of Societies, or in periodicals specially devoted to mathematical writings: the contents even of the most original treatises being generally anticipated by their authors in memoirs, which are often not wholly superseded

by the works themselves. But the number of the periodical repositories of mathematical literature has become so great, that papers consigned to them, although preserved, as we may hope, for all time, are in imminent danger of passing out of sight within a few years after their first appearance. They are preserved from destruction, but not from oblivion; they share the fate of manuscripts hidden in the archives of some great library from which it is in itself a work of research to disinter them. This "mislaying," if it may be so termed, of important memoirs is not only a loss to the history of science, but interferes seriously with the discovery of new knowledge. For notwithstanding the ardour with which mathematical investigation is at present pursued in every direction, a much longer time than is perhaps sometimes supposed elapses before a mathematical work of genuine originality, be it a brief note, or an elaborate treatise, becomes antiquated. It would be out of place in this connexion to mention the Principia of Newton, which stands apart by itself, and of which not only the methods and results, but even the very words have become the common property of all men of science. Nor need we even refer to works which have marked the beginning of a new epoch in their respective departments, such as the Mécanique Céleste, the Mécanique Analytique, the Disquisitiones Arithmeticæ, the Traité Analytique de la Chaleur, the Fundamenta Nova, or the Systematische Entwickelung der Geometrischen Gestalten—the freshness of which time has hardly impaired, while the superstructures which have been based upon them have added incalculably to their importance. But leaving out of count these and other great classics of the science, the trains of thought hidden in the opuscula of Euler, of Lagrange, of Gauss, of Poisson, of Cauchy, of Abel, and Jacobi, are still unexhausted; and, far from having lost their value by the lapse of time, have in many cases acquired an increased suggestiveness from the light which the more extended knowledge of recent times has thrown upon them.

Such a prospect of future and long-continued usefulness, we may venture to hope, awaits many of Clifford's memoirs; and, even more than the immediate interest attaching to them, justifies, if any justification be needed, their appearance in their present collected form.

It might be interesting to enquire why it is that mathematical writings retain a scientific (as opposed to a merely historical) value for a longer time than memoirs recording researches in the sciences of experiment and observation. Among many partial answers which might be given to this question, one is suggested by the character of many of Clifford's papers, and has its foundation in the nature of the subjects with which they deal. Speculation in pure mathematics resembles metaphysical speculation in this, that the whole universe of thought to which it refers is so closely inter-dependent, that a clear-sighted and powerful thinker cannot fix his mental vision (however keen his effort after concentration may be) on any one region in it, without catching glimpses of something that lies beyond, and without discerning, more or less dimly, new relations to be examined, and new lines of research, which may perhaps have no immediate relevancy to the particular enquiry in which he is engaged. And these glimpses, if recorded, or even if only half unconsciously indicated, in the account which he afterwards gives of his work, are not unlikely to suggest a wholly new departure to some kindred spirit in a future time. On this ground, more strongly perhaps than on any other, we may venture to commend the present volume to the rising generation of English mathematicians. The collection includes papers—some of them youthful efforts—some suggested, one might say casually, by the researches of scientific friends—which relate to special problems, and which nevertheless would be sufficient of themselves to establish a considerable mathematical reputation. There are others, again, the work of a maturer time, and planned with a wider scope, which are models of artistic perfection, in respect both of

the clearness and depth of the thought, and of the manner of its presentation. Lastly, besides these finished pieces there are others, rough-hewn and imperfect in execution, but conveying a still stronger impression of the fertility of invention, and of the far-reaching power of mental vision, with which Clifford was endowed. Some of these fragmentary records are full of great ideas, shadowed forth in outlines, not always free from indistinctness, but always suggesting long vistas of future discovery, the path to which seems for the moment to lie clear before his mind. Their very incompleteness reminds us how much the world has lost by losing him; and brings home to us the melancholy feeling that, however highly we may estimate the work which he actually accomplished during his brief life-time, he must nevertheless be counted among "the inheritors of unfulfilled renown."

But if the republication of Clifford's papers stands in no need of any justification, some apology is wanted for an Introduction which can offer but little interest to those who do not intend to study the volume itself, and which to those who do, must seem at the best superfluous. Perhaps, however, even in these days of increasing specialisation, there may still be found an intermediate class of readers, who are not mathematicians by profession, and who nevertheless do not regard analysis and geometry as volumes sealed except to the initiated few, but as belonging, in their results at least, to the whole world of science. Some persons, one is willing to believe, partly from a recollection of their own early studies, and partly from a general sympathy with all branches of intellectual activity, are disposed to follow with an appreciative, or at least an indulgent curiosity, the exposition of new mathematical ideas. For such friendly readers these pages are intended; and it would be strange if the class of persons, among whom they are to be found, has not been considerably increased by the admirable lectures[1] in which Clifford

[1] *Lectures and Essays*, by W. K. Clifford, Vol. I. pp. 254—340.

has himself analysed, in popular phraseology but with the utmost scientific precision, the fundamental principles of geometry and arithmetic, as they appear in the "fierce light" which has been turned upon them by a controversy in which both metaphysicians and mathematicians have taken part.

All then that can with any propriety be attempted here is, in the first place, to characterise some one or more of the principal trains of thought which seem to have exercised an abiding influence on Clifford's mind; and then to classify his memoirs in a few main groups, pointing out the central ideas of each group, and shewing, as far as possible, the interdependence of these ideas upon one another.

Clifford was above all and before all a geometer. Nine-tenths, and more, of the contents of this volume, including nearly all the systematic researches recorded in it, are geometrical. It is true that in the treatment of geometrical questions he shows a marked preference for symbolical methods; and, as might be expected, a marvellous command over analytical expression. It may even be true that the limitations involved in a scrupulous adherence to the methods of pure geometry would have been distasteful to him. Of his skill in the use of these special methods the "Problems and Solutions" so liberally contributed by him to the *Educational Times* afford abundant proof. But among his more elaborate papers there is perhaps but one, the "Geometry on an Ellipsoid," which would satisfy purists of the school of Poncelet and Chasles, as being wholly free from the contamination of analytical methods; and even in this beautiful application of the method of stereographic projection—in the generalized sense in which that method is used in modern pure geometry—the "imaginary circle at infinity" occurs in the first sentence. But, whatever the method employed—and in variety of method Clifford takes an evident pleasure—the properties of space remain the perpetual subject of his contemplation. Purely analytical researches undertaken without any impulse from or

reference to geometry, are few and far between. For even the Elliptic and Abelian functions were approached by Clifford from the side of geometry. His early note "On some Porismatic Problems," relating to the theorem of Poncelet, which asserts that "given two conics, a polygon of a given number of sides, all whose vertices shall lie on one of them, and all whose sides shall touch the other, can either not be drawn at all, or else can be drawn in an infinite number of different ways," led him to the study of the connexion, established by Professor Cayley, between this theorem and the addition of elliptic functions of the first species; and thus to the discovery of a geometrical theory of the transformation of elliptic functions, which forms the subject of one of his most brilliant investigations (XXII.). But in his further prosecution of the subject the elliptic functions again disappear from view, and he returns to the geometrical doctrine of correspondences, and to the theory of the polyhedra, of which the faces osculate a skew cubic curve.

In like manner it would appear that he was attracted to the consideration of the Abelian Integrals by their relation to another and widely different part of geometry, the Geometry of Situation, as it has been termed. His memoir on the "Canonical Dissection of a Riemann Surface," founded on the researches of Clebsch and Lüroth, contains the simplest account which has yet been given of this important chapter of a great theory; and the reduction of a Riemann surface to the surface of a solid having a certain number of holes through it, presents to the mind what is perhaps the clearest image which it is possible to obtain of the space of two dimensions upon which a many-valued algebraical function can be mapped with the same distinctness with which a one-valued function can be mapped on a plane. But the study of the Abelian Integrals—however geometrical may have been the form in which he first envisaged this theory, led him by an inevitable sequence of ideas to the Theta functions, which form the indispensable

basis for a study of the relations subsisting between the upper limits of a system of algebraical integrals, and the values of the integrals themselves. His posthumous and unfinished memoirs on the Double and Multiple Theta functions (XXXVIII., XXXIX. and XL.) form a not unimportant contribution to pure analysis; and, though now published long after their proper date, are, it may be hoped, neither too late, nor too incomplete, to exercise some influence on the development of this rapidly growing theory. The "Algebraical Introduction to Elliptic functions," which is in fact a treatise on the single Theta functions, probably took its rise in connexion with these ulterior researches; Clifford desiring to obtain a complete command over the manipulation of these series in the simplest cases, before proceeding to apply them to the general problem of the inversion of algebraical integrals.

Enough has been said to show that Clifford's predilection for geometry lay deep. But to this his favourite science he attributed the widest imaginable scope, and at times regarded it as co-extensive with the whole domain of nature. He was a metaphysician (though he would only have accepted the name subject to an interpretation) as well as a mathematician; and geometry was to him an important factor in the problem of "solving the universe[1]". Thus he was a geometer of a type peculiarly his own; and his dealings with the science were characterized by an amount of scepticism and an amount of faith which one would hardly expect to find combined in a mathematician. He had early read and translated Riemann's celebrated discourse (IX.) "On the Hypotheses which lie at the Basis of Geometry," and had imbibed the views set forth in it as a part of his intellectual nature. Some men who have an ardent love for new knowledge find it

[1] "At the Savile I met C.... and solved the universe with great delight until A. came in and wanted to take him off... Of course I would not let him go... We walked about in the New Road solving more universe." *Lectures and Essays*, Introduction, p. 30.

difficult to maintain an unflagging interest in geometry, because they regard it as a purely deductive science, of which the first principles (axioms, postulates and definitions), whether derived from experience or not, are unquestionable, and contain implicitly in themselves all possible propositions concerning space. Thus the unknown, or at least the unforeseen, seems to be excluded from geometry, because whatever may be found out hereafter must be latent in what is already known. But upon the view put forward by Riemann and adopted by Clifford, the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection, and not as axioms to be accepted on the authority of universal experience, or of the inner consciousness.

These speculations had so much influence on a great part of Clifford's work that it may not be out of place to pursue the subject a little further. In his lecture "On the Postulates of the Science of Space[1]", he has stated his own views on the question with singular clearness and brilliancy; and the pages in which he has expressed them are likely to be remembered, as marking an important moment in the controversy concerning the nature of space and the origin of our knowledge of it, which is likely to last as long as metaphysical enquiries have any interest for mankind. In this lecture he enumerates four fundamental postulates on which the ordinary conception of space is founded, (1) its continuity, (2) its flatness in its smallest parts, (3) its similarity to itself at every point, or, which is the same thing, the possibility of the existence of the same figure in any two different places, (4) the possibility of the existence of figures similar to one another, but on different scales of magnitude. The second of these postulates requires some comment to make it intelligible. Perhaps the simplest account that can be given of a space which is flat in its smallest parts is, that if anywhere

[1] *Lectures and Essays*, Vol. I. pp. 295—325.

in it we take three points very near to one another and join them by the shortest lines that can possibly be drawn, the triangular figure so formed will lie very nearly in a plane; the mathematical equivalent of this statement being that the square of the distance between any given point and any other infinitely near to it can always be expressed as a homogeneous quadratic form, in which the indeterminates are the infinitesimal differences between the co-ordinates of the two points, and the coefficients are functions of the co-ordinates of the given point. If we go further and join one of the three infinitely near points by the shortest lines possible to every point on the line already joining the other two, the assemblage of these lines will form a triangular surface-element: if this surface-element is absolutely plane, whatever be the three infinitely near points which we have taken, the space is flat; if the surface-element has a finite curvature, the space, while retaining the property of elementary flatness, is said to have curvature; and this curvature is measured, for the surface direction determined by the three points, by the curvature of the surface-element which we have constructed.

As to the first postulate, Clifford indicates his readiness to adopt either of the two opposite hypotheses that space is continuous or that it is discontinuous, while admitting fully that no phenomena have yet been observed which point to its discontinuity.

Of the second postulate, in this respect following Riemann, he speaks in the same general terms; we must not shrink from rejecting it, if its rejection should be found to assist us in the explanation of natural phenomena. The postulate is not inconsistent with a hypothesis which at one time was a great favourite with him, and which he has described in a remarkable communication (v.), presented to the Cambridge Philosophical Society, in 1870. In this brief note, comprised within a single page, he appears to adopt the hypothesis (for his language on the point is not quite free from ambiguity), that space

has everywhere a finite curvature, but that this curvature is continually changing, and that all the phenomena of the universe may possibly consist in changes of the curvature of space. A finite curvature, it will be remembered, is consistent with, and indeed implies elementary flatness. Unfortunately Clifford, though in earlier days he was fond of discussing this theory, no doubt as one possible mode of "solving the universe," has left no memoranda relating to it, perhaps because the efforts which he made to work it out in detail, led him to no satisfactory conclusion. In the note of 1870 he speaks of it with a confidence which must not be taken too literally. He would probably have allowed that Lord Bacon's criticism on Gilbert, "postquam in contemplationibus magnetis se laboriosissime exercuisset, confinxit statim philosophiam consentaneam rei apud ipsum præpollenti," admitted of an application, *mutatis mutandis*, to his own effort to resolve all philosophy into geometry; though he would no doubt have maintained with the utmost depth of conviction that, for aught we know to the contrary, the properties of space may change with time.

But whatever importance he may have temporarily attached to the opinion that space may not be independent of time, this idea has left no other perceptible traces in his mathematical writings. Very different is the case with another hypothesis as to the nature of space, which is somewhat less widely divergent from ordinary conceptions, and to which Clifford appears at all times to have turned with peculiar favour.

This hypothesis admits the first three of the postulates enumerated above, as expressing true properties of space, but rejects the fourth, substituting for it the new postulate that space has a finite but very small curvature, which is approximately the same for any two points, and for any two surface directions at the same point. Admitting this postulate, we find ourselves in the presence of two alternatives, between which we have to choose. For we may imagine either that

the curvature of all surface-elements, constructed in the manner above described, has the same positive value, or that it has the same negative value; understanding by a positive curvature a curvature such as that of the outer portion of the surface of an anchor-ring, where the tangent plane at any point just meets the surface and does not cut it; and by a negative curvature a curvature such as that of the inner portion of the same surface, where the tangent plane cuts the surface at the point of contact. To the hypothesis that space has a constant negative curvature considerable historical interest attaches. For this hypothesis was first arrived at, not by following out such general views as those indicated by Riemann, but in a much more elementary manner. The celebrated twelfth axiom, as is well known, is the basis of Euclid's theory of parallel lines; and the assertion made in it is in fact equivalent to an assumption of the fundamental proposition of plane geometry, that the three angles of a triangle are equal to two right angles. It is now universally allowed that all efforts to demonstrate Euclid's axiom have failed; but the Russian mathematician, Lobatchewsky, appears to have been the first person to whom the idea occurred of dispensing with the axiom altogether, and trying to see what would become of geometry without it. The idea was obvious, but it was also profound; and Lobatchewsky was rewarded by the discovery that it is possible to construct a consistent and complete system of geometry upon the hypothesis that the three angles of a triangle are less than two right angles. Till the discovery of Lobatchewsky, the only substantial addition that had been made to Euclid's theory of parallel lines was a demonstration by Legendre, that the angles of a triangle cannot be greater than two right angles. As a matter of fact the demonstration of Legendre depends on the assumption that space is infinite; an assumption which, from the point of view taken by Riemann, cannot be regarded as justified by experience: but the considerations upon which the demonstration rests decided Lobat-

chewsky, as between the two alternative hypotheses that the angles of a triangle are less, and that they are greater, than two right angles, to adopt the former. This was in effect to adopt the hypothesis (though it does not appear to have occurred to Lobatchewsky in that light) that a plane has negative curvature. It was reserved for an Italian mathematician, Beltrami, to show that the plane geometry of Lobatchewsky is identical with the geometry of a pseudo-spherical surface, *i.e.* of a surface of constant negative curvature.

What Clifford thought of the philosophical importance of the work of Lobatchewsky the following quotation may serve to show:

"Each of them [Copernicus and Lobatchewsky] has brought about a revolution in scientific ideas so great that it can only be compared with that wrought by the other. And the reason of the transcendent importance of these two changes is that they are changes in the conception of the Cosmos. Before the time of Copernicus men knew all about the universe. They would tell you in the schools, pat off by heart, all that it was, and what it had been, and what it would be.... In any case the universe was a known thing. Now the enormous effect of the Copernican system, and of the astronomical discoveries that have followed it, is that, in place of this knowledge of a little, which was called knowledge of the universe, of Eternity and Immensity, we have now got knowledge of a great deal more; but we only call it the knowledge of Here and Now.... This then was the change effected by Copernicus in the idea of the universe. But there was left another to be made. For the laws of space and motion implied an infinite space and an infinite duration, about whose properties as space and time everything was accurately known. The very constitution of those parts of it which are at an infinite distance from us, 'geometry upon the plane at infinity,' is just as well known, if the Euclidean assumptions are true, as the geometry of any portion of this room.... So that here we have real knowledge of something at least that concerns

the Cosmos; something that is true of the Immensities and the Eternities. That something Lobatchewsky and his successors have taken away. The geometer of to-day knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or a future eternity. He knows, indeed, that the laws assumed by Euclid are true with an accuracy that no direct experiment can approach...; but he knows this as of Here and Now; beyond his range is a There and Then, of which he knows nothing at present but may ultimately come to know more. So, you see, there is a real parallel between the work of Copernicus and his successors on the one hand, and the work of Lobatchewsky and his successors on the other."—*Lectures and Essays,* Vol. I. pp. 298—300.

But in spite of this eulogium, the conception of space which has left the deepest traces in Clifford's writings, is not that of Lobatchewsky, but that founded on the alternative hypothesis (rejected by the Russian geometer) of a constant positive curvature. This conception lies at the bottom of Clifford's theory of biquaternions, to which he devoted much continuous thought, and which was the origin of his researches into the classification of geometric algebras. A space of constant positive curvature is most easily represented to the mathematician (in the absence of any possibility of imaging it to the mind) as the locus of an equation of the form

$$x^2 + y^2 + z^2 + w^2 = \text{constant}$$

in a flat space of four dimensions in which $xyzw$ are rectangular co-ordinates. It is related to the two dimensional surface of a sphere, just as in ordinary geometry space of three dimensions is related to a plane surface. The following description of a space of this kind is taken from the Lecture "On the Postulates of the science of Space." It can hardly be necessary to point out, that in the last sentence Clifford is half laughing at himself.

"I cannot perhaps do better than conclude by describing to you as well as I can what is the nature of things on the supposition that the curvature of all space is uniform and positive.

"In this case the universe, as known, becomes again a valid conception; for the extent of space is a given number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever and move in that direction in a perfect straight line according to the definition of Leibnitz; after travelling a most prodigious distance, to which the parallactic unit 200,000 times the diameter of the earth's orbit would be only a few steps, you would arrive at—this place.... Upon this supposition of a positive curvature the whole of geometry is far more complete and interesting: the principle of duality, instead of half breaking down over metrical relations, applies to all propositions without exception. In fact, I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."—*Lectures and Essays*, Vol. I. pp. 322—3.

A third line of thought, different from those followed by Lobatchewsky and by Riemann, had no doubt a large share in determining Clifford to regard the hypothesis of constant positive curvature with special favour. One of the earliest geometrical enquiries of wide scope which interested him was the connexion between the descriptive and metrical properties of figures. In two unfinished memoirs "On Analytical Metrics" (XI.), and "On the Theory of Distances" (XVI.), he applied himself to work out the conception, which he justly attributed to Poncelet, that the metrical properties of any figure are in reality descriptive properties of the figure considered in relation to certain fixed geometrical elements, which Professor Cayley has termed the Absolute. In the ordinary geometry of a plane, the Absolute consists of two fixed imaginary points, and of the real straight line containing them, [the imaginary circular points, and the straight line, at an infinite distance].

For these two imaginary points, in the geometry of a spherical surface, we have to substitute the imaginary circle in which the sphere is cut by the plane at an infinite distance. In ordinary space of three dimensions the Absolute is the same imaginary circle and the plane at an infinite distance in which it lies. Professor Cayley in his celebrated "Sixth Memoir on Quantics", generalized this conception by substituting any quadric whatever for the imaginary circle at an infinite distance (which may be regarded as a quadric surface of which one dimension has vanished). The effect of this substitution is to change the metric properties of space, the nature of the change depending on the nature of the quadric chosen as the Absolute. If we wish the space which we thus bring under contemplation to possess one of the most obvious properties which we know by experience to characterize the space in which we live, (viz. that the rotation round a fixed axis, which brings a body from any given position back into the same position again, is a finite and not an infinite operation), our choice of the form of the Absolute is limited to three hypotheses, (1) the Absolute is an imaginary quadric, (2) the Absolute is a real umbilical quadric (*i.e.* a quadric not having real right lines on it) and the space considered is internal to the quadric, (3) the Absolute is an imaginary quadric which has degenerated into a conic section by losing one of its dimensions. Of these three hypotheses the last corresponds to the ordinary conception of space: the spaces characterized by the suppositions (1) and (2) have been termed elliptic and hyperbolic respectively by Professor Klein, who succeeded in showing that in each of them the curvature is constant, being positive in the elliptic, and negative in the hyperbolic space. Thus the geometry of Lobatchewsky is the geometry of hyperbolic space: and Professor Klein's discovery of the identity of the two has thrown a wholly new light upon the researches of the former geometer. Of Clifford's study of the details of the system of Lobatchewsky only one brief note

is preserved (Appendix, p. 531). Indeed he seems to have quickly abandoned hyperbolic for elliptic geometry, influenced no doubt by the reason indicated in the passage which we have quoted—the perfect duality of the properties of elliptic space.

In the geometry of Lobatchewsky every straight line has two real points on it at an infinite distance, viz. the two real points in which it intersects the Absolute. Again, among the planes which pass through a given straight line there are two which belong to the Absolute, and which therefore are to be regarded as planes at an infinite distance. But these two planes are imaginary, being the two planes which can be drawn through the given line to touch the Absolute. Thus in the hyperbolic geometry there is no perfect duality, because when we compare the points which lie along a line, and the planes which pass through it, the absolute elements are real in the one case, and imaginary in the other: in fact, the space which is the dual correlative of an hyperbolic space is not itself a similar space, but is analogous to the space *outside* the Absolute of the hyperbolic space. On the other hand, in elliptic geometry all the elements of the Absolute, whether points or planes, are imaginary, and the duality is as perfect as it is on the surface of a sphere. It follows at the same time that all distances as well as all rotations are finite, and that a point moving on a straight line (or more properly on a shortest line) will come round after a finite journey to the point from which it set out, just as a plane revolving round a straight line returns after describing a finite angle of 360° to its original position.

We proceed to give an enumeration of the memoirs contained in this volume, grouped according to their subjects. Perfect accuracy is not important, nor indeed attainable, in such a classification, of which the only object is to convey a general impression of what Clifford has done in each department

of mathematical science. The grouping, however rough, will of itself serve to distinguish the problems with which he occupied himself habitually, and by deliberate preference, from those which had only a temporary interest for him, and were suggested by some accidental circumstance. The enumeration is followed by a few remarks on the methods or results of some of the more important papers.

A. ANALYSIS.

(*a*) *Mathematical Logic.*

(1) On the types of compound statement involving four classes, (I.).

(2) Enumeration of the types of compound statement, (II.).

(*b*) *Theory of Equations and of Elimination.*

(1) Proof that every equation has a root, (IV.).

(2) On a case of Evaporation in the order of a Resultant, (XVII.).

(*c*) *Abelian Integrals and Theta Functions.*

(1) On the Canonical form and Dissection of a Riemann's surface, (XXVII.).

(2) On groups of Periodic Functions, (XXXVIII.).

(3) Theory of Marks of Multiple Theta Functions, (XXXIX.).

(4) Double Theta Functions, (XL.).

(*d*) *Invariants and Covariants.*

(1) Remarks on the Chemico-Algebraical Theory, (XXVIII.).

(2) Notes on Quantics of Alternate Numbers, (XXIX.).

(3) Binary forms of Alternate Variables, (XXXI.).

(e) *Miscellaneous.*

(1) Remarks on a Theory of the Exponential Function, (XLV.).

(2) On Bessel's Functions, (XXXVII.).

(3) On a Canonical form of Spherical Harmonics, (XXV.).

(4) A Fragment on Matrices, (XXXV.).

B. GEOMETRY.

(a) *Projective and Synthetic Geometry.*

(1) On some Porismatic Problems, (III.).

(2) A Synthetic proof of Miquel's Theorem, (VIII.).

(3) Analogues of Pascal's Theorem, (X.).

(4) Analytical Metrics, (XI.).

(5) On the Theory of Distances, (XVI.).

(6) Geometry on an Ellipsoid, (XIX.).

(7) On the general Theory of Anharmonics, (XII.).

(8) A Geometrical Theorem, (XLVII.). [An early note on the Properties of the Quadrilateral].

(9) Triangular Symmetry, (XLVIII.).

(10) On the Powers of Spheres, (XXXIV.).

(11) Tricircular Sextics, (XXXVI.).

(12) On the Triple Generation of Three Bar Curves, (XLVI. ii.).

(13) On the Mass-centre of an Octahedron, (XLVI. iii.).

(14) On some extensions of the Fundamental Proposition in M. Chasles's Theory of Characteristics, (XLIX.).

(b) *Applications of the Higher Algebra to Geometry.*

(1) On Jacobians and Polar Opposites, (VI.).

(2) On a Generalization of the Theory of Polars (XIII.).

(3) On Syzygetic relations among the powers of Linear Quantics, (XIV.).

(4) On Syzygetic relations connecting the powers of Linear Quantics, (XV.).

(5) On a Theorem relating to Polyhedra, (XVIII.).

(6) On Mr Spottiswoode's Contact Problems, (XXXII.)

(c) *Geometrical Theory of the Transformation of Elliptic Functions.*

(1) On the Transformation of Elliptic Functions, (XXII.).

(2) Note on the preceding communication, (XXIII.).

(3) On In- and Circum-scribed Polyhedra, (XXIV.).

(d) *Kinematics.*

(1) On the principal Axes of a Rigid Body, (VII.).

(2) On a Graphic representation of the Harmonic Components of a Periodic Motion, (XXI).

(3) On Vortex Motion, (XLVI. i.). [A quaternion solution of a Kinematical Problem].

(4) Instruments used in Measurement.—Instruments illustrating Kinematics, Statics and Dynamics, (L. and LI.). [From the Hand-book to the Special Loan Collection of Scientific Apparatus].

(e) *Generalized Conceptions of Space.*

(1) On the Hypotheses which lie at the basis of Geometry, (IX.) [Translation of a discourse of Riemann].

(2) Preliminary Sketch of Biquaternions, (XX.). [Elliptic space].

(3) Further note on Biquaternions, (XLII.). [Elliptic space].

(4) Motion of a Solid in Elliptic space. (XLI.).

(5) On the Theory of Screws in a space of constant positive curvature, (XLIV.). [Elliptic space].

(6) Applications of Grassmann's extensive Algebra, (XXX.).

(7) On the classification of Geometric Algebras, (XLIII.).

(8) On the Free Motion under no forces of a rigid system in an n-fold Homaloid, (XXVI.).

(9) On the Classification of Loci, (XXXIII.).

A (*a*). The paper (I.) on the types of compound statement involving four classes, and the note (II.) upon the general problem of compound statements involving any number of classes, belong to the theory of combinations, which at no time appears to have engaged Clifford's attention continuously. Indeed the only other problems of combinatorial analysis which he has treated are those relating to the marks or systems of indices of the multiple Theta functions.

He appears to have undertaken a discussion of the problem of compound statements rather from an interest casually awakened by its acknowledged difficulty (see the Editor's note, p. 16), than from any predilection for the mathematical theory of Logic, to which (so far as can be gathered from his remaining papers) he never returned again.

A (*b*). The proof given in (IV.) of the fundamental theorem that every equation has a real quadratic factor, is very remarkable because it depends solely on the theory of elimination. A proof substantially the same was afterwards obtained independently by Mr J. C. Malet, of Trinity College, Dublin, and is published in the *Transactions of the Royal Irish Academy* for 1878. The principle of the demonstration is so simple and natural that, when once it has been pointed out, the only wonder is it should not have occurred to any one before. It turns on what Clifford calls an obvious remark that the order of a resultant is reduced when the weight of the coefficients, in the two equations from which the elimination takes place, falls, not by successive units, but by some other constant integer. The development of this remark forms the subject of XVII.

A (*c*). We have already referred to the canonical form of a Riemann's surface imagined by Clifford. His theorem in effect is that a Riemann's surface having n sheets or leaves, and w spiral or branch points, by winding around which the different sheets are connected with and pass into one another, may, if the material of which it is made is sufficiently extensible, be

transformed without tearing into the surface of a body with $\frac{1}{2}w-n+1$ holes through it. For example, an anchor ring is a body with one hole through it; and into the surface of an anchor ring a Riemann's surface with two sheets and four spiral points can be transformed. Again, in chains of a not uncommon pattern, each link is of the form of a flattened ellipsoid with two circular holes through it; the surface of such a body would represent the equation $y^2=f(x)$, where $f(x)$ is a rational and integral function of the sixth order, and might therefore be used as the geometrical scaffolding for the theory of hyper-elliptic integrals. A closed box with p tubes, open at each end, carried through it from its lid to its bottom, is a simple example of a body with p holes through it. Such a surface would serve to represent (in a certain sense) all the real and imaginary points of a curve of deficiency p; or, which is the same thing, all the pairs of values of two variables which satisfy an algebraical equation of that deficiency. What however is represented in this way is not the quantitative relation between the two variables, but the *connexion* between the different values of one of them corresponding to one and the same given value of the other; viz. if z is the independent and s the dependent variable in an algebraical equation between z and s, and if s_0, s_0' are two different values of s corresponding to the same value z_0 of z, these two values are *connected* in the sense that it is always possible, by making z pass through a continuous cycle of values beginning and ending with z_0, to cause s, though it may have set out with the value s_0, to end with the value s_0'. Each point of any Riemann's surface representing the equation corresponds to a pair of values of z and s; and the different tracks on the surface, irreducible to one another by any continuous change, along which we can pass from the point (z_0, s_0) to the point (z_0, s_0'), answer precisely to the different courses of values by which we can pass with analytical continuity from the pair of values (s_0, z_0) to the pair of values (s_0', z_0). Remodelled as it has been by Clifford, the Riemann's

surface enables us to form a distinct conception of these different tracks, as well as of the systems of curves or "cross-cuts" which have to be drawn in order to reduce the surface to one simply connected; *i.e.* to a surface in which but one set of irreducible tracks can be drawn from any one given point to another. The advantage resulting from Clifford's simplification will be acknowledged by students approaching the theory for the first time, who generally find considerable difficulty in eliciting from Riemann's description a clear image either of the surface itself or of the complicated systems of curves which he directs to be drawn upon it.

The unfinished memoir on the Double Theta Functions (XL.) relates to the problem of the inversion of hyper-elliptic integrals, solved, so far as its main outlines are concerned, by Rosenhain and Goepel. Except in a few matters of detail Clifford does not seem to have added anything to the results of his predecessors, and the papers may be regarded as a continuation of the "Algebraical Introduction to Elliptic Functions" (Appendix, p. 443). The memoir on groups of periodic functions (XXXVIII.) deals with the multiple Theta functions. In it Clifford considers the different Theta functions of the same arguments as multiples by an exponential of any one of them, in which the arguments are increased by *quadrants* (i.e. by multiples of the halves of the periods and of the quasi-periods): and he determines the number of even and uneven Theta functions respectively Leaving these known results, he proceeds to investigate the differential coefficient of the quotient of two Theta functions (p. 354). The method is the same as that employed in the "Algebraical Introduction;" viz. two Theta series are multiplied together, and the sums of two squares which appear as exponents in the product are transformed by the formula $(m-n)^2+(m+n)^2=2(m^2+n^2)$. But the research is left unfinished, and it does not appear whether Clifford had succeeded in completing it. In the memoir "on the Marks of Multiple Theta Functions" the point of view is somewhat

different, and was evidently suggested by the researches of Riemann and of Dr Weber on the Triple Theta Functions. The expression of a Theta function of multiplicity μ involves μ pairs of indices, each of which is either 0 or 1; thus there are $2^{2\mu}$ different Theta functions of the same arguments, each having its own system of indices, termed by Riemann and Weber the *characteristic*, by Clifford the *mark* of the function. The memoir of Weber[1] contains a complete theory of the marks of the triple Theta functions; and this theory Clifford endeavours to extend to the Theta functions of any multiplicity. What place the fragmentary, but not unimportant, results obtained by him will one day take in a complete theory of the Theta functions must be left to the future to decide. We may observe however that Clifford, in the latest note which we have of his (see p. 329), intimates that he had found the true generalization, for the most general Abelian Integrals, of the beautiful theorem by which Riemann has established a correspondence between the 28 double tangents of a quartic curve and the 28 uneven Theta functions of multiplicity 3. The details of the "Theory of Marks" may perhaps be found dry and repulsive, as indeed is the case with many questions relating to the combinatorial analysis; but the generalization to which we have referred shews what great importance they may hereafter be found to possess for the theory of algebraical integrals.

A (*d*). The study of the geometrical methods of Grassmann, so long neglected even in Germany, made a great and enduring impression on Clifford's mind. Speaking of the *Ausdehnungslehre* in the paper (xxx.) entitled "Applications of Grassmann's Extensive Algebra," he says "I may be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science."

[1] *Theorie der Abelschen Functionen vom Geschlecht* 3. Berlin, 1876.

A straight line, of which the end-points are A and B, has been, from the days of the ancient mathematicians to our own, universally denoted by the symbol AB. To this notation modern geometry has added the convention $AB = -BA$, which serves to distinguish between the two directions in which the line can be drawn; from A to B, or from B to A. The notations of the Barycentric Calculus of Moebius may perhaps have suggested to Grassmann the idea (which at first sight seems both paradoxical and unpromising) of considering the two points A and B as *factors* of the symbol AB. He was thus led to consider symbolical quantities (which he termed *extensive magnitudes*) $a, b, c, \ldots$, capable of combining with one another by a species of multiplication, subject to the special laws $ab = -ba$, $aa = bb = cc = \ldots = 0$. Such symbolic quantities Clifford, following Dr Sylvester, calls *polar* quantities, and their multiplication *polar* multiplication; the term *alternate* which is certainly less appropriate, had previously been used by him in the same sense. It had been observed (by Grassmann and by Cauchy) that the product of n linear and homogeneous functions of n polar quantities is the determinant of the linear functions multiplied by the product of the polar quantities; *e.g.* $[a\lambda_1 + b\lambda_2] \times [c\lambda_1 + d\lambda_2] = (ad - bc) \times \lambda_1\lambda_2$. And this observation suggested to Clifford the attempt to form invariants and covariants in the same manner. For this purpose he considered in the first place multipartite binary forms in which the indeterminates are polar quantities, and found that by multiplying them together in such a manner that each pair of variables occurs in two factors he obtained an invariant; if, for example, $f = a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2$, the polar multiplication of f by itself gives

$$f \times f = 2\,(a_{12}\,a_{21} - a_{11}\,a_{22}) \times \lambda_1\lambda_2\,\mu_1\mu_2\,;$$

and this, if we suppose, as we may do, that $\lambda_1\lambda_2 = 1$, $\mu_1\mu_2 = 1$, is the double of the determinant. Invariants obtained in this way have the characteristic property of invariance even when

the different sets of variables are transformed by different linear transformations: but invariants which require that the transformations should be the same may be obtained by a somewhat similar process; *e.g.* if we multiply f by $\lambda_1\mu_2 - \lambda_2\mu_1$ we find the invariant $a_{12} - a_{21}$. In these simple cases the relation of the method to that of "contravariant differentiation" is evident; viz. if in f we write

$$\frac{d}{d\lambda_2}, -\frac{d}{d\lambda_1}; \ \frac{d}{d\mu_2}, -\frac{d}{d\mu_1}; \text{ for } \lambda_1\lambda_2; \ \mu_1\mu_2;$$

respectively, and apply to f the operator thus obtained, we get the same invariant which arises from the polar multiplication of f by itself. Clifford has not dwelt on the connexion of the two methods; but it must have been present to his mind, since, as we shall presently see, he had already employed the notation of Grassmann to express contravariant differentiation.

Still in the first instance confining himself to multipartite binary forms, Clifford next found ready-made to his hand a geometrical mode of representing the invariants or covariants obtained by polar multiplication. Let a bipartite linear form, such as f, be represented by -o-, a tripartite form by -ȯ-, the number of rays or bonds attached to the small circle denoting the number of pairs of indeterminants, and the direction of the rays being indifferent. Then the symbol o=o will serve to represent $f \times f$; *i.e.* the invariant $a_{12}\,a_{21} - a_{11}\,a_{22}$; or, if we suppose that the two small circles represent two different forms, the same symbol will represent their joint invariant. But this symbol is precisely one of the "graphs" which, according to a theory developed by Dr Sylvester and Professor Cayley, may be employed to represent a saturated chemical compound; viz. if the two circles represent two diatomic elements the symbol represents a compound in which the two atomicities are saturated. In this way a parallelism or correspondence is established between the invariants of a system of forms on the one hand and the formulæ of saturated chemical compounds on the other.

In this singularly beautiful series of conceptions it is difficult to say how much belongs to Clifford and how much to Dr Sylvester: the more so because each of them was ready to attribute to the other the larger share. (See the letter to Dr Sylvester entitled "Remarks on the Chemico-algebraical Theory," and the note of Dr Sylvester at p. 257).

A point of considerable interest in itself appears to have had a special attraction for Clifford. "The part," he writes to Dr Sylvester, "of the theory which astonished me most is its application to intergredient variables when the number in a set is greater than three, such as the six coordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, then intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes we get one intergredient form representing their line of intersection, etc."

The theory of the *intervariants* appertaining to intergredient variables has been so little developed in detail that it is a matter of regret that no other record remains of Clifford's meditations on the subject.

It may perhaps be right to point out that simple and beautiful as are the methods of the formation of invariants by polar multiplication, and of their representation by graphs, the application of these methods to individual cases is not exempt from difficulties. One source of these difficulties Clifford indicates when, in the letter just quoted, he says "of course the main thing is to pass from this system of separate variables [in the multipartite binary forms] to that in which the same variables occur to higher orders in the same form, or back again—what you call unravelment."

Thus in the papers XXIX. and XXXI. in which alone we have any detailed applications of the method to the actual determination of invariants and covariants, a large amount of space has to be devoted to the consideration of the distinction between symmetric and unsymmetric forms. The difficulties

of the method are perhaps even more apparent upon an examination of the lithographed fragments to which reference is made in the note on XXXI. The interpretation of each of the graphs recorded in these fragments is not difficult to decipher. But the method does not afford any answer to the question whether the corresponding invariant or covariant is really existent, or is identically evanescent and therefore nugatory. Thus it will be noticed in Dr Sylvester's note on XXVIII., that there is no difficulty in finding the 'algebraical content' of the graph which represents the discriminant of a cubic; but to establish the non-evanescence of this algebraical content, considerations are required which Dr Sylvester supplies from another source.

A (*e*). Of the notes included under this head perhaps one only—that on Bessel's functions—is purely analytical. To these functions Clifford does not appear to have devoted any continuous attention, though he has again considered them from the same point of view in the paper 'on the multiplication of two infinite series.' (Appendix, p. 474). The note on the exponential function is intended to shew that an exponential operator applied to vectors in one line, in a plane, and in space, leads successively to the conceptions of a ratio having a sign, of a complex ratio, and of a quaternion. The note on spherical harmonics points out that Laplace's equation may be interpreted as signifying that the curve represented by any spherical harmonic equated to zero stands in a certain covariant relation to the absolute imaginary circle upon the sphere. Lastly, the fragment on matrices contains (1) a geometrical investigation of the effect of transformation by a matrix of which the determinant vanishes, (2) a geometrical investigation of the condition that two matrices should be commutative in multiplication.

B (*a*). The papers VIII., XLVIII., XI., XVI., XIX. may perhaps be classed together as all turning on the derivation of the metrical from the descriptive properties of figures by the intro-

duction of the imaginary circular points at infinity. Of these XLVIII. (a note from the *Educational Times*), VIII. "The Synthetic Proof of Miquel's Theorem," XIX. "The Geometry on an Ellipsoid," relate to special problems, but are remarkable for their great elegance. The generalization of Miquel's theorem, and the projection of the lines of curvature of an ellipsoid into confocal anallagmatics may be instanced as results which have obtained considerable and well-deserved celebrity. The papers (XI. and XVI.) on "Analytical Metrics," and on "The Theory of Distances," contain systematic attempts to work out the theory of the connexion between metrical and descriptive properties upon the principles of Poncelet and Professor Cayley. Neither of these memoirs is complete; the former, which is of a more elementary character, was probably discontinued by Clifford in favour of the methods adopted in the second, which remained unpublished at the time of his death. This latter paper, in addition to the results which it contains relating to the foci of curves of any order, is remarkable for the method employed in it, which may be described as an extension of the notation of Grassmann so as to include the contravariant differentiation of Dr Sylvester. This paper indeed might without impropriety have been placed in the division B (*b*); and as in one of its sections the absolute is considered as an imaginary conic instead of a pair of imaginary points, it has close relations with the group B (*d*).

Among the other papers under this head which have a character of great generality, we may call attention to the extension of Chasles's theory of characteristics (XLIX.) and to the admirable memoir on the "General Theory of Anharmonics" (XII.), in which the definition of the anharmonic ratio of four points on a line was extended for the first time to systems of points in a plane and in space. Among papers on special problems we may notice one (perhaps more properly belonging to B, *b*) "on the Analogues of Pascal's Theorem," which, dealing chiefly with the case $n=4$, relates to the figure formed by two

sets of n lines such that $n(n-1)$ of their n^2 intersections lie on a curve of order $n-1$; and that "on some Porismatic Problems," which contains a proof and an extension by the method of correspondences alone, of Poncelet's celebrated porism of the polygon inscribed in one and circumscribed about another conic section. This last paper has an additional interest, because to this porism of Poncelet and to the connexion established by Professor Cayley between it and the theory of Elliptic Functions, we owe Clifford's subsequent researches on the geometrical representation of Elliptic Transformation.

B (b). The theory of contravariant differentiation is the leading idea in the paper "on a Generalization of the Theory of Polars," and in the two papers "on the Syzygetic Relations among the Powers of Linear Quantics." The first of these contains the extension (which perhaps was already known) of the theory of the polar curves of a point with regard to a curve of order n: viz. for the point we substitute a curve $f_m(\xi\eta\zeta)$ of the class m $(m<n)$; and operating on $f_n(xyz)$ by

$$f_m\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$$

we obtain the polar curve of f_m with regard to f_n. For this conception Clifford subsequently provided (as has been already said) a convenient notation by the extension of the meaning of Grassmann's symbols. The note in the Appendix "on the Polar theory of Cubics" may serve as an illustration of the purposes to which he proposed to apply the definitions of this paper in connexion with Grassmann's Geometric Analysis. The geometrical interpretation of the results of the higher algebra was at one time a subject of great interest to him, and I remember his expressing to me a confident expectation that such an interpretation could in all cases be obtained by the combination which he was then employing of the method of contravariant differentiation with that of Grassmann's *Ausdeh-*

nungslehre. How far, at the time when he thus expressed himself, geometers were from having kept pace with the rapid development of algebra, may be inferred from the fact, to which Clebsch often adverted in conversation, that the geometric interpretation of the simplest known linear covariants—those of the binary quintic—has not yet been given. In the two papers "on syzygetic relations" (which do not imply any reference to the methods of Grassmann), Clifford generalizes in various directions the methods employed and the results obtained by M. Paul Serret in his *Géométrie de Direction.*

Among researches of a somewhat more special kind, we must especially mention (1) "The Theorem relating to Polyhedra" (XVIII.) enunciated at p. 169, which is the analogue of an equally beautiful property of polygons discovered by the late T. Cotterill; and (2) the investigations relating to Mr Spottiswoode's problem of "the contact of conics and of quadrics with surfaces."

B (*e*). Passing over the important papers on Elliptic Transformation, the general character of which we have already indicated, and the few notes on kinematical subjects (which are of less importance as Clifford has left a complete work on dynamical science), we come to the class of researches in which he seems to have found the fullest scope for his mathematical imagination.

To the translation of Riemann's discourse we have already referred; and, as Clifford has added no notes of his own to it, we may pass it over here, only observing that the Lecture on the Postulates of the Science of Space is not only a popular account of the theory, but a commentary, which any young mathematician reading Riemann's discourse for the first time will find invaluable. Of the other papers enumerated under this head, all except the last agree in the employment of symbolic methods, founded on those of Grassmann and Sir William Hamilton, but modified and enlarged so as to include within

the domain of symbolic geometry the theory of screws invented by Dr Ball. And it must be allowed that Clifford has succeeded in combining these, in form if not in substance, somewhat heterogeneous elements into a theory of great beauty and wide scope, which is certainly his own. He appears to have shared the conviction of Grassmann * that Sir William Hamilton's quaternions lie entirely within the four corners of the *Ausdehnungslehre.* Some exception may perhaps be taken to this view. In the first place Clifford and Grassmann do not altogether agree as to the way in which the symbols of Grassmann are to be identified with those of Hamilton; and each of them has to assume a law of multiplication for the fundamental units, which is at variance with the law actually adopted in the *Ausdehnungslehre,* though included as a special case under one of the general types of multiplication described by Grassmann in his Memoir "Sur les différents genres de Multiplication" (Crelle's *Journal,* Vol. XLIX. p. 123). In fact, throughout the *Ausdehnungslehre* of 1862 only one species of multiplication (beside that of ordinary algebra) is employed: viz. the polar multiplication (see above, p. lvi), characterized by the equations $[ab] = -[ba]$, $[a^2] = [b^2] = \ldots = 0$, where $a, b, \ldots$ are any extensive magnitudes whatever, and the square brackets are employed to signify that the multiplication is polar and not algebraical. It is true that Grassmann uses a great number of different terms to describe what he considers to be different kinds of multiplication (outer, inner, progressive, regressive, planimetric, stereometric); but these terms really define the interpretation to be given to the symbols in certain combinations, and not the law of the multiplication itself; for example, the *inner* product of an unit a by another unit b is the polar (or, to use Grassmann's expression, the *combinatorial*) product of a, not by b, but by a certain extensive magnitude (viz. the product taken with a definite sign of all the fundamental units

* See his memoir "Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre," *Mathematische Annalen,* Vol. XII. p. 375.

which are not factors of b), termed by Grassmann the *complement* of b. To bring quaternions within the methods of the *Ausdehnungslehre,* Grassmann has to suppose a law of multiplication, such that the product of two factors is equal to the complement of their outer or combinatorial product, diminished by their inner product*; and Clifford has to suppose a system of three fundamental units, of which the squares are -1 instead of zero. Each of these suppositions seems to lie wholly outside the particular system of symbolic geometry developed in the *Ausdehnungslehre;* though it is not contended that either of them lies outside the general principles of the memoir "Sur les différents genres de Multiplication." So far as Clifford's own work is concerned, the question of the true relation between the quaternion algebra and that of extensive magnitudes is not perhaps very important: it is certain however that a comparison of the points of difference and of agreement between the two theories exercised a marked influence on the course of his own speculations.

The conception of a vector in space, and of the addition and subtraction of vectors, is common to both systems, and was no doubt familiar to geometers before the invention of either. The idea of a quotient of two vectors is peculiar to the calculus of quaternions; and the expression of such a quotient as the sum of a scalar and a vector is the central point in the whole theory. There is nothing properly corresponding to this in the *Ausdehnungslehre.* We do indeed meet in the work of Grassmann with quotients of vectors†; but these quotients are operators, converting simultaneously three given vectors into three other given vectors; and thus affording a symbolic representation of the homographic transformation of space. Nor again are the quotients of Grassmann themselves extensive magnitudes; thus they do not possess the characteristic property of the vector-quotients

* *Mathematische Annalen,* Vol. XII. p. 378.

† *Ausdehnungslehre,* Arts. 377—390. See also *Math. Ann.* Vol. XII. p. 381—383.

of Sir William Hamilton—that of admitting of expression in terms of the same units as the vectors which are compared. On the other hand, the calculus of quaternions is a calculus of vectors only; *i.e.* it is a geometric calculus in which equal lengths in parallel directions are regarded as equivalent. But the analysis of Grassmann carefully distinguishes between vectors (*Strecken*) and rotors (*Linientheile*), *i.e.* vectors of which the position is limited to some one indefinite straight line; viz. if A and B are two extensive magnitudes denoting simple points, their product $[AB]$ and their difference $B-A$ are respectively the rotor and vector AB. The statical theorems that the sum of any number of rotors can be reduced in an infinite number of ways to the sum of two rotors, and in one way to a rotor and an area of which the axis coincides with that of the rotor, are shewn by Grassmann to follow, upon the principles of his calculus, from the definition of a rotor as the product of two points. Seeing then that rotors as well as vectors could be brought within the range of a geometric calculus, Clifford was naturally led to enquire whether the quotient of two rotors might not be capable of similar expression; and this enquiry led him to the conception of a biquaternion. One step in the way (and an important one) had already been made by Dr Ball's theory of screws. Any composite geometrical quantity consisting of a rotor and of a vector having the same (or any parallel) axis is of the nature of a screw; the pitch of the screw being the quotient of the absolute length of the vector divided by that of the rotor. Such a composite quantity Clifford proposes to call a *motor*; the name being suggested by the statical theorem just referred to, which in fact asserts that any system of forces (*i.e.* of rotors) in space is equivalent to a rotor and a parallel vector (the latter representing an area). It was obvious that the quotient of two rotors is a quantity of the screw or motor type—a tensor-twist: but to complete the symbolic geometry of space an expression for the quotient of two motors had to be obtained; and for this an entirely new conception

had to be invented, that of the Biquaternion*. It is possible that Clifford might not have been guided to this conception if he had not considered the problem with reference to elliptic instead of parabolic space. In an elliptic space a motor can in general be represented in an infinite number of ways as the sum of two rotors, and in one way, and one way only, as the sum of two rotors reciprocal to one another with regard to the absolute. If we represent by ω a rectangular twist of unit pitch, Clifford shews (1) that all such twists are equivalent to one another; (2) that if α be any rotor, $\omega\alpha$ is the rotor reciprocal, and equal to it; (3) that every motor can be expressed in the form $\alpha+\omega\beta$, where α and β are rotors passing through the origin; and (4) that the ratio of any two motors can be exhibited in the form $s+\omega t$ where s and t are quaternions. The symmetry of this system is very striking; in parabolic space the definition of the symbol ω is far less simple and natural (see p. 186)†.

These ideas are developed (though only too briefly) in the "Preliminary Sketch of Biquaternions," the "Further Note on Biquaternions," and the note "On the theory of Screws in a space of Constant Curvature." The paper "On the motion of a body in elliptic space" contains a singularly beautiful application of the whole theory.

* The word Biquaternion had already been employed by Sir W. R. Hamilton. But his biquaternions are entirely different from those of Clifford, and are simply quaternions of which the coefficients are complex quantities containing the imaginary unit of ordinary algebra.

† Dr Ball, who kindly allowed me to submit to him a proof of the above very imperfect account of Clifford's theory of biquaternions, writes to me as follows: "I think it might be well to add a line or two with a view of prominently bringing out Clifford's conception of the 'vector' in elliptic space. A 'right vector' is the result of two equal rotations about a pair of conjugate polars with regard to the absolute; a 'left vector' is the result of two equal *and opposite* rotations about a pair of conjugate polars. Clifford shows beautifully that this is the legitimate generalization of the Hamiltonian vector, and he enunciates the splendid theorem that *any* motor must be *in one way* the sum of a right vector and a left vector."

Of the papers entitled "Applications of Grassmann's extensive Algebra" and "On the classification of Geometric Algebras," no clearer account can be given than that contained in Clifford's letter to Dr Sylvester: "I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of combination; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a compound (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras."

The fundamental units $\iota_1, \iota_2 \dots \iota_n$ in these papers, in accordance with the *Ausdehnunglehre*, are points or vectors; vectors being equivalent to points of weight zero at an infinite distance. But these units differ essentially from those of Grassmann, inasmuch as $\iota_r^2 = \pm 1$, instead of $\iota_r^2 = 0$, although $\iota_r \iota_s = -\iota_s \iota_r$. It must be admitted that in the combination of these conditions there is something paradoxical, for the equation $\iota_r \iota_s = -\iota_s \iota_r$, if we suppose it to hold when $r = s$, gives necessarily $\iota_r^2 = 0$: and Clifford has nowhere clearly explained why the square of a symbol denoting a point should be a positive or negative unit in the same symbolical system in which the product of two points represents a distance.

The only application of this geometrical algebra which he has left consists in his solution of the problem of the motion of a rigid body round a fixed point in a flat space of n dimensions. The paper (XXVI.) containing this solution is further remarkable as sketching the way in which Theta functions of $n-r$ arguments might be employed to integrate the equations of motion in the case in which no forces are acting; just as Jacobi has expressed as Theta functions of the time the nine direction cosines of a rigid body, rotating under the action of no forces round a fixed point in space of three dimensions.

The last paper under this head, that "on the Classification of Loci," is of a different character from those which have preceded; and is perhaps, even in the unfinished state in which we have it, the most profoundly interesting of all Clifford's mathematical writings. No use is made in it of any special symbolical methods: the only apparatus employed is that of ordinary algebra. And, if the language of the geometry of many dimensions is adopted throughout, yet the results are capable of immediate translation into the language of algebra. Thus the Theorem A, p. 307, "Every proper curve of the n^{th} order is in a flat space of n dimensions or less," is equivalent to the following algebraical proposition:

"If there be a system of $n+m+1$ quantities x connected by $n+m-1$ homogeneous equations; and if this system be such that, upon the addition to it of one equation more, linear and homogeneous in the quantities x and having arbitrary coefficients, it gives n sets of values for the ratios of the quantities x, these $n+m+1$ quantities can always be expressed as linear and homogeneous functions of $n+1$ quantities."

The value of such an algebraical proposition cannot be questioned, because the study of the properties of systems of algebraical equations is of importance for every part of analysis. But the advantage of the geometrical statement in point of clearness and precision is unmistakeable. And our sense of this advantage would be yet further quickened if we were to attempt to render into pure algebra the proof, in three lines only, which Clifford has given of the Theorem. As an example, having no immediate connexion with Clifford's memoir, of the use which has been made by other mathematicians of a similar extension of geometrical language, we may refer to the researches on elimination contained in Lesson XVIII. of the Higher Algebra of Dr Salmon, who cannot be justly accused of having shewn any undue partiality for space of more than three dimensions.

The fundamental principle of Clifford's investigation has been so clearly explained by Professor Henrici in his note on p. 307, that we need not dwell on it here. It may however be worth while to observe that Grassmann (whose footsteps Clifford is certainly not following in this paper) had already proposed to regard the coefficients in the Cartesian equation of a curve as coordinates; and as an example of this method had worked out the geometry of a system of circles in a plane, shewing its analogy to the geometry of the points of space of three dimensions: viz. if $XYZW$ are the equations of four circles, the equation of any given circle may be expressed in the form $lX + mY + nZ + rW$, and l, m, n, r may be regarded as the homogeneous coordinates of this circle. These coordinates of Grassmann's are closely related to the 'power coordinates' of a circle and of a sphere employed in XXXIV. and in the Appendix, p. 546; Clifford's power coordinates of a circle being in fact linear functions of Grassmann's. It is remarkable, however, that Grassmann, at least in his general definition, contemplates only a single equation between his generalized coordinates. Thus the circles of which the coordinates satisfy the homogeneous relation $f(l, m, n, r) = 0$, form the only "Kurvengebilde" of circles which he defines. This according to Clifford would be a "two spread" of circles, and a "one spread" or "one way locus" would be formed by the circles of which the coordinates satisfy two equations. All that remains to us of Clifford's work on the subject relates to "one-way loci."

The application of Abelian functions to this new aspect of geometry awakened all Clifford's enthusiasm. He spoke to me of this part of his theory as opening a boundless field for new researches—as "altogether too big a thing" for one man to manage; and, with the instinct for companionship so characteristic of his nature, he expressed an earnest desire to get others to join him in the work. All that remains to indicate the nature of the discoveries which he had made in this direction is (1) a brief but masterly summary of the known results of the

theory of Abelian functions, stated in a form suitable to the applications which he contemplated; (2) a generalization, to which we have already referred, of an important result obtained by Riemann; and (3) a remarkable theorem limiting, in certain cases, the number of dimensions requisite for the existence of a curve of given order and deficiency. How much may have perished unrecorded we cannot tell. But, however this may be, no geometer will look for a more splendid monument of Clifford's genius, or for a more touching memorial of his early death, than is to be found in the unfinished pages 'On the Classification of Loci' which embody the last and perhaps the greatest effort of his inventive powers.

MATHEMATICAL PAPERS.

I.

ON THE TYPES OF COMPOUND STATEMENT INVOLVING FOUR CLASSES*.

PROFESSOR STANLEY JEVONS has enumerated† the types of compound statement involving three classes, among which the premises of a syllogism appear as a type of four-fold statement. He propounded at the same time the corresponding problem of enumeration for four classes, which is solved in the present communication. The reader is referred to the paper or the book just mentioned for further explanation of the nature and purpose of the problem than is to be found in Art. 1. It may, however, be premised that the letters A, B, C, D, denote four *classes* or *terms* (for example, hard, wet, black, nice), and that, according to a convenient notation of De Morgan's, the small letters, a, b, c, d denote the complementary classes or contrary terms (not hard, not wet, not black, not nice). A *simple* statement is of the form $ABCD = 0$ (no hard, wet, black, nice things exist, or, which is the same thing, all hard, wet, black things are nasty). The statement $ABC = 0$ (no hard, wet, black things exist, or all hard, black things are dry) is to be regarded as made of these two, $ABCD = 0$, $ABCd = 0$ (no hard, wet, black, nice things exist, and no hard, wet, black, nasty things exist) and so is called a *compound* (in this case a *two-fold*) statement. The notion of *types* is defined in Art. 1.

1. Four classes, or terms, A, B, C, D, give rise to sixteen cross-divisions or *marks*, such as $AbCd$. A denial of the existence of one of these cross-divisions, or of anything having its mark (such as $AbCd = 0$), is called a simple statement. A

* [From the *Memoirs of the Literary and Philosophical Society of Manchester*, Session 1876—77. Vol. XVI. No. 7, pp. 88—101. Communicated January 9th, 1877.]

† *Proceedings of the Manchester Philosophical Society*, vol. vi. pp. 65—68, and *Memoirs*, Third Series, vol. v. pp. 119—130. *The Principles of Science*, vol. i. pp. 154—164. [New Edition, pp. 134—143.]

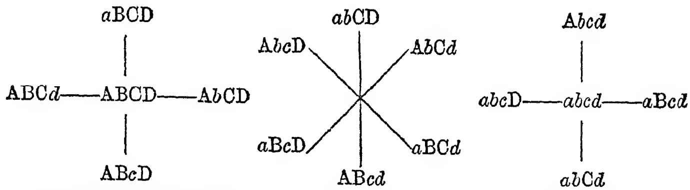
denial of two or more cross-divisions is called a compound statement, and, moreover, two-fold, three-fold, etc., according to the number denied.

When two compound statements can be converted into one another by interchange of the classes A, B, C, D with each other or with their complementary classes *a, b, c, d,* they are called *similar;* and all similar statements are said to belong to the same *type.* The problem before us is to enumerate all the types of compound statement that can be made with four terms.

2. Two statements are called *complementary* when they deny between them all the sixteen marks without both denying any mark; or, which is the same thing, when each denies just those marks which the other permits to exist. It is obvious that when two statements are similar, the complementary statements will also be similar; and, consequently, for every type of n-fold statement there is a *complementary type* of $(16-n)$ fold statement. It follows that we need only enumerate the types as far as the eighth order; for the types of more-than-eight-fold statement will already have been given as complementary to types of lower orders. Every eight-fold statement is complementary to an eight-fold statement; but these are not necessarily of the same type.

3. One mark ABCD may be converted into another A*b*C*d* by interchanging one or more of the classes A, B, C, D with its complementary class. The number of such changes is called the *distance* of the two marks. Thus in the example given the distance is 2. In two similar compound statements the distances of the marks denied must be the same; but it does not follow that when all the distances are the same, the two statements are similar. There is, however, as we shall see, only one example of two dissimilar statements having the same distances. When the distance is 4, the two marks are said to be *obverse* to one another, and the statements denying them are called *obverse* statements—as ABCD, *abcd*; or, again, A*b*C*d*, *a*B*c*D. When any one mark is given (called the *origin*), all the others may be grouped in respect of their relations to it as follows:—Four are at distance *one* from it, and may be called *proximates;* six at distance *two,* and may be called *mediates;*

four at distance *three*, and may be called *ultimates*. Finally, the obverse is at distance *four*.



Origin and 4 proximates. 6 mediates. Obverse and 4 ultimates.

It will be seen from the above table that the four proximates are respectively obverse to the four ultimates, and that the mediates form three pairs of obverses. Every proximate or ultimate is distant 1 and 3 respectively from such a pair of mediates. Thus each proximate or ultimate divides the mediates into two classes; three of them are at distance 1 from it, and three at distance 3. Two mediates which are not obverse are at distance 2. Two proximates, or two ultimates, or an ultimate and a proximate which are not obverse, are also at distance 2.

This view of the mutual relations of the marks is the basis of the following enumeration of types.

4. There is clearly only one type of simple statement. But of two-fold statements there are four types; viz. the distance may be 1, 2, 3, or 4; and so, in general, with n classes there are n types of two-fold statement.

5. A compound statement containing no pair of obverses is called *pure*. In a three-fold statement there are three distances; one of these must be not less than either of the others. If this be 2, the remaining mark must be at odd distance from both of these or at even distance from both; thus we get the types 1, 1, 2, and 2, 2, 2. If the not-less distance be 3, the remaining distances must be one even and the other odd; the even distance must be 2, the odd one either 1 or 3; and the types are 1, 2, 3; 2, 3, 3. Thus there are 4 pure three-fold types. With a pair of obverses, the remaining mark must be at odd or even distance from them; 1, 3, 4; 2, 2, 4. In all six three-fold types observe that there is necessarily one even distance.

6. *A fortiori*, in a four-fold statement there must be one even distance. In a pure four-fold statement this distance is 2. From this pair of marks let both the others be oddly distant; then they must be evenly distant from one another, *i.e.* at distance 2, obverses being excluded. The odd distances are 1 or 3; and it will be easily seen that the following are all the possible cases:

$$\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline 1 & 3 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline 3 & 3 \end{array} \qquad \begin{array}{c|c} 1 & 3 \\ \hline 3 & 1 \end{array} \qquad \begin{array}{c|c} 1 & 3 \\ \hline 3 & 3 \end{array} \qquad \begin{array}{c|c} 3 & 3 \\ \hline 3 & 3 \end{array}$$

In these figures the dots indicate the four marks, the cross lines indicate distance 2, and the other figures the distances between the marks on either side of them. Next, from the pairs of marks at distance 2 let one of the others at least be evenly distant, *i.e.* at distance 2. Then we have three marks which are all at distance 2 from one another; and it is easy to show that they are all proximates of a certain other mark. For, select one of them as origin; then the other two are mediates which are not obverse, and which consequently are at distance 1 from some one proximate. With this proximate as origin, therefore, all three are proximates. We have therefore only to enquire what different relations the fourth mark can bear to these three. It may be the origin, its obverse, the remaining proximate, its obverse, or one of two kinds of mediates, viz. at distance 1 or 3 from the remaining proximate. Thus we have 6 types, in which the distances of the fourth mark from the triad are respectively 111, 333, 222, 222, 133, 113. The third and fourth of these are especially interesting, as being distinct types with the same set of distances; I call them *proper* and *improper groups* respectively: viz., a proper group is the four proximates of any origin; an improper group is three proximates with the obverse of the fourth. On the whole we get 12 types of pure four-fold statement.

7. In a four-fold statement with *one* pair of obverses, take one of them for origin; the remaining two marks must then be either a pair of proximates or ultimates, a proximate and an ultimate, a pair of mediates, or a proximate or ultimate with one of two kinds of mediate—in all, 5 types, with the distances

$13^2,13$; $13^2,31$; $22^2,22$; $13^1,22$; $13^3,22$. With *two* pairs of obverses they must be either at odd or even distances from one another; two types Altogether $12 + 5 + 2 = 19$ four-fold types.

8. In a *pure* five-fold statement there is always a triad of marks at distance 2 from one another. For there is a pair evenly distant; if there is not another mark evenly distant from these, the remaining three are all oddly distant, and therefore evenly distant from one another. First, then, let the remaining two marks be both oddly distant from the triad. In regard to the origin of which these are proximates, the two to be added must be either two mediates, like (of two kinds) or unlike, or a mediate of either kind with the origin or the obverse; 7 types. Next, if one of the two marks be evenly distant from the triad, it must form with the triad either a proper or an improper group of four. To a proper group we may add the origin, the obverse, or a mediate; to an improper group, the origin or the obverse (the mediates give no new type), 5 types; or in all 12 pure five-fold types.

9. In a five-fold statement with one pair of obverses there must be another pair of marks at distance 2. We have therefore to add one mark to each of the following three types of fourfold statement,—a pair of obverses together with (1) two proximates, (2) a proximate and an ultimate, (3) two mediates. To the first we may add another proximate, an ultimate, or a mediate of three kinds, viz. at distances 11, 13, 33 from the two proximates; 5 types. To the second if we add a proximate or an ultimate, we fall back on one of the previous cases; but there are again three kinds of mediates, at distances 11, 33, 13 from the proximate and ultimate; 3 types. To the third we may add another mediate, whereby the type becomes a proper group together with the obverse of one of its marks, which is the same thing as an improper group together with the obverse of one of its marks—or a proximate or ultimate which are of three kinds, at distances 11, 13, 33 from the two mediates; 4 types. Thus there are 12 five-fold types with one pair of obverses. With two pairs of obverses at odd distances, there is only one type, all the remaining marks

being similarly related to them; at even distance the remaining mark may be evenly or oddly distant from them; 2 types. On the whole we have $12+12+3=27$ types of five-fold statement.

It is to be remarked that there is no pure five-fold statement in which all the distances are even, and that, if there is only one pair of obverses with all the distances even, the type is a proper group together with the obverse of one of its marks.

10. We may now prove, as a consequence of the last remark, that a pure six-fold statement either contains a group of four with a pair oddly distant from it or consists of two triads oddly distant from one another. For there must be a pair at distance 2; if the other four are all oddly distant from these, they form a group; if one is evenly distant and three oddly distant, we have the case of the two triads; if two are evenly distant, we again have a group. We must add, then, first to a proper group, and then to an improper group, a pair oddly distant from it. To a proper group consisting of the proximates to a certain origin we may add the origin or its obverse with a mediate, or two mediates; 3 types. An improper group is symmetrical; that is to say, if we substitute for any one of its marks the obverse of that mark, we shall obtain a proper group. In this way we shall get four origins distant 1113 from the group, and four obverses distant 1333; if we add to these the obverses of the marks in the group itself, we have described the relation of the twelve remaining marks to the group. To form, therefore, a pure six-fold statement we may add either two origins or two obverses or an origin and an obverse; 3 types.

In the case of the two triads, since they are oddly distant from one another their origins must be oddly distant; that is, they must be distant either 1 or 3. If they are distant 1, neither, both, or one of the origins may appear in the statement; if they are distant 3, neither, both, or one of the obverses; 6 types. Thus we obtain 12 types of purely six-fold statement.

11. If a six-fold statement contains one pair of obverses, the remaining four marks cannot all be evenly distant from this pair. For in that case they would constitute a group; and it is easy to see that the marks evenly distant from a group, whether proper or improper, do not contain a pair of obverses. We have therefore only these four cases to consider:—

(1) The four marks are all oddly distant from the obverses.

(2) One is evenly distant and three oddly distant.

(3) Two are evenly distant and two oddly.

(4) Three are evenly distant and one oddly.

In the first case the four marks form a group. If this is a proper group, the pair of obverses must be either the origin and obverse of the group, or a pair of mediates; 2 types. If the group is improper, the pair must be an origin and an obverse; 1 type. In the second case, we have an origin, an obverse, and a mediate, to which we must add three marks taken out of the proximates and ultimates. We may add 3 proximates distant respectively 113 or 133 from the mediates (2 types),—or 2 proximates distant respectively 11, 13, 33 from the mediate, and with each of these combinations an ultimate distant either 1 or 3 (6 types). To interchange proximates with ultimates clearly makes no difference; so that in reckoning the cases of 1 proximate and 2 ultimates or 3 ultimates, we should find no new types. In the third case we have an origin, an obverse, and two mediates distant 2 from each other; and to these we have to add either two proximates or a proximate and an ultimate. The two proximates may be distant from the two mediates 11, 13, or 11, 33, or 13, 13, or 13, 33; 4 types. The proximate or ultimate must not be respectively distant 11, 33, or 33, 11; for then they would form a pair of obverses; there remain the cases 11 with 11 or 13, 13 with 13, and 33 with 13 or 33; 5 types. In the fourth case we have an origin, obverse, and three mediates distant 2 from one another; the remaining mark must be distant either 1 or 3 from these mediates; 2 types. This makes twenty-two types of six-fold statement with one pair of obverses.

12. If a six-fold statement contains two pairs of obverses, these must be either evenly or oddly distant. If they are evenly distant we have an origin, obverse and two obverse mediates, to which two other marks are to be added. These may be both evenly distant; taking one of them as origin, it is associated with 5 mediates, so that there is 1 type only. Or both oddly distant; here there are two cases, according as the distances are 11, 33 or 13, 13. Or one oddly and one evenly distant; the latter is any one of the four remaining mediates, and then the former is distant 1 or 3 from it; 2 types. If the two pairs of obverses be oddly distant they form an aggregate which is related in the same way to all the remaining twelve marks; viz. any one of these being taken as origin, we have a pair of mediates and a proximate with its obverse ultimate. The thing to be considered, therefore, is the distance between the two marks to be added, which may be 1, 2 or 3, and each in two ways; 6 types.

A six-fold statement with three pairs of obverses is one of two types only; viz. these are all evenly distant when they are the mediates to one origin, or two evenly distant and one oddly distant from both of them.

13. A pure seven-fold statement must consist of a group and a triad; for it must contain a triad, by the same reasoning by which this was proved for a five-fold statement; and then either all the other four marks are oddly distant from this, and so form a group by themselves, or else one of them is evenly distant from the triad and so forms a group with it. If the group is proper, being the proximates to a certain origin, the triad must consist of two mediates and either the origin, the obverse or another mediate; and in the latter case the three mediates are distant 111 or 333 from some proximate; 4 types. If the group is improper, the triad is either all origins or all obverses, or two origins and an obverse, or an origin and two obverses; 4 types. In all, 8 types of pure seven-fold statement.

14. A seven-fold statement with one pair of obverses must consist either of four marks evenly distant from one another

and three oddly distant from them; or of five marks evenly distant from one another and two oddly distant from them. In the former case the pair of obverses may be in the four or in the three. If they are in the four, the three form a triad which are proximates to one origin; and then the pair may be the origin and obverse or a pair of mediates. If the pair are origin and obverse, the other two (at distance 2) are mediates, distant 11, 13 or 33 from the proximate which is not in the triad; if the pair are mediates, the two may be the origin or obverse with a mediate distant 1 or 3 from that proximate (4 types) or two mediates distant 11, 13, 33 from it (3 types). If the pair of obverses are in the set of three marks, the four form a group, which may be proper or improper. If proper, the three may be origin and obverse with a mediate, or a pair of mediates with origin, obverse, or another mediate; 4 types. If improper, the three must be two origins and an obverse, or an origin and two obverses; 3 types.

Five marks evenly distant containing only one pair of obverses, must be a proper group with the obverse of one of its marks; see end of Art. 9. To these we may add the origin or obverse of the proper group with a mediate distant 1 or 3 from the extra mark, or else two mediates distant 11, 13 or 33 from that mark; 7 types.

15. A seven-fold statement with two pairs of obverses may have six marks evenly distant from one another and one oddly distant from them; in this case the six are an origin and five mediates in two different ways, or say two pairs and a two; the remaining mark may be distant 11, 13 or 33 from the two, which gives 3 types.

Otherwise the seven-fold statement must subdivide (as in the last case) into five and two or into four and three. If it subdivide into five and two, the two may be a pair or not. In the first case we have a proper group and the obverse of one of its marks, together with the origin and obverse of the group or a pair of mediates; 2 types. In the second case we have five mediates of an origin or its obverse, to which we may add two proximates distant 11, 13 or 33 from the odd mediate,

or a proximate and an ultimate distant 11, 13 or 33 respectively from the odd mediate; 6 types.

If the seven-fold statement subdivide into four and three, the two pairs may be both in the four, or one in the four and one in the three. In the former case we have a triad, to which may be added the origin and obverse and a pair of mediates, or two pairs of mediates; 2 types. In the latter case the four consist of an origin and obverse and two mediates; we must add a pair consisting of a proximate and an ultimate, which may be distant 11, 33 or 13, 13 from the two mediates, and then another proximate or ultimate which may be distant 11, 13, or 33 from the two mediates; 6 types.

16. Three pairs of obverses in a seven-fold statement may be all evenly distant, or two evenly and the other pair oddly distant from each. If they are all evenly distant they are the mediates to a certain origin or its obverse, and the seventh mark may be the origin or a proximate; 2 types. In the other case we have an origin, obverse, and pair of mediates, together with a proximate and its obverse ultimate; we may add a proximate or a mediate; 2 types.

17. A pure eight-fold statement must consist of two groups, either both proper or both improper, or one of each. Two proper groups may have their origins distant 1 or 3; 2 types. To an improper group we may add a proper group made of one origin and three obverses, or of three origins and one obverse, or an improper group made of four origins or four obverses, or two origins and two obverses; 5 types. Altogether there are 7 types of pure eight-fold statement.

18. An eight-fold statement with one pair of obverses must subdivide into four and four, or into five and three. In the former case we have a pair of obverses, viz. an origin and its obverse, and two mediates; to which we must add a group formed out of the proximates and ultimates. This group may be proper, (1 type,) or improper, the mediates being in regard to it two origins, two obverses, or an origin and an obverse; 3 types. In the latter case the five marks must be a proper group with the obverse of one mark, to which we must add a triad made out of the origin, obverse, and mediates of the

group. This triad may be the origin or obverse together with two mediates distant 11, 13, 33 from the ultimate; 6 types; or else it may be three mediates distant 111, 113, 133, 333 from the ultimate; 4 types.

19. An eight-fold statement with two pairs of obverses must subdivide into four and four, or into five and three, or into six and two. In the first case the two pairs of obverses may be evenly distant, when the remaining marks form a group either proper, with its origin, obverse, and pair of mediates, or two pairs of mediates, or else improper; 3 types; or oddly distant, when the remainder form one of the six pure four-fold statements enumerated Art. 6. Two marks distant 2 from each other may be distant 11, 33 or 13, 13 from the pair of obverses which are oddly distant from them; thus each of the six four-fold statements gives 3 types of eight-fold statement, except the third, which gives 4; in all, 19. In the second case the three may be a triad, or may contain a pair of obverses. If it is a triad, the five are mediates to one origin and its obverse, and we may add three proximates distant 113 or 133 or two proximates distant 11, 13 or 33, with an ultimate distant respectively 11 or 33 from the odd mediate; 6 types. If the three contain a pair of obverses, the five make a proper group with obverse of one mark; to this we may add the origin and obverse of the group with mediate distant 1 or 3 from the ultimate, or a pair of obverse mediates with a mediate distant 1 or 3 as before; 4 types. In the third case the six must be an origin and five mediates, and we may add two proximates distant 11, 13, 33 from the odd mediate, or a proximate and an ultimate, or two ultimates, distant as before; 9 types.

20. In an eight-fold statement with three pairs of obverses these may be either all evenly distant, or two of them evenly distant and the other oddly distant from both. In the first case they are mediates to a certain origin and its obverse, and we may add the origin with a proximate or ultimate, two proximates, or a proximate and ultimate; 4 types. In the second case take the oddly distant pair for origin and obverse; then these are associated with two proximates and their ob-

verse ultimates, and we may add the two other proximates, a proximate and an ultimate, a proximate and a mediate (distant 11, 13, 31, 33 from this proximate and the remaining one), or two mediates distant 11, 33 or 13, 13 from the two proximates; 8 types.

Lastly, in an eight-fold statement with four pairs of obverses they may be all evenly distant, or the statement may subdivide into six and two, or into four and four; in the latter case there are 2 types.

21. To obtain the whole number of types, we observe that for every less-than-eight-fold type there is a complementary more-than-eight-fold type (Art. 2); so that we must add the number of eight-fold types (78), to twice the number of less-than-eight-fold types (159); the result is 396.

Art	TABLE.			
4	1-fold			1
	2-fold, distance 1, 2, 3, 4			4
5	3-fold, pure, distance 112, 222, 123, 233		4	
	1 pair obv., dist., 134, 224		2	
			6	6
6	4-fold, pure, two and two—			
	$\frac{1\,\vert\,1}{1\,\vert\,1}$ $\frac{1\,\vert\,1}{1\,\vert\,3}$ $\frac{1\,\vert\,1}{3\,\vert\,3}$ $\frac{1\,\vert\,3}{3\,\vert\,1}$ $\frac{1\,\vert\,3}{3\,\vert\,3}$ $\frac{3\,\vert\,3}{3\,\vert\,3}$	6		
	three and one	4		
	group, proper or improper	2		
		12	12	
7	,, 1 pair obv.		5	
	,, 2 pair obv., dist. odd or even		2	
			19	19
8	5-fold, pure, three and two	7		
	four and one	5		
			12	
9	,, 1 pair obv. + two prox.	5		
	+ prox. + ult.	3		
	+ two med.	4		
		12		
			12	
	2 pair obv., odd dist., 1; even, 2		3	
			27	27

TABLE (*continued*).

Art.				
10	6-fold, pure, three and three	6		
	four and two	6		
		12	12	
11	,, 1 pair obv., two and four	3		
	three and three	8		
	four and two	9		
	five and one	2		
		22	22	
12	,, 2 pair obv., odd dist., 6; even, 5		11	
	3 pair obv.		2	
			47	47
13	7-fold, pure; proper group, 4; improper, 4		8	
14	,, 1 pair obv., four and three	10		
	three and four	7		
	five and two	7		
		24	24	
15	,, 2 pair obv., six and one	3		
	five and two	8		
	four and three	8		
		19	19	
16	,, 3 pair obv.		4	
			55	55
	Total of less-than-eight-fold statements			159
	Complementary more-than-eight-fold statements			159
17	8-fold, pure		7	
18	,, 1 pair obv., four and four	4		
	five and three	10		
		14	14	
19	,, 2 pair obv., four and four	22		
	five and three	10		
	six and two	9		
		41	41	
20	,, 3 pair obv., all evenly dist.	4		
	two evenly dist.	8		
		12	12	
	4 pair obv.		4	
			78	78
	Grand Total			396

* II.

ENUMERATION OF THE TYPES OF COMPOUND STATEMENTS.

A SET of n classes A, B, C, ... involve a number 2^n of cross-divisions A, b, C, d, ... etc.; these shall be called simply *divisions*. A *statement* about these classes affirms the non-existence of one or more divisions; and it is called a one-fold, two-fold, etc., statement, according to the number of divisions whose existence it denies. A one-fold statement about a set of n classes is a 2^m-fold statement about a set of these and m other classes. Two statements are called *inverse* when each denies the divisions whose existence is allowed by the other; thus the inverse of a p-fold statement about n classes is a $(2^n - p)$ fold statement.

One division may be converted into another by the operation of changing certain classes into their complementary classes (A into a which is not-A). The number of these changes required is called the *distance* of the two divisions. Thus in four classes, the division $AbCD$ is distant two changes from the division $abcD$.

Two statements are said to be of the same type when one may be obtained from the other by a repetition of two processes: (1) the simultaneous substitution of a class and its complementary for another class and its complementary (A for B and a for b), (2) the interchange of a class and its complementary (A for a). It is clear that this amounts only to a use of new symbols for the classes involved.

If the same substitutions are performed upon any two divisions their distance will remain unaltered. Hence *if two statements are of the same type the relative distances of the divisions denied by them will be the same;* but it does not follow that whenever these distances are the same the statements are of the same type.

The whole number of statements that can be made about n classes is $2^{2^n} - 2$, when we exclude the absence of any statement and its inverse the denial of the universe. The problem of enumeration consists in the distribution of these under a finite number of types; it requires an exhaustive method of describing the types, and may be checked by counting the number of statements belonging to each.

There is only one type of one-fold statement about n classes, namely, $ABC...N = 0$; and as the equation of any division to zero gives a statement of this type it follows that there are 2^n such statements.

There are n types of two-fold statement; for the two divisions equated to zero may have any distance from 1 to n. If this distance is r, the number of statements included in the type is $2^{n-1} \dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$. The whole number of two-fold statements is of course $2^{n-1}(2^n - 1)$.

To determine the number of types of three-fold statements, let α, β, γ be the three divisions denied by such a statement, and suppose the distance $\beta\gamma$ to be not less than either of the distances $\alpha\beta$, $\alpha\gamma$. Let $\alpha\beta = r$, $\alpha\gamma = s$, and let t changes be common to the r and the s; then $\beta\gamma = r + s - 2t$. Let r not be less than s; we have $r + s - 2t \not< r$, therefore $s \not< 2t$, or t must not be greater than half the least of the numbers r, s. With the distances r, s then we shall have distinct types for the values of $t = 0, 1, 2, \ldots I\left(\frac{s}{2}\right)$ (I meaning *integral part of*); that is $I + I\left(\frac{s}{2}\right)$ types, provided that $r + s$ is not greater than

n; in this case t cannot be less than $r+s-n$. First let r be even and equal to $2p$, where $4p$ is not greater than n. Then the number of types is

$$1+2+2+3+3+\ldots+p+p+\overline{p+1}$$
$$=p+p(p+1)=p(p+2)$$
$$=\tfrac{1}{2}r(r+1),\quad 2r \not> n.$$

Next let $r=2p+1$, $2r \not> n$; then the

*   *   *   *   *

[Prof. Cayley remarks that Prof. Clifford considers, in this fragment, the further question, how many are the statements of each type. In the case of 4 classes, the results up to the 3-fold statement, see Table, p. 12, are as follows:—

	No. of Statements.	
1-fold	16	16
2-fold, dist. 1	32	
,, 2	48	
,, 3	32	
,, 4	8	120
3-fold, dist. 112	96	
,, 222	64	
,, 123	192	
,, 233	96	
,, 134	64	
,, 224	48	
		560

where the totals 16, 120, 560, &c. are, of course, the numbers for the combinations of 16 things 1, 2, 3, &c. together.

Prof. Jevons in *The Principles of Science* (Third Edition, p. 143) writes: "In the first edition (vol. I. p. 163), I asserted that some years of labour would be required to ascertain even the precise number of types of law governing the combination of four classes of things. Though I still believe that some years' labour would be required to work out the types themselves, it is clearly a mistake to suppose that the *numbers* of such types cannot be calculated with a reasonable amount of labour, Professor W. K. Clifford having actually accomplished the task." It was on reading Prof. Jevons's original statements that Prof. Clifford said laughingly, he couldn't let anyone say that and not do it straight off ..it was 'his luck' to do quickly difficult things which were usually got at by long processes only.

Dr Hopkinson has drawn my attention to the fact that the problems treated of by Clifford and Jevons are different in this respect, that Clifford's result includes what Jevons calls 'inconsistent statements'.]

III.

ON SOME PORISMATIC PROBLEMS *.

The Problem :—To draw a polygon of a given number of sides, all whose vertices shall lie on one given conic, and all whose sides shall touch another given conic : is either not possible at all, or possible in an infinity of ways. This remark, originally made by Poncelet, has been shewn by Professor Cayley to depend in a very beautiful manner upon the theory of elliptic functions; and in this way he has proved that an analogous theorem holds good wherever a (2, 2) correspondence exists: that is to say, whenever two things are so related that to every position of either there correspond two positions of the other. Two points, x, y for instance, in a conic U, which are connected by the relation that the line xy touches a second conic V, have a correspondence of this kind : for if the point x be taken arbitrarily, two tangents can be drawn from it to V, determining two positions of y: and conversely, the point y being fixed determines two positions of x. The theorem is then that in a (2, 2) correspondence there is either no closed cycle of a given order, or an infinite number. In the present communication I propose first to prove this result by the method of correspondence alone, and then to extend the proof to higher orders of correspondence.

In a (2, 2) correspondence there are $4\ (= 2 + 2)$ united points, that is to say, four points, each of which coincides with one of its correspondents. In fact, if two numbers x and y are

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Nov. 9, 1868, pp. 120—123]

connected by an equation of the second degree in each of them, then when we make x and y coincide, there results an equation of the fourth degree (Chasles, *Comptes Rendus*, 1864). I call these united points the points a. Each point a has one of its correspondents coinciding with it; it has also another correspondent b. Each point b again has another correspondent c, and so on. There are also four points α, each of which is such that its two correspondents coincide in a point β. For let q be a correspondent of p, and r a correspondent of q; then the relation between p and r is a (2, 2) correspondence (since to each position of p there are two positions of r and *vice versa*), and therefore has four united points, viz. the points β. Each of these points β has another correspondent γ, and so on. We have thus two series of points, $abcd\ldots$ $\alpha\beta\gamma\delta\ldots$ each letter indicating a set of four generally distinct points.

Let us now endeavour to obtain a closed cycle of an odd order: for distinctness' sake we will try to draw a pentagon inscribed in one conic, U, and circumscribed to another, V*. Start with a point x on the outer; pass to one of its correspondents, y; y has another correspondent, z; from z we go to u, from u to v, from v to w. If now w were the same point as x, we should have succeeded in our object. But the relation between w and x is a (2, 2) correspondence, for we might have started from x in either of two directions. The united points of this correspondence should therefore apparently give solutions of our problem.

But these united points are no other than the four points c. For starting with one of these, we get the cycle $cbaabc$, which is a sufficient solution of the correspondence problem last enunciated. But it is *not* a solution of the original problem: for the series will go on $cbaabcde\ldots$ and not repeat itself, so that the points $cbaab$ do not form a *proper* in-and-circumscribed pentagon. Thus the problem is in general impossible. If however there is any proper solution, the equation of the fourth degree

* [Supposing as before that the corresponding points x, y in the conic U are such that the line xy touches the conic V, then, as is easily seen, the points a are in fact the points of contact with U of the common tangents of U and V; and the points a are the points of intersection of U and V.—C.]

(which determines the improper solutions) will have more than four roots, and will therefore be identically satisfied by any number whatever; so that whatever point x we start with, the point w will come to coincide with it.

Precisely similar reasoning is applicable to the cycles of an even order. Thus, *e.g.* for a quadrilateral we get the four improper solutions $\gamma\beta\alpha\beta$, got by starting from the points γ. I pass to the consideration of correspondences of higher orders.

In an (r, r) correspondence there are

$$2r \text{ united points } a;$$

their remaining correspondents form

$$2r(r-1) \text{ points } b;$$

to these again correspond

$$2r(r-1)^2 \text{ points } c, \text{ and so on.}$$

Similarly, there are

$$2r(r-1) \text{ points } \alpha,$$

each of which is such that two of its correspondents coincide; viz. these are

$$2r(r-1) \text{ points } \beta,$$

to which also correspond

$$2r(r-1)^2 \text{ points } \gamma, \text{ and so on.}$$

Now if we attempt to form a closed cycle of the n^{th} order, we shall be led to a correspondence

$$\{r(r-1)^{n-1},\quad r(r-1)^{n-1}\},$$

which has $2r(r-1)^{n-1}$ united points. From this number we shall have to subtract the number of improper solutions as given by our previous reasoning; thus we shall find

$$(n = 2m+1),\ \{2r(r-1)^{2m} - 2r(r-1)^m\} \text{ proper solutions,}$$
$$(n = 2m),\ \{2r(r-1)^{2m-1} - 2r(r-1)^m\} \text{ proper solutions.}$$

For example, the problem to inscribe in a conic a triangle whose sides shall touch a given curve of the third class admits of twelve proper and twelve improper solutions. If the number of proper solutions exceeds this number, the problem becomes porismatic: that is to say, there is an infinite number of solutions.

IV.

PROOF THAT EVERY RATIONAL EQUATION HAS A ROOT*.

(*Abstract.*)

THE proof contained in the present communication depends on the determination of a quadratic factor of the rational integral expression

$$x^{2s}+a_1x^{2s-1}+a_2x^{2s-2}+\dots+a_{2s}.$$

On dividing this expression by $x^2+p_1x+p_2$, we obtain by the ordinary algebraic rules a remainder of the form $M_{2s-1}x+N_{2s}$, where M_{2s-1} and N_{2s} are functions of p_1 and p_2 whose weights are $2s-1$ and $2s$ respectively, and which may accordingly be written in the forms

$$M_{2s-1}=b_{2s-1}+p_2b_{2s-3}+\dots\dots+p_2^{s-1}b_1,$$
$$N_{2s}=c_{2s}\quad+p_2c_{2s-2}+\dots\dots+p_2^{s},$$

where the b, c are of an order in p_1 indicated by their suffixes. On writing down (by Professor Sylvester's Dialytic method) the result of eliminating p_2 between these equations, it is at once apparent that this resultant is of the order $s(2s-1)$. Thus the determination of a quadratic factor of an expression of degree $2s$ is reduced to the solution of an equation of order $s(2s-1)$. But this number is *one degree more odd* than the original number $2s$; that is to say, if the number $2s$ is 2^k multiplied by an odd number, then $s(2s-1)$ is 2^{k-1} multiplied by an odd number. Hence by a repetition of this process we shall ultimately arrive at an equation of odd order, which, as is well known, must have a real root. By then retracing our steps the existence of a quadratic factor of the original expression is demonstrated.

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Feb. 21, 1870, pp. 156, 157.]

V.

ON THE SPACE-THEORY OF MATTER*.

(*Abstract.*)

RIEMANN has shewn that as there are different kinds of lines and surfaces, so there are different kinds of space of three dimensions; and that we can only find out by experience to which of these kinds the space in which we live belongs. In particular, the axioms of plane geometry are true within the limits of experiment on the surface of a sheet of paper, and yet we know that the sheet is really covered with a number of small ridges and furrows, upon which (the total curvature not being zero) these axioms are not true. Similarly, he says although the axioms of solid geometry are true within the limits of experiment for finite portions of our space, yet we have no reason to conclude that they are true for very small portions; and if any help can be got thereby for the explanation of physical phenomena, we may have reason to conclude that they are not true for very small portions of space.

I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold in fact

(1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average

* [From *Cambridge Philosophical Society's Proceedings*, II. 1876. Read Feb. 21, 1870, pp. 157, 158.]

flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

I am endeavouring in a general way to explain the laws of double refraction on this hypothesis, but have not yet arrived at any results sufficiently decisive to be communicated.

[In Paper XV. (infra) reference is made to a *proof that every rational equation has a root*, but I have not found any MS. of which IV. could be looked upon as the Abstract. Prof. Clifford once remarked to me, I think, that a paper on the same subject by Mr J. C. Malet (read before the Mathematical Society, June 14th, 1877, and printed in the *Transactions of the Royal Irish Academy*, Vol. XXVI. No. xiv.) treated the question from a somewhat similar point of view. The subject of V. was introduced at greater length to English mathematicians in a translation (IX. infra) of Riemann's *Habilitationsschrift*, 1854; see the *Gesammelte mathematische Werke*, pp. 254—269.]

VI.

ON JACOBIANS AND POLAR OPPOSITES*.

I. THE word Jacobian is commonly understood to mean a determinant formed from the n^2 differential coefficients of n functions, each of n variables. For instance, given two homogeneous functions of the second degree in x and y, as U, V, each representing two points; the Jacobian is

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy} \\ \frac{dV}{dx}, & \frac{dV}{dy} \end{vmatrix},$$

which is known to represent the foci of the involution determined by U, V. I propose to extend the meaning of the term so as to include *systems* of determinants, formed in precisely the same way from a number of functions *not* equal to the number of variables. For instance, if $U=0$ is the trilinear equation to a conic section, and $L=0$ that to a straight line, then the system of determinants

$$\left\Vert \begin{matrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dL}{dx}, & \frac{dL}{dy}, & \frac{dL}{dz} \end{matrix} \right\Vert = 0,$$

will be found to represent the pole of the line L with respect to the conic U; and I propose to call this system the Jacobian

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. II. pp. 229—239.]

of U and L. In all cases I shall denote the Jacobian of a number of functions A, B, C, ... whether equal or unequal to the number of variables, by the symbol $J(A, B, C, ...)$.

II. As the theory of polar opposites is of constant occurrence in the interpretation of Jacobians, I here set down the heads of it. If the polars of a point A with respect to two conics meet in B, then it is clear that the polars of B will meet in A. On account of this reciprocal relation, I call the points A and B *polar opposites* with respect to the two conics. Consider first one conic; it is known that any line through a point A is harmonically cut by its polar at B and by the conic. The line AB, then, is harmonically divided by both conics, and therefore by any conic through their intersections; *e.g.* by any pair of common chords. It is thus evident that *any* straight line has a pair of polar opposites upon it, which are in fact the foci of the involution in which the line is cut by the conics.

In the same way it may be seen, that if a straight line A joins the poles of B with respect to two conics, B will join the poles of A, and the two lines may be called polar opposites of one another. The point AB is subtended by the two conics and the six intersections of common tangents in an involution of ten rays, of which A and B are the double or sibi-conjugate rays. And through any point we can draw a pair of polar opposite lines, which are in fact the double lines of the involution determined by the four tangents drawn from the given point to the conics.

III. Consider next the case of two conicoids: here the polar planes of any points will intersect on a line, which may be called the polar opposite line of the point. Let A be the point, and BC its opposite line, then it is easy to see that the plane ABC will cut either conicoid in a section, with respect to which A is the pole of BC, and that the same will be true of any conicoid passing through the curve of intersection of two given ones; *e.g.* of the four cones which can be so drawn. This naturally suggests the idea of a triangle, each side of

which is the polar opposite of the corresponding vertex; such a triangle is merely the common self-conjugate triad of the two sections made by its plane. Again, the line BC joining the two poles of any plane A, may be called the polar opposite line of the plane; and the cones drawn from the point ABC touching the two conicoids, will have BC and A for polar line and plane; and the same will be true of the cone drawn touching any conicoid inscribed in the same developable as the two given ones. This again suggests a system of three planes, each the polar opposite of the line of intersection of the other two; but such a system is merely the self-conjugate triad of the two tangent cones drawn to the conicoids from the point of intersection of the planes.

IV. I pass to the case of *three* conicoids. Here a point will have three polar planes, meeting in its polar opposite. Let A and B be opposites, then it is easy to see, as before, that the line AB is harmonically divided by each of the conicoids, and therefore by any conicoid through their eight points of intersection; *e.g.* by any pair of planes which can be drawn to contain the eight points. Similarly, the three poles of a plane A are joined by its polar opposite plane B, and the line AB is subtended by the conicoids in an involution, of which A and B are the double or sibi-conjugate planes.

V. I proceed now to the interpretation of some Jacobians, and commence with those of binary quantics. I shall throughout use the letters $L, M, N, \ldots$ for equations of the first degree, and generally $U, V, \ldots$ for those of the second.

1. $J(L, M)$ is obviously the distance between the points L and M*. In general, if L, M are linear functions of any number of variables, $J(L, M) = 0$ is the condition of their coincidence. The distance between two points may also be expressed thus: if Δ stands for $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)$, then ΔL is the

* Strictly, the distance is $\frac{J(L, M)}{(l_1+m_1)(l_2+m_2)}$, between l_1x+m_1y and l_2x+m_2y.

distance between L and (ξ, η), and generally, with any number of variables, ΔL is proportional to the distance between L and the point $(\xi, \eta, \zeta \ldots)$.

2. $J(U, L)$, where U is a quadric, is the harmonic conjugate of L with respect to U. This may also evidently be expressed by ΔU. For instance, the identity

$$\Delta(\Delta U . \Delta V) = \Delta^2 U . \Delta V + \Delta U . \Delta^2 V$$

may be interpreted thus, if we remember that $\Delta^2 U$ is the product of the distances of (ξ, η) from the two points denoted by U: "Let P and Q be the harmonic conjugates of a point O with respect to A, B and C, D respectively, and let R be the harmonic conjugate of the same point with respect to P, Q; then

$$\frac{RP}{OA . OB} + \frac{RQ}{OC . OD} = 0."$$

3. $J(U, V)$, U and V being quadrics, is well known to represent the foci of the involution determined by U and V.

4. $J(S, L)$, where S is a cubic, is the polar quadric of L with regard to S; that is to say, if S represents the three points A, B, C, then $J(S, L)$ represents two points P, each of which possesses the property

$$\frac{LA}{PA} + \frac{LB}{PB} + \frac{LC}{PC} = 0.$$

They might also evidently have been represented by $\Delta S = 0$. The polar *point*, $\Delta^2 S$, is the harmonic conjugate of L with respect to the two points P; and if we call it R, it possesses the property

$$\frac{RA}{LA} + \frac{RB}{LB} + \frac{RC}{LC} = 0.$$

5. The interpretation of all Jacobians not involving points is easy, when we remember that $J(A, B)$ is in fact the eliminant of ΔA, ΔB. Thus $J(S, U)$, S being a cubic and U a quadric, represents the three points, whose polar points with respect to the cubic, are the same as their harmonic

conjugates with respect to the quadric. If the quadric is the Hessian of the cubic, their Jacobian is no other than the cubi-covariant J. (Salmon's *Higher Algebra*, p. 99.) The Hessian represents two points, each of which is the polar quadric of the other; and if we take the harmonic conjugate of each point of the cubic with respect to its Hessian, we shall obtain the covariant J. Let A, B, C be the points of the cubic, and D, E its Hessian; then the three anharmonic ratios

$$[ADBE], \quad [BDCE], \quad [CDAE],$$

are all equal to one another, as appears readily from the canonical form of the cubic. The polar point of any of the points of J with regard to the cubic, being the same as its harmonic conjugate with respect to the Hessian, must be one of the points of the cubic itself; and in this case, the relation at the end of (4), shews that the four points form an harmonic range. We are thus led to Dr Salmon's construction for the covariant J, viz. it contains the harmonic conjugate of each point of the cubic relatively to the other two. In general the Jacobian of a cubic and a quadric does not represent the harmonic conjugates of the cubic with respect to the quadric. $J(S, T)$, S and T being cubics, of course represents the four points whose polar points are the same with regard to the two cubics. The Jacobian of a cubic and its covariant J is the discriminant multiplied by the square of the Hessian.

VI. I put together the most analogous forms of Jacobians in three and four variables, reserving part of the latter for separate consideration.

1. $J(L, M, N) = 0$ and $J(L, M, N, R) = 0$ are known to be respectively the conditions that the three lines and four planes may meet in a point*.

* This Jacobian enables us to express many metrical functions of lines and planes. Let $X=0$ denote the line or plane at infinity, $\phi(L)$ the condition that the line L shall pass through either circular point at infinity, and let

$$\phi(L+\kappa M) \equiv \phi(L) + 2\psi(L, M) \,.\, \kappa + \phi(M) \,.\, \kappa^2.$$

Consider now three straight lines L, M, N, and put J for the Jacobian,

2. $J(U, L, M)$, U being a conic, and L, M lines, is the polar of the point LM with respect to U, or the locus of the pole of $lL + mM = 0$. Similarly, if U is a conicoid, and L, M, N planes, $J(U, L, M, N)$ may be interpreted either as the polar plane of the point LMN, or as the locus of the pole of $lL + mM + nN = 0$, l, m, n being arbitrary. $J(U, L)$ means, as before, the pole of L with respect to U.

3. $J(U, V, L)$ is, first, the locus of polar opposites of points on the line L; and, secondly, the locus of the pole of L with respect to $lU + mV = 0$.

If we write U, V in the canonical forms

$$a_1x^2 + b_1y^2 + c_1z^2 = 0,$$
$$a_2x^2 + b_2y^2 + c_2z^2 = 0,$$

then, putting A for the determinant $\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}$, &c., and

$$lx + my + nz = 0$$

for the equation to the line L, we find the Jacobian to be

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} = 0,$$

P for $J(X, M, N) . J(L, X, N) . J(L, M, X)$, and Π for $\phi(L)\ \phi(M)\ \phi(N)$, then

$$\text{area of triangle } LMN \equiv \frac{J^2}{P},$$

$$\text{product of sides} \equiv 2\sqrt{2}\frac{J^3\sqrt{(\Pi)}}{P^2},$$

$$\frac{(\text{area})^2}{\text{product of sides}} \equiv \frac{J}{2\sqrt{(2\Pi)}}$$

(a function of some little importance),

$$\text{radius of circumscribed circle} \equiv \frac{J}{P}\sqrt{(2\Pi)},$$

$$\text{product of sines of angles} \equiv \frac{P}{\Pi}.$$

It is to be noticed that (as is indeed evident) $\phi(L)\ \phi(M) - \psi(L, M)^2$ is identically equal to $J(L, M, X)^2$. See Mr Greer's valuable "Notes," (*Quarterly Journal* for March, 1864).

Similarly, volume of tetrahedron $LENR \equiv \frac{J^3}{P}$, and so on. In all these results there is a factor present, which is easily determined by reference to the fundamental triangle or tetrahedron.

shewing that it always circumscribes the common self-conjugate triangle. The discriminant of this is $ABClmn$, whose evanescence is the condition that L shall pass through one of the vertices of the triangle. Thus we see that *the discriminant with respect to x, y, z of the Jacobian of two conics and the line $\xi x + \eta y + \zeta z = 0$, is a contravariant representing the vertices of the common self-conjugate triangle.* So with four variables, $J(U, V, W, L)$ is both the locus of polar opposites of points in the plane L with regard to U, V, W, and the locus of the poles of L with regard to all the conicoids $lU + mV + nW = 0$. And with a similar notation to that employed above, in case the three conicoids have a common self-conjugate tetrahedron, the Jacobian may be written

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} + \frac{rD}{w} = 0,$$

shewing that it contains the edges of the tetrahedron and has double points at the vertices. The discriminant is $ABCDlmnr$, which gives, as before, the theorem, that *when three conicoids have a common self-conjugate tetrahedron, the discriminant with respect to $xyzw$ of the Jacobian of the three conicoids and the line $\xi x + \eta y + \zeta z + \omega w = 0$ is a contravariant representing the vertices of the tetrahedron.* When, in the plane case, the line L is at infinity, the Jacobian represents, as remarked by "Lanivicensis" in the last *Messenger*, the nine-point conic of the quadrangle which is the intersection of U and V; and we see that the polars of any fixed point on this conic, with regard to any conic $lU + mV = 0$, are parallel to a fixed line. In the solid case, it must be remembered that when three conicoids have a common self-conjugate tetrahedron, their eight points of intersection will form a figure which is the "projection" of a parallelepiped, and so $lU + mV + nW$ may in six ways be made to represent two planes. By sending the plane L to infinity, then, we learn that eight points so situated determine a surface of the third order, which bisects the line joining any two of the points, and contains the lines in which intersect the six pairs of planes containing the eight points. Moreover, the section of this surface made by any one of the planes consists of the

line where it meets the corresponding plane, and the nine-point conic of the quadrangle formed on it by the other four planes. And the polar planes of any fixed point on this surface, with respect to all conicoids $lU + mV + nW$, will be parallel to a fixed line.

4. $J(U, V)$ represents the vertices of the common self-conjugate triangle or tetrahedron, according as U and V are conics or conicoids.

5. $J(U, V, L, M)$ is the polar opposite of the point LM with regard to the conics U, V. It is clearly the point common to the Jacobians of U, V, and all lines through the point LM. Also the form of the Jacobian shews that it is common to the polars of LM with respect to all the conics $lU + mV = 0$. So, in Solid Geometry, $J(U, V, L, M, N)$ is the polar opposite line of the point LMN; and so also $J(U, V, W, L, M, N)$ is the polar opposite *point* of LMN with respect to U, V, W.

6. $J(U, V, W)$, the Jacobian of three conics, will be found explained at the end of Dr Salmon's *Conic Sections* and in his *Higher Algebra*. I have only to add that it is the locus of the vertices of all triangles self-conjugate to two conics of the forms

$$lU + mV + nW = 0, \quad \lambda U + \mu V + \nu W = 0,$$

and that lines whose three poles lie in a straight line, meet the lines joining their poles on the Jacobian. So if U, V, W, X are four conicoids, $J(U, V, W, X)$ is the locus of points whose polar planes meet in a point, this latter being also a point on the Jacobian; and the locus of the vertices of all cones which can be represented by

$$lU + mV + nW + rX = 0;$$

and consequently the locus of the lines of intersection of all pairs of planes which can be represented by this equation. Moreover it contains the vertices of all tetrahedra self-conjugate with regard to two conicoids of the above form; and the *edges* of all tetrahedra self-conjugate with regard to *three* conicoids of the same form. And obviously, any line whose

polar lines are identical with regard to the four conicoids, will lie on the Jacobian; as is implied by the property last stated.

VII. There are some Jacobians in four variables which have no analogues* in three variables. The simplest is:

1. $J(U, L, M)$, the polar line of LM.

2. $J(U, V, L)$ is the locus of points whose polar opposite lines lie in the plane L, and the locus of the poles of L with regard to all conicoids $lU+mV=0$. It is a twisted cubic passing through the vertices of the tetrahedron self-conjugate with respect to U and V, and cutting the plane L in the vertices of the self-conjugate triad of the sections in which it meets U, V.

3. $J(U, V, L, M)$ contains the polar opposite lines of all points in the line LM, and the polar lines of LM with regard to all conicoids $lU+mV=0$. It also contains the poles of all planes of the form $\lambda L+\mu M=0$ with respect to $lU+mV=0$. It is clearly a conicoid passing through the vertices of the tetrahedron self-conjugate with regard to U, V, and containing all the twisted cubics

$$J(U, V, lL+mM)=0.$$

4. $J(U, V, W)$ is a curve of the sixth degree containing the vertices of all cones drawn through the eight intersections of U, V, W (Dr Salmon). Hence it is the locus of the vertices of tetrahedra self-conjugate with regard to two conicoids

$$lU+mV+nW, \quad \lambda U+\mu V+\nu W.$$

If the three conicoids have a common self-conjugate tetrahedron, this sextic represents its edges.

I have given no account of the Jacobians of ternary or quaternary cubics, but their interpretation will be easy with the aid of the following theorem. Consider any number of groups, $A_1, A_2, A_3, \ldots$; $B_1, B_2, \ldots$; $C_1, C_2, \ldots$, &c.; all the

* [1 and 2 seem closely analogous to the 2, 3 of p. 28. C.]

A's being quantics of the same degree, and all the B's, &c., but the A's not necessarily of the same degree as the B's; and form any number of quantics in involution with these, as

$$l_1 A_1 + m_1 A_2 + n_1 A_3 + \ldots\ldots (a_1),$$
$$l_2 A_1 + m_2 A_2 + n_2 A_3 + \ldots\ldots (a_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \&c.$$

Now suppose that we have a geometrical interpretation for $J(A, B, C)$, for $J(A_1, A_2, B, C)$, and so on; then

$$J(A_1, A_2, \ldots; \; B_1, B_2, \ldots; \; C_1, C_2, \ldots)$$

will include all the loci formed in the same way as

$$J(a, b, c)^*; \; J(a_1, a_2, \ldots; \; b_1, b_2, \ldots; \; c_1, c_2, \ldots), \; \&c.$$

VIII. I add an example or two to shew that the theory of Polar Opposites is not wholly barren of results.

1. A pair of conjugate diameters cuts the line at infinity in two conjugate points, *i.e.* two points each of which lies in the polar of the other. Hence any two conics have two pairs of conjugate diameters parallel, which cut the line at infinity in a pair of polar opposites†. A particular case is when the axes are parallel, when we see at once, by (II.), that any pair of common chords will be equally inclined to them, and so a circle will pass through the quadrangle of intersection. Similarly, if three conicoids have their principal planes parallel, the eight points of intersection will lie on a sphere.

2. If from the centre O of any circle a normal ON be drawn to a conic, and if two common tangents of the circle and conic intersect in A, and the other two in B, then AN and BN are equally inclined to ON. For it is easily seen that ON and the tangent at N are polar opposite lines. When the point O is on the conic, Professor Cayley has shewn that this

* [? $J(a_1, b_1, c_1)$.]

† [i. e. parallel to each other, and each cutting the line at infinity in one and the same pair of polar opposites. C.]

theorem is a particular case of the theorem that three conics inscribed in the same quadrilateral subtend any vertex in involution (*Educational Times* for December)*. It will be seen that Art. II. is only a development of this remark.

3. The last note on p. 34 of Dr Salmon's *Conics* will be a little plainer if stated as a property of the nine-point conic of any quadrilateral inscribable in a circle. For

$$J(U, V, L^2) \equiv L \, . \, J(U, V, L).$$

So, if U is a sphere, and L the plane at infinity, the twisted cubic $J(U, V, L)$ passes through the feet of the six normals that can be drawn from the centre of U to the conicoid V.

4. If A and B are polar opposites with respect to two circles, AB is bisected by the radical axis, and the circle on AB as diameter cuts both circles orthogonally. Hence, immediately, Dr Salmon's theorem, *Conics*, p. 344, Ex. 3, that the Jacobian of three circles is the circle cutting them orthogonally, together with the line at infinity. It is thus seen that the polars of a point A on the Jacobian meet in the opposite extremity of the diameter through A.

* [*Mathematical Questions with their Solutions.* From the *Educational Times*, Vol. I. p. 33]

VII.

ON THE PRINCIPAL AXES OF A RIGID BODY*.

THE object of this note is to simplify the manner in which the theory of principal axes is made to depend on the theory of confocal surfaces of the second order.

It is well known†, that if A, B, C are the principal moments of inertia of a rigid body at the centre of gravity, the moment of inertia about an axis through the centre of gravity, whose direction-cosines referred to the principal axes are l, m, n is

$$l^2A+m^2B+n^2C \quad\text{.....................}\quad (1).$$

But if M be the mass of the body, and we draw the ellipsoid

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=\frac{1}{M} \quad\text{.....................}\quad (2),$$

(ellipsoid of gyration), this amounts to saying that the moment of inertia is Mp^2, where p is the perpendicular from the centre on the tangent plane

$$lx+my+nz=p \quad\text{.....................}\quad (3),$$

of the ellipsoid (2); that is, if we draw a plane perpendicular to the given axis to touch the ellipsoid (2), then p is the central perpendicular on this plane.

It is also known‡ that the moment of inertia about any

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. IV. pp. 78—81.]

† Routh's *Rigid Dynamics*, p. 7.

‡ Routh, *l. c.* p. 2.

axis whatever, is equal to the moment about a parallel axis through the centre of gravity, together with the moment which the whole mass, if collected at the centre of gravity, would have about the original axis.

These things being so, the following is a construction for the moment of inertia about any axis PQ through a point P. Draw a plane PT through P perpendicular to the axis; then determine λ so that the surface

$$\frac{x^2}{A+\lambda M}+\frac{y^2}{B+\lambda M}+\frac{z^2}{C+\lambda M}=\frac{1}{M},$$

may touch this plane PT; this gives a simple equation* for λ, which has consequently only one value. Then the required moment of inertia is $M(OP^2-\lambda)$, O being the centre of gravity, which is the origin.

For, draw OT perpendicular to the plane PT; and let Ot be the perpendicular on the parallel plane which touches the ellipsoid of gyration. Then OT being a parallel axis through the centre of gravity, and PT the perpendicular distance of O from PQ; the moment of inertia about PQ is

$$M(Ot^2+PT^2).$$

But† $$OT^2=l^2\left(\frac{A}{M}+\lambda\right)+m^2\left(\frac{B}{M}+\lambda\right)+n^2\left(\frac{C}{M}+\lambda\right)$$
$$=\frac{l^2A+m^2B+n^2C}{M}+\lambda$$
$$=Ot^2+\lambda.$$

Hence $$Ot^2+PT^2=OT^2+PT^2-(OT^2-Ot^2)$$
$$=OP^2-\lambda,$$

the moment of inertia is therefore $M(OP^2-\lambda)$.

Now, consider the confocal ellipsoid which passes through P, and its tangent plane (A) at P. Let λ_1 be the value of λ for this ellipsoid. Then if the plane A be turned ever so little round P, it will begin to cut the ellipsoid in a small ellipse;

* Salmon's *Geometry of three dimensions*, p. 121.

† Salmon, *l. c.* p. 58.

and so the confocal ellipsoid which touches the plane A in its new position, will lie wholly within the other. Its axes will therefore be less than those of the other, and therefore λ will be less than λ_1. That is to say, the value of λ is a *maximum* for the plane which touches the confocal ellipsoid through P. Therefore the moment of inertia about the axis perpendicular to this plane, namely $M(OP^2 - \lambda_1)$, is a *minimum*. Now the axis of least moment at any point is a principal axis. It follows therefore, that *the normal to the confocal ellipsoid through* P *is the principal axis of least moment at* P.

In a manner precisely similar, it may be shewn that if we draw through P an hyperboloid of two sheets confocal to the ellipsoid of gyration, the value of λ for its tangent plane at P is a *minimum;* and therefore that the normal to this surface at P is the principal axis of greatest moment at P.

This being so, we know that the remaining principal axis is perpendicular to these two, and is therefore normal to the confocal hyperboloid of one sheet which passes through P.

We have proved then that the principal axes at any point P, are the normals to the three surfaces confocal to the ellipsoid of gyration which pass through P; and if λ, μ, ν are the values of λ for these three surfaces, or, as we may say, if λ, μ, ν are the curvilinear co-ordinates of P in respect of the ellipsoid of gyration, then the three moments of inertia are

$$M(OP^2 - \lambda), \quad M(OP^2 - \mu), \quad M(OP^2 - \nu).$$

This connection having been established, all the usual propositions about principal axes follow at once from the known theory of confocal surfaces. Thus, the locus of points where two of the principal moments are equal, is the focal conics of the ellipsoid of gyration. For*, of the three confocal surfaces which pass through any point on a focal conic, two coincide with the focal conic itself. For any point on the focal ellipse, the momental ellipsoid is an oblate spheroid; and for

* Salmon, *l. c.* p. 114.

any point on the focal hyperbola, the momental ellipsoid is a prolate spheroid.

Again*, two confocal surfaces can be drawn to touch any given straight line, and the two corresponding tangent planes are at right angles. Hence, on every straight line there are two points at which one principal axis is perpendicular to the line, and these two axes are at right angles. If the two points of contact coincide, then at that point two principal axes are perpendicular to the line, which is therefore itself a principal axis. The condition, therefore, that an axis may be a principal axis at some point of its length, is that the two points of contact of confocal surfaces touching it must coincide; which is obvious, for in that case the line is normal to the third surface passing through the common point of contact.

* Salmon, *l. c.* p. 127.

VIII.

SYNTHETIC PROOF OF MIQUEL'S THEOREM*.

In the note to page 235 of Dr Salmon's *Conics*, mention is made of a theorem originally given by M. Auguste Miquel, in Liouville's *Journal*, Vol. x., p. 349. The theorem may be stated as follows. It is known that we can draw exactly one parabola to touch four given lines. If now we have five lines given, we can draw five parabolas, each of which touches four of the given lines. The theorem is that the foci of these five parabolas lie all on one circle. M. Miquel's proof, reproduced by Catalan, depends on the fact that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. Since, as above remarked, a parabola can be drawn to touch any four straight lines, it follows from this that the four circles, circumscribing the four triangles which we get by leaving out each of the lines in turn, all meet in a point, the focus of the parabola. The theorem can thus be stated as a property of straight lines and circles, without any mention of parabolas; and accordingly M. Miquel proves it by using the ordinary (Euclid) geometry appropriate to such theorems.

I find, however, that the theorem connects itself in a somewhat interesting way with that general conception of geometrical facts which regards all the properties of a curve or other continuous figure as depending upon its order alone, and which it has become usual to call Synthetic Geometry. I am not

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. v. pp. 124—141.]

satisfied with the word, and use it under protest for want of a better. The *thing* should be more widely known than it is in this country; the simplicity and instructiveness of its application in the present case are my reasons for calling attention to a matter otherwise unimportant. Those who are of the mind of Herr Gretschel may substitute for the words "Synthetic Proof," "Proof by *Organic* Geometry."

I.

On the Shapes of Certain Curves.

It is very important that we should know what we mean by a *curve*. To this end I start with Plücker's mode of generating curves.

Imagine a point and a straight line passing through it to be moved about on a plane in this manner. The point always moves along the line, never stopping for any portion of time however small; and all this while the line is turning round the point and never stops for any portion of time however small. Then the point will *describe*, and the line will *envelope*, a curve. To get a clear notion of this, take the particular case of a line being rolled round a circle (without sliding) so as always to touch it. The point of contact will then be always moving along the tangent, and the tangent will always be turning round the point of contact. But, at the same time, the point will by its motion have traced out the circle; and if we imagine all that part of the plane which the line passes over to become black, there will be left a white patch bounded by this circle, *enveloped* by all the positions of the moving line.

Now here there are three things to consider:—

A. The actual curve which you see, dividing that portion of the plane which is inside it from the portion which is outside it.

B. The assemblage of all the positions of the moving point.

C. The assemblage of all the positions of the moving line.

It is very easy but at the same time very important to observe that these are three distinct things. We are accustomed to say that *A* is the *locus* of *B* (the points) and the *envelope* of *C* (the tangents). Every curve of course has an assemblage of points upon it, and an assemblage of lines touching it; but it is not *the same thing* as either of these, any more than the assemblage of points is the same thing as the assemblage of lines. I shall illustrate this further by taking the two simplest and most fundamental cases.

Namely, suppose first that the point, instead of moving, remains always at rest while the line turns round it. Then the figure *A* is merely the point itself and no longer a curve. *B* has entirely disappeared; there is no assemblage of positions of the moving point. For by an *assemblage* of positions [of a point] we mean at least a line; now a point is absolutely *no* line, as a line is *no* surface, and a surface *no* space. *C* is now the assemblage of lines through the point. So then of our three things *A* and *C* remain, while *B* has entirely disappeared; but it is exceedingly obvious that a point and the assemblage of straight lines through it are two different things. Next suppose that the line remains still while the point moves along it. Then *A* is the straight line, which we are not accustomed to call a curve. *B* is the assemblage of all the points on this line. *C* has entirely disappeared, for there is no assemblage of lines. But a straight line is not the same thing as all the points on it, though you may think so at first. To be convinced, contemplate the other case just considered; you have just as much right to say that a point is the same thing as all the lines through it. And then read S. Thomas Aquinas on this question, if you can find the reference, which I have forgotten.

An assemblage of points is said to be of a certain *order*, when a certain number of the points can be found upon an arbitrary straight line. Thus the assemblage of points lying upon a straight line is of the *first* order, because *one* of the points can be found upon another arbitrary straight line.

The assemblage of points lying upon two straight lines is of the second order, since two of them can be found upon another arbitrary straight line. And generally the assemblage of all the points lying upon n straight lines is of the n^{th} order.

Similarly, an assemblage of lines is said to be of a certain *class*, when a certain number of the lines can be drawn through an arbitrary point. Thus the assemblage of lines passing through a point is of the *first* class, because *one* of the lines can be drawn through another arbitrary point. The assemblage of lines passing through two points is of the second class, because two of the lines can be drawn through another arbitrary point. And generally the assemblage of all the lines passing through n points is of the n^{th} class.

We have now a number appertaining to the aggregate of points, viz. its *order*, and a number appertaining to the aggregate of lines, viz. its *class*. Neither of these numbers belong in strictness to the curve itself; but there is a number—the *Geschlecht-zahl* or *deficiency*—which does belong to the curve, and not immediately to the points or tangents*. It is not however my business to speak of that now. I am going to investigate certain cases in which these two things—the assemblage of points, and the assemblage of tangents—change continuously together; and in which it is very important to observe the modifications which *both* of them undergo.

A curve of the third class is, in general, of the shape represented in fig. 1; that is to say, it consists of a tricusp surrounded by an oval. It is very easy to see that from any point within the tricusp we can draw three tangents to it and none to the oval; from any point in the intermediate space we can draw one tangent to the tricusp and none to the oval; and from any point outside the oval we can draw two tangents to it and one to the tricusp. Such a curve is of the sixth order; and it is singled out among curves of the

* [This incidental remark seems noteworthy: the number in question belongs as much, and in the same way, to the tangents as to the points, that is, not peculiarly to either: but the author's point of view seems to be a different one. C.]

sixth order as having nine cusps, of which as you see only three are real. The tangents at three real cusps always meet in a point*.

Here then is an assemblage of points which is of the sixth order connected with an assemblage of lines which is of the third class by the fact that they are respectively the points and tangents of a certain curve. I am going to alter the curve, and to watch what becomes of these two assemblages.

Imagine now that two of the cusps approach very near to the oval, as in fig. 2. I call these two cusps a and b, and I want to attend particularly to two portions of the curve; namely, (1) the branch of the tricusp which is between these two cusps, and (2) the portion of the oval which is between the two points to which the cusps are very near. Suppose these two portions to become flatter and flatter, and to approach nearer and nearer to each other; what becomes of the tangents to them? All these tangents get to differ less and less from the line joining the two cusps. At last, suppose that they all coincide with it, and let us watch what becomes in this case of the curve, its points, and its tangents. First, for the curve; the two other branches of the tricusp join on to the remaining portion of the oval and form a figure like a cardioid, represented in fig. 3. The assemblage of tangents must remain of the third class; it consists just of the tangents to this cardioidal curve, *among which the line ab counts for two.* Lastly, for the points; these are in the first place the points of the cardioidal curve afore-mentioned, clearly enough. But besides these we have to account for the points on the two portions which coincided with the segment ab. These obviously pass continuously into the points upon the linear segment ab, and each of these counts for two. Moreover, there were two series of invisible points of the curve very near to all the rest of the line ab, outside this segment; and at the instant that the two visible portions united on the segment, these invisible portions started into

* Mr Cotterill is, I believe, the first person that ever *saw* a curve of the third class.

visible existence by uniting all along the rest of the straight line. I affirm this dogmatically, and there is no reason for you to believe it, unless you are acquainted with the theory of invisible points. Our assemblage of points which was of the sixth order has then broken up into the points on a cardioidal curve of the fourth order, and the points on a straight line each counting for two.

Other changes have taken place at the same time. I said that the original curve of the third class and sixth order had nine cusps, three visible and six invisible. In the transformation which the curve has undergone, we have seen it acquire a double tangent and lose two visible cusps. Now I assert—dogmatically as before—that besides these two visible cusps it has also lost four of the invisible ones; so that the cardioidal curve which is left has three cusps, one visible and two invisible. And it is a general rule (discovered by Plücker and explained in Dr Salmon's *Higher Plane Curves*, p. 73) that every curve of given class which acquires a double tangent loses thereby six cusps; exactly as a curve of given order which acquires a node loses thereby six inflexions. Of course it is now natural to ask "can we not give the curve a double tangent in such a way as to cut off all the invisible cusps, and leave only the three real ones?" We can do this; but it is necessary first to enter into explanations in regard to two possible kinds of double tangents. Precisely as a double point may be either a point at which two visible branches of a curve cross, or else a conjugate point, the limit of a very small oval; so a double tangent may either have two visible points of contact (as in fig. 3), or two invisible ones*. In the latter case it is called an *ideal* tangent (Poncelet) and appears to have—like a conjugate point—nothing to do with the curve. Now a curve of the third order (fig. 4) may have a double point given to it in two ways. Either we may visibly join together the oval and the sinuous part, making a loop (fig. 5); or we may let the oval shrink up into a conjugate point, at which two invisible

* Salmon's *Higher Plane Curves*, pp. 34, 35. I learn with great satisfaction that already the new edition of this is partly in print. [Published Jan., 1873. The work is now, August, 1879, in a third edition.]

branches of the curve cross each other (fig. 6). Precisely analogous distinctions hold between the two ways in which a curve of the third class (fig. 7) may acquire a double tangent. (Fig. 7 is the same curve as fig. 1, but it has been projected so that the line at infinity cuts the oval part, which consequently resembles a hyperbola instead of an ellipse.) We may either make this go through the process already described, by which it becomes fig. 8, acquiring a real double tangent with visible points of contact; or we may make the angle between the asymptotes larger and larger, till the two branches of the hyperbolic part coincide into a doubled straight line (fig. 9), which is then an ideal double tangent having invisible points of contact. And this kind of double tangent does what was wanted; viz. it removes six invisible cusps, leaving three visible ones.

We establish then two different kinds of curves of the third class having a double tangent. First, there is the cardioidal curve, having visible points of contact with its double tangent, one visible and two invisible cusps. Secondly, there is the simple tricusp, having invisible points of contact with a real double tangent, and three visible cusps. My main business is with the first of these kinds; but before coming to it, I shall make some remarks about two special forms, one of each kind, and mention also some apparently different general forms of curves of the third class not having a double tangent.

Of the cardioidal form the cardioid itself is a particular case; viz. it is a curve of the third class and fourth order with one visible and two invisible cusps. In general, a curve of the fourth order which has cusps at the two invisible points at infinity through which all circles pass is called a Cartesian oval. A Cartesian oval may also be defined as the locus of a point whose distances (ρ, ρ') from two fixed points satisfy an equation of the form $m\rho + n\rho' = c$. The curve has three foci, all in the same straight line, and any two of them may be taken for the fixed points. A Cartesian oval with an additional cusp (this is necessarily real) is a cardioid; the three foci coincide at the cusp. Any curve of our first kind

then may be regarded as the shadow of a cardioid; and in order to project such a curve into a cardioid, it is only necessary to project the two invisible cusps into the circular points at infinity*.

A particular case of the other form is the hypocycloid of three branches. This curve has for its double tangent the line at infinity, and touches it at the circular points. Every curve of the third class with an ideal double tangent may thus be regarded as the shadow of a hypocycloid; and in order to project it into a hypocycloïd, it is only necessary to project the invisible points of contact into the circular points at infinity.

This curve is the envelope of the asymptotes of all the rectangular hyperbolas that pass through three fixed points. (Steiner). To prove this, it is necessary to shew first that the envelope is of the third class, *i.e.* that three such asymptotes can be drawn through an arbitrary point; and secondly, that the line infinity counts twice as an asymptote, or is a double tangent to the envelope, its points of contact being the two circular points. Now a rectangular hyperbola is one which cuts the line infinity in two points which make with the circular points a harmonic range. If the hyperbola *touch* the line at infinity (become a parabola), its two points of intersection coincide; and they can only do this at one or other of the circular points. There are therefore these two cases in which the line infinity is itself an asymptote; and if we consider a hyperbola very near to one of these cases, we see that the point at which the line infinity is met by the consecutive asymptote is the circular point itself. We have therefore established the *second* of our two facts; that the line infinity is a double tangent to the envelope, [and that the] points of contact are the circular points. To find now the *class* of the envelope, let us enquire how many tangents can be drawn to it through an

* On Cartesian Ovals see Crofton, *Proceedings of the London Mathematical Society* [Vol. VI. pp. 5—18]. On the Cardioid, Purkiss, *Messenger of Mathematics*, [Vol. II. pp. 241—249], and in especial relation to the present theory, Siebeck, *Ueber die Erzeugung der Curven* 3ter *Klasse und* 4ter *Ordnung durch Bewegung eines Punktes*, Crelle, Vol. LXVI. p. 344 (1866).

arbitrary point at infinity. There is in the first place the line infinity itself, counting for two. Besides this, there is the asymptote of the *one* hyperbola of the series that can be drawn through the given point; in all *three* tangents, or the envelope is of the third class. But every curve of the third class touching the line infinity at the circular points is a three-cusped hypocycloid: therefore, &c.*

I said that *in general* a curve of the third class consists of a tricusp surrounded by an oval. But an oval is a thing of the nature of a conic section, and may at times be wholly invisible, like the conic section $x^2 + y^2 + a^2 = 0$. We have accordingly a variety in which the oval has disappeared, and the curve is represented by a tricusp only. The tricusp, however, need not be finite; I have represented in fig. 10 a curve met by every straight line in at least two real points, which cannot therefore be projected into any finite form. To satisfy yourself that it is really a tricusp very similar to fig. 9, draw on a sphere its curve of intersection with a cone, whose vertex is the centre, standing on this curve; it will consist of two equal and opposite tricusps, each with two branches longer than half a great circle.

II.

The Focus of a Double Parabola.

I shall take the liberty of using the name *Double Parabola* to denote a curve of the third class, having the line infinity for a double tangent; the points of contact being any two points whatever at infinity. The reason of the name is tolerably obvious; for the curve has two pair of parabolic branches, and may be derived from the *ensemble* of two parabolas by continuous modification. If the two points of contact are visible, the curve is a central projection of a car-

* On the three-cusped hypocycloid, see a series of papers in the *Educational Times*, Reprint, Vols. III., IV.; and a most elegant synthetic discussion by Cremona, *Sur l'hypocycloïde à trois rebroussements*, which appeared about the same time in Crelle, Vol. LXIV. p. 101 (1865).

dioid, obtained by projecting the double tangent to infinity; if invisible, the curve is an orthogonal projection of a hypocycloid. The two kinds may be distinguished as *hyperbolic* and *elliptic* respectively—names on which more light will be thrown in the sequel. The hypocycloid of three branches must then be regarded, in accordance with this nomenclature, as a *circular* double parabola.

Now a *focus* of a curve is a point such that the two lines joining it to the circular points at infinity both touch the curve. Accordingly, a double parabola has in general one focus and no more than one; for the curve is of the third class, and consequently only one tangent can be drawn from one of the circular points, besides the line infinity which counts for two. This single focus lies inside the hyperbolic curve, and outside the elliptic one, moving further away from the curve as it approaches the circular form, the focus of which is away at infinity in no particular direction.

I am going to seek for a geometrical property of this focus, which may serve as a rule for constructing the curve. To this end I form as follows the tangential equation of the curve. Let $f=0$ be the tangential equation of the focus, $i=0$, $j=0$, the equations of the circular points at infinity; $a=0$, $b=0$ the equations of the points of contact with the line at infinity, and $c=0$ the point at infinity on the remaining tangent from the focus to the curve. Then I say that the equation is

$$abf=\lambda\,.\,ijc \qquad\qquad (1),$$

where λ is a numerical constant. For the equation is of the third degree, representing therefore a curve of the third class; it is satisfied by the coordinates of the lines fi, fj, fc, shewing that fi, fj are tangents to the curve, and f consequently a focus, and that fc is the remaining focal tangent; and it shews that the three tangents from each of the points a, b (viz. ai aj, ac; bi, bj, bc) coincide with the line at infinity—the points a, b, c, i, j being all on that line—so that a and b must be points on the curve at which that line touches it, unless the line were a triple tangent, which is impossible for a curve of the third class.

To make this equation yield us a relation of lengths and angles serving for geometrical construction, let us take lines A, B, C at a finite distance passing through the points a, b, c respectively, and denote by the letter O the line at infinity. The points a, b, c may now be denoted by AO, BO, CO; viz. a is the intersection of the lines A, O and so on. Let X be a variable tangent to the curve; then substituting its coordinates in the equation (1) thus modified, we have

$$XAO \,.\, XBO \,.\, Xf = \lambda \,.\, Xi \,.\, Xj \,.\, XCO \,\ldots\ldots\ldots\, (2),$$

where XAO means the determinant formed with the coordinates of the lines X, A, O, which vanishes when they meet in a point; and Xf means the result of substituting the coordinates of the line X in the equation of the point f, or (which is the same thing) the result of substituting the coordinates of the point f in the equation of the line X. Now if we observe that*

$$\sin(X, A) = \frac{XAO}{\sqrt{(Xi \,.\, Xj \,.\, Ai \,.\, Aj)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{Xf}{\sqrt{(Xi \,.\, Xj)} \,.\, Of},$$

* These formulæ are justified as follows. The coordinates of the circular points are taken to be

$$\text{of the point } i,\ x : y : 1 = 1 : \quad \iota : 0,$$
$$\text{of the point } j,\ x : y : 1 = 1 : -\iota : 0,$$

and the equation of the line O at infinity is taken to be

$$0 \,.\, x + 0 \,.\, y + 1 = 0.$$

Then if the equation of X is $lx + my + n = 0$, we have

$$Xi \,.\, Xj = (l + m\,\iota)\,(l - m\,\iota) = l^2 + m^2,$$

and if the equation of A is $l'x + m'y + n' = 0$, then

$$XAO = \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 0, & 0, & 1 \end{vmatrix} = lm' - l'm,$$

while $Of = 1$, whatever are the coordinates of f. Making these substitutions, the formulæ become

$$\sin(X, A) = \frac{lm' - l'm}{\sqrt{(l^2 + m^2)}\,\sqrt{(l'^2 + m'^2)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{lx' + my' + n}{\sqrt{(l^2 + m^2)}}\ (x'y' \text{ coordinates of } f),$$

which are the ordinary ones.

we may transform equation (2) into the following

$$\frac{\text{perpendicular from } f \text{ on } X}{\sin(X, C)} = \frac{\mu}{\sin(X, A)\sin(X, B)} \text{......} (3),$$

where
$$\mu = \frac{\lambda \sqrt{(Ci \,.\, Cj)}}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)} \,.\, Of}.$$

Now C was any line through the point c, that is, any line parallel to the focal tangent; but I shall now regard it as the focal tangent itself. This being so, it is clear that $\dfrac{\text{perpendicular from } f \text{ on } X}{\sin(X, C)} =$ distance from f to X, measured along the focal tangent; or it is the segment on the focal tangent determined by the variable tangent X. We have arrived then at the theorem that *this distance is inversely proportional to the product of the sines of the angles which the tangent makes with the asymptotic directions* A *and* B. Let δ be the distance from the focus to the point of contact of the focal tangent; then taking X to be the focal tangent itself, we get

$$\delta = \frac{\mu}{\sin(A, C)\sin(B, C)},$$

and, eliminating μ, we may state our theorem in the form

$$\frac{\text{segment on focal tangent}}{\delta} = \frac{\sin(A, C)\sin(B, C)}{\sin(A, X)\sin(B, X)},$$

= anharmonic ratio of points a, b, c, XO.

We may simplify still further this enunciation by remembering that any squared diameter of a conic is inversely proportional to the product of the sines of the angles which it makes with the asymptotes, and at the same time directly proportional to the parallel focal chord. Finally, we arrive at the following

Construction. On any fixed line C through a focus f of a conic section let a distance fp be measured equal to a focal chord, and through p let a line X be drawn parallel to the focal chord; the envelope of the line X as the chord turns round will be a double parabola which has infinite branches

parallel to the asymptotes of the conic, the point f for focus, and the line C for focal tangent.

According as the conic is ellipse or hyperbola, the double parabola will be elliptic or hyperbolic. If the conic is a parabola, it assumes an intermediate form, the semi-cubical parabola $ay^2 = x^3$. If the conic is a circle, the envelope reduces to a point, as it ought to, being a hypocycloid whose focus is at an infinite distance compared with the dimensions of the figure.

III.

THEOREM. *The locus of the foci of all the double parabolas which touch five fixed lines is a circle.*

A double parabola is uniquely determined by six tangents. For a curve of the third class is determined in general by nine tangents; and to be given that a particular line is a double tangent is equivalent to three linear tangential conditions. The double parabolas touching five fixed lines are therefore a singly infinite series like the conics inscribed in a quadrilateral, and there is only one of them that touches another arbitrary line. Let an arbitrary line be drawn through the circular point i; there is then one double parabola of the series that touches this line. From the point j one tangent only can be drawn to the curve, for it is of the third class, and the line infinity already counts for two tangents. When therefore the tangent from i to the curve is given, the tangent from j is uniquely determined; and so likewise when the tangent from j is given, the tangent from i is uniquely determined. These two tangents are therefore corresponding rays of two homographic pencils, and the locus of their intersection (*i.e.* of the focus f) is consequently a conic section passing through the points i, j, that is to say, a circle.

COR. 1. The foci of the five parabolas which touch every four of the five lines are on this circle.

For a double parabola, being a curve of the third class, may break up into a conic and a point; namely, into an ordinary parabola and a point at infinity. Among the double parabolas which touch the five lines are to be reckoned five such degenerate cases, consisting of the point at infinity on one of the lines and the conic parabola which touches the other four. The focus of this degenerate form is (as is obvious from the definition of a focus) simply the focus of the conic parabola; whence the corollary follows, and Miquel's theorem is proved.

Cor. 2. If we take six lines to start with, we may in this way determine six circles, omitting the lines one by one. These six circles all meet in a point, the focus of the double parabola which touches the six lines.

The transformation (as to its tangents) of a cardioidal curve into a conic and a point is illustrated by fig. 11, which represents such a curve very nearly consisting (as to its visible points) of a conic and a doubled finite portion of one of its tangents.

IV.

Developments.

So far we have considered the following series of propositions:

(1). Given three lines, a circle may be drawn through their intersections. (Euc. IV. 5.)

(1′). Given four lines, the four circles so determined meet in a point. (Well known.)

(2). Given five lines, the five points so found lie on a circle. (Miquel.)

(2′). Given six lines, the six circles so determined meet in a point. (Sect. III., Cor. 2.)

I shall now shew that the series is interminable; that is, that $2n$ lines determine $2n$ circles all meeting in a point, and

that for $2n+1$ lines the $2n+1$ points so found lie on the same circle.

Connected with Prop 1, however, there is another theorem which is susceptible of generalization. If from any point in the circumscribing circle we draw perpendiculars to the sides of a triangle, the feet of these perpendiculars are in one straight line. I call this $(1p)$; the corresponding pendant to Miquel's theorem is,

$(2p)$. If from any point p on the circle in Prop. (2) we draw perpendiculars on the five lines, their feet lie on a conic passing through p.

So again we have

(3). Given seven lines, the seven points obtained as in $(2')$, are all in one circle.

$(3p)$. If from any point p on this circle we draw perpendiculars on the seven lines, their feet will lie on a cubic having a node at p.

And generally

(np). If from any point p on the circle determined by $2n+1$ lines we draw perpendiculars to them, the feet of the perpendiculars will lie on a curve of order n passing $n-1$ times through p.

To prove these results, it is necessary to consider a curve of class $n+1$, touching the line infinity n times. I shall call such a curve an n-fold parabola. It is in fact of order $2n$, and has n pairs of parabolic branches*. From any point at infinity

* The tangential equation to an n-fold parabola is always

$$fa_1a_2 \,.\, a_n = \lambda \,.\, ijc_1c_2 \,.\, c_{n-1},$$

where $a_1, a_2, \ldots$ are points of contact with the line infinity, and $c_1, c_2, \ldots$ points at infinity on tangents from the focus to the curve. From this we may deduce at once a construction analogous to (3), p. 49. In the case of the triple parabola this construction takes the simplified form

$$\text{intercept on } X \text{ made by } A_1, A_2 = \frac{\mu}{\sin(X, C_1)\sin(X, C_2)\sin(X, C_3)},$$

where A_1, A_2 are the focal tangents, and C_1, C_2, C_3 the asymptotic directions. μ may be determined by making X coincide with A_1, A_2 successively.

not a point of contact, there may be drawn one other tangent to the curve; the line infinity counting for n, and the class being $n+1$. Such a curve therefore has always one and only one focus. Now a curve of class $n+1$ is in general determined by $\frac{1}{2}(n+2)(n+3)-1$ or $\frac{1}{2}(n^2+5n+4)$ tangents; but an n-fold tangent of given position is equivalent to $\frac{1}{2}n(n+1)$ single tangents in its effect upon the determination of the curve. The number of tangents finite in position which determine an n-fold parabola is the difference of these numbers, $2(n+1)$. All the n-fold parabolas, therefore, which touch $2n+1$ fixed lines form a singly infinite series; and it is easy to see that the locus of their foci is a circle. For if we draw an arbitrary line through the circular point i, *one* curve of the series can be drawn to touch it, and this determines uniquely the tangent from the other point j. These two tangents then, as before, are corresponding rays of two homographic pencils, and their intersection must trace out a conic through the points i, j, that is to say, a circle.

Now among these n-fold parabolas are included $2n+1$ degenerate cases, each consisting of an $(n-1)$ fold parabola and a point, viz. the point at infinity on one of the lines, and the $(n-1)$ fold parabola determined by the other $2n$. The foci of these are therefore points on the circle in question, and we may enunciate the following propositions:

(n). Given $2n+1$ lines, the foci of the $2n+1$ $(n-1)$ fold parabolas each of which touches $2n$ of the lines are on the same circle.

This circle is the locus of the foci of n-fold parabolas touching the lines, and therefore

(n'). Given $2n+2$ lines, the $2n+2$ circles so determined meet in a point, the focus of the n-fold parabola touching the lines.

By successive applications of this theorem with suitable values of n, we are able to see that the series of statements at the beginning of this section may be continued indefinitely.

Let us now inquire what is the *pedal* of an n-fold parabola with regard to the focus, that is to say, what is the locus of the

foot of the perpendicular from the focus on the tangent. It is easy to prove—as was pointed out by Dr Hirst—that this pedal is the inverse of the polar reciprocal of the curve, both taken with regard to a circle whose centre is the focus. Now the reciprocal of an n-fold parabola in regard to the focus is a curve of the $(n+1)^{\text{th}}$ order passing n times through the focus (because the original curve touches n times the line infinity) and once through each of the circular points (because the original curve touches the lines fi, fj). Its inverse is then a curve of order $2(n+1)-n-2=n$, passing $n+1-2=n-1$ times through f, and not at all through the circular points*. Hence we have the proposition

(np). From any point which is the focus of an n-fold parabola touching $2n+1$ lines, if perpendiculars be drawn to the lines, their feet will lie on a curve of order n passing $n-1$ times through the point in question.

* Circular inversion (or transformation by reciprocal radii-vectores) is a particular case of triangular or quadric inversion; the pole and the two circular points forming the triangle employed. Now in general the triangular inverse of a curve of order n, passing α, β, γ times respectively through the vertices of the triangle, is a curve of order $2n-\alpha-\beta-\gamma$ passing $n-\beta-\gamma$, $n-\gamma-\alpha$, $n-\alpha-\beta$ times respectively through the vertices of the triangle. (Dr Hirst, *Proceedings of the Royal Society*, March, 1865; and *Educational Times*, Reprint, Vol. I. p. 41; Vol. III. p. 91.)

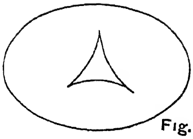


Fig. 1.

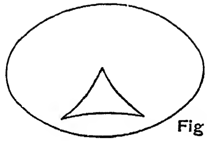


Fig 2.

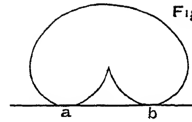


Fig. 3.

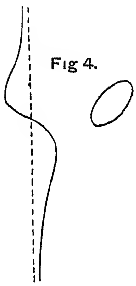


Fig 4.

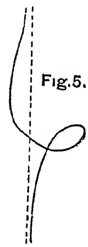


Fig. 5.

Fig 6

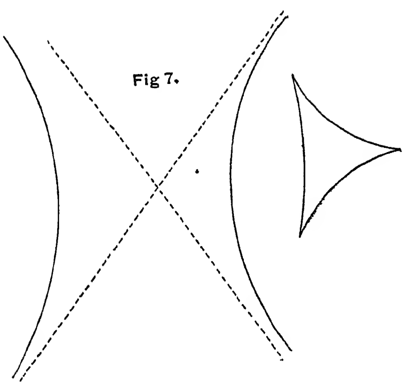


Fig 7.

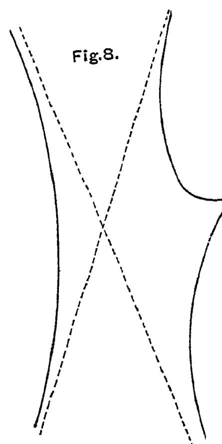


Fig. 8.

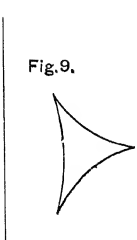


Fig. 9.

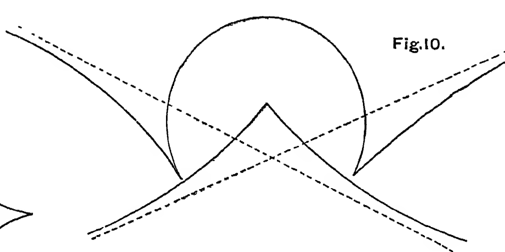


Fig. 10.

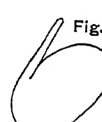


Fig. 11.

IX.

ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY*.

[Translation of a paper by Riemann, see V. supra.]

Plan of the Investigation.

IT is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it. The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magni-

* [From *Nature*, Vol. VIII. Nos. 183, 184, pp. 14—17, 36, 37. For a Bibliography of Hyper-space and Non-Euclidean Geometry, see Articles by George Bruce Halsted in the *American Journal of Mathematics, Pure and Applied*, Vol. I. pp. 261—276, 384, 385; Vol. II. pp. 65—70.]

tude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. But hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space—the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are—like all matters of fact—not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small.

I. *Notion of an* n-*ply extended magnitude.*

In proceeding to attempt the solution of the first of these problems, the development of the notion of a multiply extended magnitude, I think I may the more claim indulgent criticism in that I am not practised in such undertakings of a philosophical nature where the difficulty lies more in the notions themselves than in the construction; and that besides some very short hints on the matter given by Privy Councillor Gauss in his second memoir on Biquadratic Residues, in the *Göttingen Gelehrte Anzeige*, and in his Jubilee-book, and some philosophical researches of Herbart, I could make use of no previous labours.

§ 1. Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisa-

tions. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness: the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a *continuous* manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

Definite portions of a manifoldness, distinguished by a mark or by a boundary, are called Quanta. Their comparison with regard to quantity is accomplished in the case of discrete magnitudes by counting, in the case of continuous magnitudes by measuring. Measure consists in the superposition of the magnitudes to be compared; it therefore requires a means of using one magnitude as the standard for another. In the absence of this, two magnitudes can only be compared when one is a part of the other; in which case also we can only determine the more or less and not the how much. The researches which can in this case be instituted about them form a general division of the science of magnitude in which magnitudes are regarded not as existing independently of position and not as expressible in terms of a unit, but as regions in a manifoldness. Such researches have become a necessity for many parts of mathematics, e.g., for the treatment of many-valued analytical functions; and the want of them is no doubt a chief cause why the celebrated theorem of Abel and the achievements of Lagrange, Pfaff, Jacobi for the general theory of differential equations, have so long remained unfruitful. Out of this general part of the science of extended magnitude in which nothing is assumed

but what is contained in the notion of it, it will suffice for the present purpose to bring into prominence two points; the first of which relates to the construction of the notion of a multiply extended manifoldness, the second relates to the reduction of determinations of place in a given manifoldness to determinations of quantity, and will make clear the true character of an n-fold extent.

§ 2. If in the case of a notion whose specialisations form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forwards or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness. In a similar manner one obtains a triply extended manifoldness, if one imagines a doubly extended one passing over in a definite way to another entirely different; and it is easy to see how this construction may be continued. If one regards the variable object instead of the determinable notion of it, this construction may be described as a composition of a variability of $n+1$ dimensions out of a variability of n dimensions and a variability of one dimension.

§ 3. I shall now show how conversely one may resolve a variability whose region is given into a variability of one dimension and a variability of fewer dimensions. To this end let us suppose a variable piece of a manifoldness of one dimension—reckoned from a fixed origin, that the values of it may be comparable with one another—which has for every point of the given manifoldness a definite value, varying continuously with the point; or, in other words, let us take a continuous function of position within the given manifoldness, which, moreover, is not constant throughout any part of that manifoldness. Every system of points where the function has a constant value, forms

then a continuous manifoldness of fewer dimensions than the given one. These manifoldnesses pass over continuously into one another as the function changes; we may therefore assume that out of one of them the others proceed, and speaking generally this may occur in such a way that each point passes over into a definite point of the other; the cases of exception (the study of which is important) may here be left unconsidered. Hereby the determination of position in the given manifoldness is reduced to a determination of quantity and to a determination of position in a manifoldness of less dimensions. It is now easy to show that this manifoldness has $n-1$ dimensions when the given manifoldness is n-ply extended. By repeating then this operation n times, the determination of position in an n-ply extended manifoldness is reduced to n determinations of quantity, and therefore the determination of position in a given manifoldness is reduced to a finite number of determinations of quantity *when this is possible.* There are manifoldnesses in which the determination of position requires not a finite number, but either an endless series or a continuous manifoldness of determinations of quantity. Such manifoldnesses are, for example, the possible determinations of a function for a given region, the possible shapes of a solid figure, &c.

II. *Measure-relations of which a manifoldness of* n *dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other.*

Having constructed the notion of a manifoldness of n dimensions, and found that its true character consists in the property that the determination of position in it may be reduced to n determinations of magnitude, we come to the second of the problems proposed above, viz. the study of the measure-relations of which such a manifoldness is capable, and of the conditions which suffice to determine them. These measure-relations can only be studied in abstract notions of quantity, and their dependence on one another can only be represented by formulæ. On

certain assumptions, however, they are decomposable into relations which, taken separately, are capable of geometric representation; and thus it becomes possible to express geometrically the calculated results. In this way, to come to solid ground, we cannot, it is true, avoid abstract considerations in our formulæ, but at least the results of calculation may subsequently be presented in a geometric form. The foundations of these two parts of the question are established in the celebrated memoir of Gauss, *Disquisitiones generales circa superficies curvas.*

§ 1. Measure-determinations require that quantity should be independent of position, which may happen in various ways. The hypothesis which first presents itself, and which I shall here develop, is that according to which the length of lines is independent of their position, and consequently every line is measurable by means of every other. Position-fixing being reduced to quantity-fixings, and the position of a point in the n-dimensioned manifoldness being consequently expressed by means of n variables x_1, x_2, x_3, ... x_n, the determination of a line comes to the giving of these quantities as functions of one variable. The problem consists then in establishing a mathematical expression for the length of a line, and to this end we must consider the quantities x as expressible in terms of certain units. I shall treat this problem only under certain restrictions, and I shall confine myself in the first place to lines in which the ratios of the increments dx of the respective variables vary continuously. We may then conceive these lines broken up into elements, within which the ratios of the quantities dx may be regarded as constant; and the problem is then reduced to establishing for each point a general expression for the linear element ds starting from that point, an expression which will thus contain the quantities x and the quantities dx. I shall suppose, secondly, that the length of the linear element, to the first order, is unaltered when all the points of this element undergo the same infinitesimal displacement, which implies at the same time that if all the quantities dx are increased in the same ratio, the linear element will vary also in the same ratio.

On these suppositions, the linear element may be any homogeneous function of the first degree of the quantities dx, which is unchanged when we change the signs of all the dx, and in which the arbitrary constants are continuous functions of the quantities x. To find the simplest cases, I shall seek first an expression for manifoldnesses of $n-1$ dimensions which are everywhere equidistant from the origin of the linear element; that is, I shall seek a continuous function of position whose values distinguish them from one another. In going outwards from the origin, this must either increase in all directions or decrease in all directions; I assume that it increases in all directions, and therefore has a minimum at that point. If, then, the first and second differential coefficients of this function are finite, its first differential must vanish, and the second differential cannot become negative; I assume that it is always positive. This differential expression, then, of the second order remains constant when ds remains constant, and increases in the duplicate ratio when the dx, and therefore also ds, increase in the same ratio; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x. For Space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{\Sigma\,(dx)^2}$; Space is therefore included in this simplest case. The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically expressed; I restrict myself, therefore, to those manifoldnesses in which the line-element is expressed as the square root of a quadric differential expression. Such an expression we can transform into another similar one if we substitute for the n independent variables functions of n new independent variables. In this way, however, we cannot transform any expression into any other; since the expression contains $\frac{1}{2}n(n+1)$ coefficients which are

arbitrary functions of the independent variables; now by the introduction of new variables we can only satisfy n conditions, and therefore make no more than n of the coefficients equal to given quantities. The remaining $\frac{1}{2}n(n-1)$ are then entirely determined by the nature of the continuum to be represented, and consequently $\frac{1}{2}n(n-1)$ functions of positions are required for the determination of its measure-relations. Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form $\sqrt{\Sigma dx^2}$, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call *flat*. In order now to review the true varieties of all the continua which may be represented in the assumed form, it is necessary to get rid of difficulties arising from the mode of representation, which is accomplished by choosing the variables in accordance with a certain principle.

§ 2. For this purpose let us imagine that from any given point the system of shortest lines going out from it is constructed; the position of an arbitrary point may then be determined by the initial direction of the geodesic in which it lies, and by its distance measured along that line from the origin. It can therefore be expressed in terms of the ratios dx_0 of the quantities dx in this geodesic, and of the length s of this line. Let us introduce now instead of the dx_0 linear functions dx of them, such that the initial value of the square of the line-element shall equal the sum of the squares of these expressions, so that the independent variables are now the length s and the ratios of the quantities dx. Lastly, take instead of the dx quantities $x_1, x_2, x_3, \ldots x_n$ proportional to them, but such that the sum of their squares $= s^2$. When we introduce these quantities, the square of the line-element is Σdx^2 for infinitesimal values of the x, but the term of next order in it is equal to a homogeneous function of the second order of the $\frac{1}{2}n(n-1)$ quantities $(x_1dx_2 - x_2dx_1)$, $(x_1dx_3 - x_3dx_1) \ldots$ an infinitesimal, therefore, of the fourth order; so that we obtain a finite quantity on dividing

this by the square of the infinitesimal triangle, whose vertices are $(0, 0, 0, \ldots)$, $(x_1, x_2, x_3, \ldots)$, $(dx_1, dx_2, dx_3, \ldots)$. This quantity retains the same value so long as the x and the dx are included in the same binary linear form, or so long as the two geodesics from 0 to x and from 0 to dx remain in the same surface-element; it depends therefore only on place and direction. It is obviously zero when the manifold represented is flat, *i.e.*, when the squared line-element is reducible to Σdx^2, and may therefore be regarded as the measure of the deviation of the manifoldness from flatness at the given point in the given surface-direction. Multiplied by $-\frac{3}{4}$ it becomes equal to the quantity which Privy Councillor Gauss has called the total curvature of a surface. For the determination of the measure-relations of a manifoldness capable of representation in the assumed form we found that $\frac{1}{2}n(n-1)$ place-functions were necessary; if, therefore, the curvature at each point in $\frac{1}{2}n(n-1)$ surface-directions is given, the measure-relations of the continuum may be determined from them—provided there be no identical relations among these values, which in fact, to speak generally, is not the case. In this way the measure-relations of a manifoldness in which the line-element is the square root of a quadric differential may be expressed in a manner wholly independent of the choice of independent variables. A method entirely similar may for this purpose be applied also to the manifoldness in which the line-element has a less simple expression, *e.g.*, the fourth root of a quartic differential. In this case the line-element, generally speaking, is no longer reducible to the form of the square root of a sum of squares, and therefore the deviation from flatness in the squared line-element is an infinitesimal of the second order, while in those manifoldnesses it was of the fourth order. This property of the last-named continua may thus be called flatness of the smallest parts. The most important property of these continua for our present purpose, for whose sake alone they are here investigated, is that the relations of the twofold ones may be geometrically represented by surfaces, and of the morefold ones may be reduced to those of the surfaces included in them; which now requires a short further discussion.

§ 3. In the idea of surfaces, together with the intrinsic measure-relations in which only the length of lines on the surfaces is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines—*i.e.*, if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by mere bending, in which the intrinsic measure-relations remain, and all theorems about a plane—therefore the whole of planimetry—retain their validity. On the other hand they count as essentially different from the sphere, which cannot be changed into a plane without stretching. According to our previous investigation the intrinsic measure-relations of a twofold extent in which the line-element may be expressed as the square root of a quadric differential, which is the case with surfaces, are characterised by the total curvature. Now this quantity in the case of surfaces is capable of a visible interpretation, viz., it is the product of the two curvatures of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of the same. The first definition assumes the proposition that the product of the two radii of curvature is unaltered by mere bending; the second, that in the same place the area of a small triangle is proportional to its spherical excess. To give an intelligible meaning to the curvature of an n-fold extent at a given point and in a given surface-direction through it, we must start from the fact that a geodesic proceeding from a point is entirely determined when its initial direction is given. According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the n-fold continuum at the given point in the given surface-direction.

§ 4. Before we make the application to space, some considerations about flat manifoldnesses in general are necessary;

i.e., about those in which the square of the line-element is expressible as a sum of squares of complete differentials.

In a flat n-fold extent the total curvature is zero at all points in every direction; it is sufficient, however (according to the preceding investigation), for the determination of measure-relations, to know that at each point the curvature is zero in $\frac{1}{2}n(n-1)$ independent surface-directions. Manifoldnesses whose curvature is constantly zero may be treated as a special case of those whose curvature is constant. The common character of these continua whose curvature is constant may be also expressed thus, that figures may be moved in them without stretching. For clearly figures could not be arbitrarily shifted and turned round in them if the curvature at each point were not the same in all directions. On the other hand, however, the measure-relations of the manifoldness are entirely determined by the curvature; they are therefore exactly the same in all directions at one point as at another, and consequently the same constructions can be made from it: whence it follows that in aggregates with constant curvature figures may have any arbitrary position given them. The measure-relations of these manifoldnesses depend only on the value of the curvature, and in relation to the analytic expression it may be remarked that if this value is denoted by α, the expression for the line-element may be written

$$\frac{1}{1+\frac{1}{4}\alpha\Sigma x^2}\sqrt{\Sigma dx^2}.$$

§ 5. The theory of *surfaces* of constant curvature will serve for a geometric illustration. It is easy to see that surfaces whose curvature is positive may always be rolled on a sphere whose radius is unity divided by the square root of the curvature; but to review the entire manifoldness of these surfaces, let one of them have the form of a sphere and the rest the form of surfaces of revolution touching it at the equator. The surfaces with greater curvature than this sphere will then touch the sphere internally, and take a form like the outer portion (from the axis) of the surface of a ring; they may be rolled upon zones of spheres having less radii, but will go round

more than once. The surfaces with less positive curvature are obtained from spheres of larger radii, by cutting out the lune bounded by two great half-circles and bringing the section-lines together. The surface with curvature zero will be a cylinder standing on the equator; the surfaces with negative curvature will touch the cylinder externally and be formed like the inner portion (towards the axis) of the surface of a ring. If we regard these surfaces as *locus in quo* for surface-regions moving in them, as Space is *locus in quo* for bodies, the surface-regions can be moved in all these surfaces without stretching. The surfaces with positive curvatuıe can always be so formed that surface-regions may also be moved arbitrarily about upon them without *bending*, namely (they may be formed) into sphere-surfaces; but not those with negative curvature. Besides this independence of surface-regions from position there is in surfaces of zero curvature also an independence of *direction* from position, which in the former surfaces does not exist.

III. *Application to Space.*

§ 1. By means of these inquiries into the determination of the measure-relations of an n-fold extent the conditions may be declared which are necessary and sufficient to determine the metric properties of space, if we assume the independence of line-length from position and expressibility of the line-element as the square root of a quadric differential, that is to say, flatness in the smallest parts.

First, they may be expressed thus: that the curvature at each point is zero in three surface-directions; and thence the metric properties of space are determined if the sum of the angles of a triangle is always equal to two right angles.

Secondly, if we assume with Euclid not merely an existence of lines independent of position, but of bodies also, it follows that the curvature is everywhere constant; and then the sum of the angles is determined in all triangles when it is known in one.

Thirdly, one might, instead of taking the length of lines to be independent of position and direction, assume also an independence of their length and direction from position. According to this conception changes or differences of position are complex magnitudes expressible in three independent units.

§ 2. In the course of our previous inquiries, we first distinguished between the relations of extension or partition and the relations of measure, and found that with the same extensive properties, different measure-relations were conceivable; we then investigated the system of simple size-fixings by which the measure-relations of space are completely determined, and of which all propositions about them are a necessary consequence; it remains to discuss the question how, in what degree, and to what extent these assumptions are borne out by experience. In this respect there is a real distinction between mere extensive relations, and measure-relations; in so far as in the former, where the possible cases form a discrete manifoldness, the declarations of experience are indeed not quite certain, but still not inaccurate; while in the latter, where the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate: be the probability ever so great that it is nearly exact. This consideration becomes important in the extensions of these empirical determinations beyond the limits of observation to the infinitely great and infinitely small; since the latter may clearly become more inaccurate beyond the limits of observation, but not the former.

In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*, the former belongs to the extent relations, the latter to the measure-relations. That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is for ever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means

follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, *i.e.*, a surface which in a *flat* manifoldness of three dimensions would take the form of a sphere, and consequently be finite.

§ 3. The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends. The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo, and Newton, and used by modern physic. But in the natural sciences which are still in want of simple principles for such constructions, we seek to discover the causal relations by following the phenomena into great minuteness, so far as the microscope permits. Questions about the measure-relations of space in the infinitely small are not therefore superfluous questions.

If we suppose that bodies exist independently of position, the curvature is everywhere constant, and it then results from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes may be neglected. But if this independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great, to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three directions, provided that the total curvature of every measurable portion of space does not differ sensibly from zero. Still more complicated relations may exist if we no longer suppose the linear element expressible as the square root of a quadric differential.

Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.

This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day.

Synopsis.

PLAN of the Inquiry:

I. Notion of an n-ply extended magnitude.

§ 1. Continuous and discrete manifoldnesses. Defined parts of a manifoldness are called Quanta. Division of the theory of continuous magnitude into the theories,

(1) Of mere region-relations, in which an independence of magnitudes from position is not assumed;

(2) Of size-relations, in which such an independence must be assumed.

§ 2. Construction of the notion of a one-fold, two-fold, n-fold extended magnitude.

§ 3. Reduction of place-fixing in a given manifoldness to quantity-fixings. True character of an n-fold extended magnitude.

II. Measure-relations of which a manifoldness of n-dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other.

§ 1. Expression for the line-element. Manifoldnesses to be called Flat in which the line-element is expressible as the square root of a sum of squares of complete differentials.

§ 2. Investigation of the manifoldness of n-dimensions in which the line-element may be represented as the square root of a quadric differential. Measure of its deviation from flatness (curvature) at a given point in a given surface-direction. For the determination of its measure-relations it is allowable and sufficient that the curvature be arbitrarily given at every point in $\frac{1}{2}n(n-1)$ surface directions.

§ 3. Geometric illustration.

§ 4. Flat manifoldnesses (in which the curvature is everywhere = 0) may be treated as a special case of manifoldnesses with constant curvature. These can also be defined as admitting an independence of n-fold extents in them from position (possibility of motion without stretching).

§ 5. Surfaces with constant curvature.

III. Application to Space.

§ 1. System of facts which suffice to determine the measure-relations of space assumed in geometry.

§ 2. How far is the validity of these empirical determinations probable beyond the limits of observation towards the infinitely great?

§ 3. How far towards the infinitely small? Connection of this question with the interpretation of nature.

X.

ANALOGUES OF PASCAL'S THEOREM*.

1. A SYSTEM of $2n$ right lines will in general have $n(2n-1)$ intersections. If we divide them into two systems containing n lines each, and only consider the intersections of one system by the other, the number is reduced to n^2. Suppose further that $n(n-1)$ of these lie on a curve of the $(n-1)$th degree. Then we call the system a $2n$-lateral $n(n-1)$-gon totally inscribed in the $(n-1)^{\text{ic}}$. The $n(n-1)$ points which lie on the curve we call the *ineunts* of the $2n$-lateral; and the other n intersections we call the *exeunts*. The $2n$-lateral is said to be *totally* inscribed, because all the points in which any side is cut by the curve are ineunts of the figure. Such a figure we denote by the square symbol

$$\left\{\begin{matrix} 1, & A_1, & B_1, & C_1 \ldots\ldots N_1 \\ N_2, & 2, & A_2, & B_2 \ldots\ldots\ldots \\ M_3, & N_3, & 3, & A_3 \ldots\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots (1),$$

where the rows denote one system of n right lines, and the columns the other; and the n exeunts are ranged on the diagonal line from the left-hand upper corner to the right-hand lower corner. A_1, B_1, ... &c. are the ineunts.

We propose, first, to prove in various ways that the n exeunts always lie on a right line; and, secondly, to demon-

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strate one or two other properties of the figure. The present communication is confined for the most part to the case in which $n=4$; in a second we hope to notice some peculiarities of the higher plane cases, and to state the *true* analogues of Pascal's Theorem in Geometry of Higher Dimensions

2. *Of the mn-intersections of two curves of the* mth *and* nth *degrees, if pn lie on a curve of the* pth *degree, the remaining* $(m-p)\,n$ *will lie on a curve of the* $(m-p)$th *degree* (p being less than m, and n not greater than m).

Let U_m, U_n, U_p be three curves of degrees m, n, p, respectively. Since U_m passes through all the intersections of U_n and U_p, U_m must be of the form

$$\phi U_n + \psi U_p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

but U_m is of the mth degree and U_p of the pth, therefore ψ must be of the degree $m-p$; but $\psi=0$ passes through all the remaining intersections of U_m and U_n. Therefore, &c. Q.E.D.

This includes (when $m=n$) the well known theorem of Salmon's *Higher Plane Curves*, Chap. II. Sect. 1, Art. 24. And the latter at once gives the property that the n exeunts of (1) lie on a right line. For we have two systems of n right lines, and a curve of the $(n-1)$th degree passing through $n(n-1)$ of their intersections; therefore the other n intersections lie on a right line. In its general form the theorem shews that we may erase any of the rows of the symbol (1) (which we call the "*index*" of the figure) without destroying the property of the exeunts.

3. We shall have occasion to use Prof. Cayley's theorem, that "*every curve of the* mth *order,* (m not being less than n or p, nor greater than $n+p-3$), *which passes through all but* $\frac{1}{2}(n+p-m-1)(n+p-m-2)$ *of the intersections of two curves of the* nth *and* pth *orders, passes also through the remaining intersections.*" The form (2) contains $\frac{1}{2}(m-n)(m-n+3)$ constants in ϕ, $\frac{1}{2}(m-p)(m-p+3)$ in ψ, and one constant of mul-

tiplication. Therefore it is the general equation of a curve of the mth degree through

$$\Phi \equiv \tfrac{1}{2} m (m+3) - \tfrac{1}{2} (m-n)(m-n+3) - \tfrac{1}{2} (m-p)(m-p+3) - 1$$

points of intersection of two curves of the nth and pth degrees. It is easily shewn that

$$np - \Phi \equiv \tfrac{1}{2} (n+p-m-1)(n+p-m-2),$$

and the equation (2) evidently represents a curve passing through *all* the intersections of U_n and U_p. The limitations to the value of m are obvious.

4. We now consider the case of an octolateral dodecagon inscribed in a cubic. Given two sets of four right lines a, b, c, d, and $\alpha, \beta, \gamma, \delta$, it follows from the last proposition that a cubic through eleven of their intersections will pass through a twelfth ($m = n = 4$, $p = 3$). Take then any straight line a meeting the cubic in three points A, B, C; through these draw straight lines β, γ, δ, meeting the curve in A, K, M; B, D, N; C, E, G respectively; join DE, GK, MN by straight lines b, c, d, meeting the curve again in F, H, L respectively; then F, H, L lie on a right line α. For, in the first place, join F, H by a line α; then the quartic $(\alpha\beta\gamma\delta)$ passes through all but one of the intersections of the quartic $(abcd)$ with the given cubic, and so it must pass through the remaining one, or L lies on FH. We write down the "index" of this figure

$$\left\{\begin{matrix} 1, & A, & B, & C \\ F, & 2, & D, & E \\ H, & K, & 3, & G \\ L, & M, & N, & 4 \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots (3).$$

Here the rows are the straight lines a, b, c, d, and the columns are $\alpha, \beta, \gamma, \delta$. The points $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ are the exeunts 1 2 3 4, which lie on a right line.

5. We give another proof on account of its consequences. With the notation of *Higher Plane Curves*, Chap. III., Sec. 6,

we form the equation $A+B+C=0$ to express that the points A, B, C lie on a right line, and so for the rest. Now let us take a as an abbreviation for the *quantity* $A+B+C$, and so on. Thus $a+b+c+d$ includes all the twelve points, and so does $\alpha+\beta+\gamma+\delta$. Hence

$$a+b+c+d \equiv \alpha+\beta+\gamma+\delta,$$

and therefore if seven of these quantities vanish, the eighth must vanish also. But we know that a, b, c, d, β, γ, δ all $=0$, therefore $\alpha=0$ also, or FHL is a straight line.

6. *An octolateral has six diagonals which intersect by pairs in three points of the curve lying on a right line.*

With the notation of the last article, each of the quantities a, b, c, d; α, β, γ, δ is identically zero. So therefore is any quantity formed from them by addition or subtraction. Now consider the identities

$$\left.\begin{array}{l}(a+b+\alpha+\beta)-(c+d+\gamma+\delta)\equiv(A+F)-(N+G)\\(a+c+\alpha+\gamma)-(b+d+\beta+\delta)\equiv(B+H)-(E+M)\\(a+d+\alpha+\delta)-(b+c+\beta+\gamma)\equiv(C+L)-(D+K)\end{array}\right\}\ldots\ldots(4).$$

If AF meet the curve in P, so that $A+F+P=0$, we must have also $N+G+P=0$, since $A+F=N+G$ identically. That is, AF and NG meet the curve in the same point. Similarly for BH and EM, CL and DK. Call these points P, Q, R. Then, if we form the six equations like $A+F+P=0$, and add them all together, we have

$$2(P+Q+R)+A+B+C+D+\ldots=0;$$

therefore $P+Q+R=0$, or PQR is a straight line.

7. By aid of this property we may consider the octolateral from an entirely different point of view.

The cubic passes through all the intersections of AF, γ, δ with NG, a, b. Therefore its equation may be written

$$AF.\gamma\delta+NG.ab=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5),$$

where, for shortness, AF represents the perpendicular from a current point on AF, so that $AF=0$ is the equation to AF. A constant multiplier is of course supposed.

Now if we consider this equation generally, it contains *thirteen* independent constants, viz. two for each of the six lines, and one constant of multiplication. But the general equation of the third degree contains *nine* constants; we may therefore, as the cubic is given, assume *four* points in (5) and the rest will be determined. Assume then P, the intersection of AF and NG, and the points A, N. These determine the straight lines PAF, PNG, and we have still one point at our disposal. Choose any point B for the intersection of α and γ; join AB, which is α, and then C, the third point where it cuts the curve, will be a point on δ. To find γ, δ we have a choice of two constructions; we may join either BN, CG, or BG, CN; we adopt the former. If BN meet the curve in D, and CG in E, then the form of equation (5) shews that FDE is a straight line, namely, b. We have constructed then the following portion of the index (3),

$$\left\{\begin{matrix} *, & A, & B, & C \\ F, & *, & D, & E \\ & & *, & G \\ & & N, & * \end{matrix}\right\} \text{ .... } (6).$$

The point P is of course not represented.

Now choose another point M, and use it as B was used before to obtain three more points H, K, L corresponding to ECD. We have then a second equation to the cubic

$$AF \, . \, cd + NG \, . \, \alpha\beta = 0 \text{.................... } (7),$$

and we may complete the symbol (6) in the form

$$\left\{\begin{matrix} 1, & A, & B, & C \\ F, & 2, & D, & E \\ H, & K, & 3, & G \\ L, & M, & N, & 4 \end{matrix}\right\},$$

the exeunts 1 2 3 4 being now determined.

8. *The four exeunts lie on a right line.*

We write the equivalent equations (5) and (7) in the forms

$$-\frac{AF}{NG}=\frac{ab}{\gamma\delta}=\frac{\alpha\beta}{cd} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

and as these are satisfied for every point on the cubic, it follows that the equation

$$abcd-\alpha\beta\gamma\delta=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

must have the cubic as a factor. But (9) is of the fourth degree; therefore the other factor must be of the first degree, which represents a straight line.

Or we may arrange the proof as follows: we have

$$AF.\gamma\delta+NG.ab \equiv lU=0,$$

and $$AF.cd+NG.\alpha\beta \equiv mU=0;$$

therefore

$$AF(p.\gamma\delta+q.cd)+NG(p.ab+q.\alpha\beta) \equiv (pl+qm)U=0.$$

Now suppose $p:q$ so taken that $(p\ \gamma\delta+q.cd)$ may represent the pair of lines NG, 3 4. Then NG is a factor of the left-hand side of the identity, but not of the right; therefore both sides must vanish identically. Thus we have

$$AF.NG.34 \equiv -NG(p.ab+q.\alpha\beta),$$

but, the left-hand side of this identity denoting three right lines, the right must also. But these can only be NG, AF, 1 2; therefore 1 2 is identical with 3 4, or 1 2 3 4 is a straight line. This straight line we call the *axis* of the octolateral.

9. The method of (7) presents an octolateral as the aggregate of a pair of diagonals Δ, and two quadrangles, X, Y, formed from Δ. Suppose we form from Δ another quadrangle Z; then we have three octolaterals, ΔXY, ΔYZ, ΔZX, and *the*

axes of these three meet in a point. For we write down the equations

$$AF.\gamma\delta + NG.ab = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (X),$$
$$AF.cd + NG.\alpha\beta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (Y),$$
$$AF.xy + NG.zw = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (Z),$$
$$-\frac{NG}{AF} = \frac{xy}{zw} = \frac{cd}{\alpha\beta} = \frac{\gamma\delta}{ab}.$$

The last statement expressing the property in question.

10. The proposition proved in (6) may be extended. *Any six diagonals which include all the ineunts meet the cubic in six new points which lie on a conic.* This is proved precisely as in (6), the addition of six equations giving the result

$$P + Q + R + S + T + U = 0,$$

which is the condition. The number of these conics we have not had the courage to count. There are nine quadrangles of ineunts which yield three pairs of diagonals each, and by combinations of these we obtain 96 conics, including the straight line PQR, Art. 6. But there are a great many other arrangements which cannot be so divided into quadrangles. The diagonals may be classified; there are 6 of the class $\alpha\beta.b\alpha$, 12 of the class $\alpha\beta.c\delta$, and 24 of the class $\alpha\beta.b\gamma$. The property of the six points of intersection of a conic and a cubic is this: *if they be divided into three pairs, the lines joining these will meet the cubic in three points on a right line.* For since

$$P + Q + R + S + T + U = 0,$$

if

$$P + Q + X = 0,$$
$$R + S + Y = 0,$$
$$T + U + Z = 0,$$

we must have $X + Y + Z = 0$. *To every such six points pertain* 15 *of these lines;* for if we take any pair PQ, the remaining four may be divided into pairs in three ways, RS, TU; RT, SU; RU, ST; and so for each of the pairs PR, PS, PT, PU. Suppose now that P, Q are the points where the diagonals

BE, KL meet the curve again; and let BK, EL meet it in P_1, Q_1, and BL, EK in P_2, Q_2. Then $R+S+T+U$ is constant, while the sum of the six is always zero; thus

$$P+Q=P_1+Q_1=P_2+Q_2,$$

or *the lines* PQ, P_1Q_1, P_2Q_2, *meet on the curve.* If then we were counting all the lines XYZ, we must [not*] take the number of conics and multiply it by 15.

11. If at any point A we draw a tangent AB, then $2A+B=0$. Prof. Cayley calls the point B the *Satellite of* A. It is easily seen that the Satellites of an octolateral form another octolateral, and that any points connected with the second are Satellites of points similarly connected with the first. Many other properties may easily be proved.

Nov. 20, 1863.

* [*Not* is introduced on the authority of a pencilled correction in Prof. Clifford's own handwriting.]

XI.

ANALYTICAL METRICS*.

I. *Introduction.*

1. Any one must have observed that there are two kinds of theorems in Geometry; one kind having reference to *position* only, the other kind having reference to *magnitude*. Pascal's theorem is an example of the first, or *graphic* geometry; Euclid I. 47 is an example of the second, or *metric* geometry. It may be possible to state the same theorem in two ways, so as to make it either metric or graphic. In such a case the graphic statement may be distinguished by the fact that it is *unaltered by projection*. In fact, the word *graphic* is co-extensive with *projective*. And so, bearing in mind the properties of projection, we may define Metric Geometry as that science which has to do with the magnitudes of angles, distances, areas, and volumes. It seems, at first sight, as if the method of coordinates, especially in its more complete homogeneous form, were almost wholly applicable to Graphic Geometry, and altogether unfit for the study of Metrics. And this idea is strengthened by the fact that nearly the whole of Graphic Geometry is due to the method of coordinates, while the science of Metrics has hitherto benefitted by it very little indeed. But there are two considerations which go against this view. The first is derived from Poncelet's discovery of the true nature of metrics. Poncelet shewed that all circles pass through the same two points at infinity, and that

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all angles, lengths, &c. may be expressed as *graphic* functions of these two points. For example, the angle between two straight lines is a certain function of the anharmonic ratio in which they cut the line joining the circular points. The general principle may be thus stated:—Whenever we speak of the metric properties of loci, we consider the loci, not by themselves, but in connection with another *fixed* locus, called the Absolute. Thus, in the case of Plane Geometry, the Absolute is the two circular points at infinity.

The second consideration is derived from the Higher Algebra. M. Magnus has proved that linear transformation is virtually equivalent to projection, and that graphic properties are in fact those which are unaltered by linear transformation. Now it is one of the propositions of the Higher Algebra, that when we consider any set of loci, the invariants, or functions of the coefficients, and covariants, or connected loci, which are unaltered by linear transformation, are *limited in number.* All graphic properties, therefore, may be stated in terms of a *finite number* of expressions. Now combine these considerations; first, that Metric Geometry may be analytically reduced to Graphic; and, secondly, that Graphic Geometry is necessarily of finite extent and exhaustible; and it will, I think, be abundantly evident that Analytical Metrics ought to be studied.

As to the mode of proof, I have freely made use of known results both in Pure Geometry and in Cartesian Coordinates, though the formulæ are to be regarded as referring primarily to systems of coordinates in which the equations are homogeneous. It is, of course, possible to *start* Coordinate Geometry from definitions or axioms, without referring at all to any other science, such as Pure Geometry; and in this way one may arrive at the results even of Metrics. But the course here adopted seemed to be, in existing circumstances, shorter. Those who wish to see the subject handled scientifically, must refer to Prof. Cayley's Sixth Memoir upon Quantics.

2. I give here two sample modes of proof, which together will suffice to prove most of the propositions which follow. Proofs which are similar to these will hereafter be omitted.

(*A*) *To find the area of the triangle contained by three points whose equations are given in the form*

$$lx + my + nz = 0,$$

(the area of the fundamental triangle being 1).

The area vanishes only when the points are in a straight line, and becomes infinite only when one of them is at infinity. Now the condition that they shall be in a straight line is

$$J \equiv \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0,$$

and the product of the conditions that they shall be at infinity is

$$P \equiv (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3).$$

Now the expression for the area must be of no dimensions in the coefficients of either point; and when two points and the area are given, the locus of the other is a straight line. The required expression is therefore $m \cdot \frac{J}{P}$ where m is a constant. By reference to the fundamental triangle, we get $m = 1$; therefore the area is $\frac{J}{P}$.

(*B*) *To find the cosine of the angle between two straight lines* $A = 0$, $B = 0$.

Let $\phi(A) = 0$ be the condition that A pass through one of the circular points; we know from Cartesians that this is of the second order in the coefficients of A. And because it is of the second order $\phi(\lambda A + \mu B)$ must be of the form

$$\lambda^2 \phi A + 2\lambda\mu\psi AB + \mu^2 \phi B \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now when we put for A and B their Cartesian equivalents, the coefficients of powers of λ, μ in (1) cannot alter in ratio, because the evanescence of (1) gives the two values of $\lambda : \mu$, for which $\lambda A + \mu B = 0$ represents a line through one of the circular points, and these lines are the same in both cases. But in Cartesian coordinates, the expression $\frac{\psi AB}{\sqrt{(\phi A \cdot \phi B)}}$ is equal to the cosine of

the angle between A and B; this is therefore its value in all other systems of coordinates.

The function ϕ is called the Absolute, and the determination of its form is an important part of our subject.

NOTE. (α) The function used in proof (A), and called J, will occur very often in the sequel. Suppose we have a number of linear equations equal to the number of variables in each, and that we form the determinant which is the result of eliminating the variables from these equations; then this determinant is called their Jacobian, and will be denoted by $J(LMN\ldots)$, where $L=0$, $M=0$, &c., ... are the linear equations:

(β) The abbreviation $\infty=0$ will be used to represent the equation to the line, &c. at infinity; so that $J(LM\infty)$ means the result of eliminating the variables between $L=0$, $M=0$, and the equation to the line at infinity.

(γ) In the demonstration (B) it is important to remember, that when we put for A and B their Cartesian equivalents, these expressions are transformed by the *same* substitution; so that if A is changed to A', and B to B', $lA+mB$ will be changed to $lA'+mB'$. The proof depends on this. It would not be sufficient to know that $A=0$, in a homogeneous system, and $A'=0$, in Cartesians, represent the same straight line, because then a constant factor might be introduced, of which we should know nothing.

(δ) The method of comparison of dimensions, made use of almost exclusively in the following sections, requires a little explanation. In the first place, all the functions employed are integral functions; so that it is quite legitimate to say "Because A always vanishes when B does, therefore A contains B as a factor." Secondly, every expression must contain some factor depending on the coefficients of the "Absolute;" but these factors will be systematically disregarded, as they are only numerical, and can easily be determined by reference to any simple particular case. These things being so, the method of comparison of dimensions amounts simply to this; when we have proved the existence of factors enough to account for all the dimensions

of an expression, we may be sure that it has no other factor. And, an analytical equivalence being thus proved, it only remains to divide both sides by such quantities as will make it interpretable geometrically.

II. *Geometry of One Dimension. Graphometrics.*

3. Geometry of one dimension treats of a series of points on a straight line, which are determined by the ratios of their distances from two fixed points $xy = 0$, given by equations of the form $lx + my = 0$. The equation to the point at infinity on the line is clearly $x - y = 0$; and so the condition that the point $lx + my$ may be at infinity is $l + m = 0$. The condition that two points $A \equiv l_1x + m_1y$ and $B \equiv l_2x + m_2y$ may be coincident is

$$J(AB) \equiv \begin{vmatrix} l_1, & l_2 \\ m_1, & m_2 \end{vmatrix} = 0.$$

So the condition that A may be at infinity may be written $J(A\infty) = 0$.

The distance between A and B is $\dfrac{J(AB)}{J(A\infty).J(B\infty)}$. (Proof A.)

Graphometrics.

4. Let A, B, C, D be four points in a straight line; then it appears, by Art. 3, that

$$\frac{AB.CD}{AC.DB} = \frac{J(AB).J(CD)}{J(AC).J(DB)} \quad\text{.................}\quad (2).$$

Now it is known that in linear transformation the function $J(AB)$ is only affected with a constant numerical factor. This factor disappears in the expression (2); the expression, therefore, is unaltered by linear transformation. It is, in fact, an anharmonic ratio of the points $ABCD$.

Now this function belongs to Metric Geometry, because it is expressed by means of *lengths*, viz. $\dfrac{AB.CD}{AC.DB}$. But it also belongs to Graphic Geometry, because it is unaltered by projection

or linear transformation. On these accounts I propose to call it a *Graphometric* function. The reason why a new name is needed, is that there are other Graphometric functions, which are naturally presented by analytical metrics. For instance, it will be proved that if $ABCDEF$ are six points in a plane, the ratio $\frac{(ABC)(DEF)}{(ABD)(CEF)}$, between the products of the triangles they contain, is graphometric, or independent of projection though expressed in terms of areas. And it is natural to suppose that these and similar functions will be important in Solid Geometry, because the analogous function, namely, anharmonic ratio, is so important in Plane Geometry. For this reason I shall, in what follows, pay particular attention to Graphometrics.

III. *Points, Lines, and Circles.*

5. In respect of one line $L=0$, the expression considered is $\phi(L)$, whose evanescence is the condition that the line shall pass through one of the circular points. If the line be, in Cartesians, $lx+my+n=0$, then

$$\phi L \equiv l^2+m^2,$$

that is, it is of the second order in the coefficients of L, and its discriminant vanishes. We have, also,

$$\phi(\lambda L+\mu M) \equiv \lambda^2(l_1^2+m_1^2)+2\lambda\mu(l_1l_2+m_1m_2)+\mu^2(l_2^2+m_2^2);$$

or $$\lambda^2\phi L+2\lambda\mu\psi(L, M)+\mu^2\phi M \text{..............} (3),$$

so that in any system of coordinates the function $\psi(LM)$, thus formed, is the condition that L and M may be at right angles. In the same way, or from the nature of the case, it is seen that the condition that (3) may be a perfect square is the square of the condition that L and M may be parallel, or may meet on the line at infinity; that is

$$\phi L.\phi M-(\psi LM)^2 \equiv J(LM\infty)^2 \text{..........} (4)^*.$$

In respect of three lines the function $J(LMN)$ will be considered.

* The equation to the line at infinity will be found from ϕ.

The principal function of a point is $(A\infty)$, the condition that it may be at infinity. Given two points $(l_1m_1n_1)$, $(l_2m_2n_2)$, the line joining them is

$$\begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} x + \begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix} y + \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} z = 0;$$

from this it will be easy to understand ϕAB, the condition that AB may pass through a circular point, and $\psi(AB, CD)$, the condition that AB and CD may be at right angles.

Angles.

6. The cosine of the angle between L and M has been proved to be $\dfrac{\psi LM}{\sqrt{(\phi L \,.\, \phi M)}}$; by (4) the sine is $\dfrac{J(LM\infty)}{\sqrt{(\phi L \,.\, \phi M)}}$.

If then we consider four lines, $LMNR$, we have

$$\frac{\sin(LM) \,.\, \sin(NR)}{\sin(LN) \,.\, \sin(RM)} \equiv \frac{J(LM\infty) \,.\, J(NR\infty)}{J(LN\infty) \,.\, J(RM\infty)}.$$

From the identity*

$$J(LM\infty) \,.\, J(NR\infty) + J(LN\infty) \,.\, J(RM\infty) + J(LR\infty) \,.\, J(MN\infty) = 0,$$

we may therefore deduce

$$\sin(LM) \,.\, \sin(NR) + \sin(LN) \,.\, \sin(RM) + \sin(LR) \,.\, \sin(MN) = 0 \quad\text{.........(5)}.$$

If we make N perpendicular to R, this becomes the formula

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

Next, consider the determinant $\begin{vmatrix} \psi LN, & \psi LR \\ \psi MN, & \psi MR \end{vmatrix}$.

This vanishes when L is parallel to M, for then the two rows become identical, and also when N is parallel to R, for then the

* A method of proving these identities will be given afterwards. This case may also be treated in the same way as the next.

columns become identical. Hence, by a comparison of dimensions, we can see that

$$\begin{vmatrix} \psi LN, & \psi LR \\ \psi MN, & \psi MR \end{vmatrix} \equiv J(LM\infty) \,.\, J(NR\infty),$$

or, which is the same thing,

$$\cos(LN) \,.\, \cos(MR) - \cos(LR) \,.\, \cos(MN)$$
$$= \sin(LM) \,.\, \sin(NR) \;\ldots\ldots\ldots(6).$$

If we make M parallel to R, this becomes the formula

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Since $\psi(LM)$ is of the *first order* in the coefficients of each of the lines, we must have

$$\psi(D,\, lA + mB + nC) \equiv l\psi AD + m\psi BD + n\psi CD,$$

so that, if l, m, n are regarded as variables, the tangential equation to the point at infinity in a direction perpendicular to D is

$$l \,.\, \psi AD + m \,.\, \psi BD + n \,.\, \psi CD = 0.$$

We have similar equations for points so connected with E and F. But these three points, being all at infinity, are in the same straight line; we must therefore have identically

$$\begin{vmatrix} \psi AD, & \psi BD, & \psi CD \\ \psi AE, & \psi BE, & \psi CE \\ \psi AF, & \psi BF, & \psi CF \end{vmatrix} \equiv 0;$$

that is to say

$$\begin{vmatrix} \cos(AD), & \cos(BD), & \cos(CD) \\ \cos(AE), & \cos(BE), & \cos(CE) \\ \cos(AF), & \cos(BF), & \cos(CF) \end{vmatrix} \equiv 0 \;\ldots\ldots\ldots\ldots(7),$$

where A, B, C, D, E, F are any six lines. By making A, B, C coincide with D, E, F respectively, we obtain the relation which exists among the cosines of the angles of a triangle; but the signs are undetermined.

The theorem (7) may also be proved indirectly as follows. Form the reciprocal determinant, expressing the minors by the help of (6); it will be found to vanish identically.

Distances.

7. The distance between two points A and B is $\dfrac{\sqrt{\phi AB}}{A\infty \,.\, B\infty}$.

The length of the perpendicular from a point A on a line L is $\dfrac{AL}{A\infty \,.\, \sqrt{\phi L}}$; where AL denotes the result of substituting the coefficients of A for the variables in L, or *vice versâ.*

So the length of a line N, cut off by two lines L and M, is $\dfrac{\sqrt{\phi N} \,.\, J(LMN)}{J(LN\infty) \,.\, J(MN\infty)}$.

And the length of the perpendicular from C on AB is

$$\frac{J(ABC)}{C\infty \,.\, \sqrt{\phi AB}}.$$

The point where the line AB cuts any line Π is of course represented by $\dfrac{A}{\Pi A} = \dfrac{B}{\Pi B}$. So that if L and M are two lines, the evanescence of the determinant

$$\begin{vmatrix} LA, & LB \\ MA, & MB \end{vmatrix}$$

is the condition that L, M, AB may meet in a point; in fact, by comparing dimensions, we may see that it is the same as $J(L, M, AB)$ or $J(LM, A, B)$. The interpretation of this will be given afterwards. Now let Π, Π' be two lines joining the point C to the circular points at infinity. Then it is clear that the product

$$\begin{vmatrix} \Pi A, & \Pi B \\ A\infty, & L\infty \end{vmatrix} \times \begin{vmatrix} \Pi' A, & \Pi' B \\ A\infty, & B\infty \end{vmatrix}$$

must vanish when $\phi AB = 0$, and also when $C\infty = 0$; in fact, by comparison of dimensions, it is seen to be $\phi AB \,.\, C\infty^2$. Now $\Pi A \,.\, \Pi' A = \phi AC$, and

$$\Pi A \,.\, \Pi' B + \Pi' A \,.\, \Pi B = 2\psi\,(AC, BC);*$$

* Suppose A and B to be $(l_1 m_1 n_1)$ and $(l_2 m_2 n_2)$, and write Bd_A for the symbol $l_2 d_{l_1} + ..$, then $2\psi(AC, BC) = Bd_A \,.\, \phi AC = \Pi A \,.\, \Pi' B + \Pi' A \,.\, \Pi B$.

thus we have

$$C\infty^2 . \phi AB \equiv A\infty^2 . \phi BC + 2A\infty . B\infty . \psi(AC, BC) \\ + B\infty^2 . \phi AC \ldots\ldots(8),$$

or $$AB^2 = BC^2 + CA^2 + 2BC . CA \cos(BC, CA).$$

By means of the identity (8) I propose to find the value of the determinant

$$\begin{vmatrix} \phi AE, & \phi BE, & \phi CE, & \phi DE, & E\infty^2 \\ \phi AF, & \phi BF, & \phi CF, & \phi DF, & F\infty^2 \\ \phi AG, & \phi BG, & \phi CG, & \phi DG, & G\infty^2 \\ \phi AH, & \phi BH, & \phi CH, & \phi DH, & H\infty^2 \\ A\infty^2, & B\infty^2, & C\infty^2, & D\infty^2, & 0 \end{vmatrix}.$$

Take any point O, and substitute for the several constituents by the formula (8). Then by subtracting the last row and column, with proper multipliers, from each of the others, we may reduce the determinant to the form

$$\begin{vmatrix} \psi(OA, OE), & \psi(OB, OE), & \psi(OC, OE), & \psi(OD, OE), & E\infty \\ \psi(OA, OF), & \psi(OB, OF), & \psi(OC, OF), & \psi(OD, OF), & F\infty \\ \psi(OA, OG), & \psi(OB, OG), & \psi(OC, OG), & \psi(OD, OG), & G\infty \\ \psi(OA, OH), & \psi(OB, OH), & \psi(OC, OH), & \psi(OD, OH), & H\infty \\ A\infty, & B\infty, & C\infty, & D\infty, & 0 \end{vmatrix}$$

multiplied by $\dfrac{A\infty . B\infty . C\infty . D\infty . E\infty . G\infty . H\infty}{O\infty^6}$. But by (7) every term of this vanishes identically. Hence we obtain the relation which connects the distances of four points, 1234, from four other points, 5678 :

$$\begin{vmatrix} 15^2, & 25^2, & 35^2, & 45^2, & 1 \\ 16^2, & 26^2, & 36^2, & 46^2, & 1 \\ 17^2, & 27^2, & 37^2, & 47^2, & 1 \\ 18^2, & 28^2, & 38^2, & 48^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} \equiv 0.$$

When 1234 are identical with 5678, this gives the relation between the distances of any four points in a plane.

Triangles.

8. The area of the triangle formed by three straight lines L, M, N is the same as that formed by the points MN, NL, LM; that is

$$\frac{J(MN,\, NL,\, LM)}{MN\infty\,.\,NL\infty\,.\,LM\infty} \equiv \frac{J(LMN)^2}{J(MN\infty)\,.\,J(NL\infty)\,.\,J(LM\infty)}.$$

This I call $\dfrac{J^2}{P}$. I write also Π for $\phi L\,.\,\phi M\,.\,\phi N$. In this notation it is easy to see that

$$a = \frac{J}{P}\sqrt{\phi L\,.\,J(MN\infty)}, \text{ \&c.}; \quad abc = \frac{J^3}{P^2}\sqrt{\Pi};$$

$$\sin A \sin B \sin C = \frac{P}{\Pi}; \quad \frac{a}{\sin A} = \frac{J}{P}\sqrt{\Pi};$$

and so on.

In the case of six *points*, 123456, it is clear that, if Δ (123) denote the area of the triangle 123,

$$\frac{\Delta(123)\,\Delta(456)}{\Delta(124)\,\Delta(356)} = \frac{J(123)\,.\,J(456)}{J(124)\,.\,J(356)},$$

and therefore that this function is graphometric. But the same proposition is not true in respect of lines. To find the graphometric function of three lines, we observe that it must be J multiplied by some power of Π, since P is obviously inadmissible. Now Π is of two dimensions in each of the lines, and J of one; the function must therefore be $\dfrac{J}{\sqrt{\Pi}}$. Expressing this in terms of the sides and angles, we find for it the values

$$\frac{(\text{area})^2}{abc}, \quad \tfrac{1}{4} a \sin B \sin C, \quad \frac{\text{area}}{2\dfrac{a}{\sin A}},$$

$$\frac{a}{2\sin A}\sqrt{\{\Sigma\,(\Sigma - \sin A)\,(\Sigma - \sin B)\,(\Sigma - \sin C)\}},$$

if $$2\Sigma = \sin A + \sin B + \sin C.$$

This function I call for convenience the *Projector* of the triangle, and denote by $P(ABC)$. We have, of course,

$$\frac{P(ABC)\,.\,P(DEF)}{P(ABD)\,.\,P(CEF)}=\frac{J(ABC)\,.\,J(DEF)}{J(ABD)\,.\,J(CEF)},$$

and this ratio is therefore unaltered by projection.

By means of the transformation (8), and its reciprocal

$$\phi O\,.\,J(AB\infty)^2=J(BO\infty)^2\,.\,\phi A$$
$$+2J(BO\infty)\,.\,J(AO\infty)\,.\,\psi AB+J(AO\infty)^2\,.\,\phi B,$$

the following theorems may be easily proved:

$$(\alpha)\quad \begin{vmatrix} \phi AD, & \phi AE, & \phi AF, & A\infty^2 \\ \phi BD, & \phi BE, & \phi BF, & B\infty^2 \\ \phi CD, & \phi CE, & \phi CF, & C\infty^2 \\ D\infty^2, & E\infty^2, & F\infty^2, & 0 \end{vmatrix}$$

$$\equiv J(ABC)\,.\,J(DEF)\,.\,A\infty\,.\,B\infty\,.\,C\infty\,.\,D\infty\,.\,E\infty\,.\,F\infty$$

(in points).

$$(\beta)\quad \begin{vmatrix} J(AD\infty)^2, & J(AE\infty)^2, & J(AF\infty)^2, & \phi A \\ J(BD\infty)^2, & J(BE\infty)^2, & J(BF\infty)^2, & \phi B \\ J(CD\infty)^2, & J(CE\infty)^2, & J(CF\infty)^2, & \phi C \\ \phi D, & \phi E, & \phi F, & 0 \end{vmatrix}$$

$$\equiv J(BC\infty)\,.\,J(CA\infty)\,.\,J(AB\infty)\,.\,J(EF\infty)\,.\,J(FD\infty)\,.\,J(DE\infty)$$

(in lines).

And, in the process of proving them, we obtain

$$(\gamma)\quad \begin{vmatrix} \psi(OA,\ OD), & \psi(OA,\ OE), & \psi(OA,\ OF), & A\infty \\ \psi(OB,\ OD), & \psi(OB,\ OE), & \psi(OB,\ OF), & B\infty \\ \psi(OC,\ OD), & \psi(OC,\ OE), & \psi(OC,\ OF), & C\infty \\ D\infty, & E\infty, & F\infty, & 0 \end{vmatrix}$$

$$\equiv J(ABC)\,.\,J(DEF)\,.\,O\infty^4 \text{ (in points).}$$

$$(\delta)\quad \begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & \dfrac{\phi A}{J(OA\infty)} \\ \psi BD, & \psi BE, & \psi BF, & \dfrac{\phi B}{J(OB\infty)} \\ \psi CD, & \psi CE, & \psi CF, & \dfrac{\phi C}{J(OC\infty)} \\ \dfrac{\phi D}{J(OD\infty)}, & \dfrac{\phi E}{J(OE\infty)}, & \dfrac{\phi F}{J(OF\infty)}, & 0 \end{vmatrix}$$

$$\equiv \phi O^2 \frac{J(BC\infty).J(CA\infty).J(AB\infty).J(EF\infty).J(FD\infty).J(DE\infty)}{J(OA\infty).J(OB\infty).J(OC\infty).J(OD\infty).J(OE\infty).J(OF\infty)}$$

(in lines).

To these may be added

$$(\epsilon)\quad \begin{vmatrix} AL, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv J(ABC).J(LMN)$$

(in points and lines).

The interpretations of these are :

$$(\alpha)\quad \begin{vmatrix} 14^2, & 15^2, & 16^2, & 1 \\ 24^2, & 25^2, & 26^2, & 1 \\ 34^2, & 35^2, & 36^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \equiv \text{area } (123) \times \text{area } (456)$$

(where 14 denotes the distance between the points 1 and 4).

$$(\beta)\quad \begin{vmatrix} \sin^2 14, & \sin^2 15, & \sin^2 16, & 1 \\ \sin^2 24, & \sin^2 25, & \sin^2 26, & 1 \\ \sin^2 34, & \sin^2 35, & \sin^2 36, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

$$\equiv \sin 23 . \sin 31 . \sin 12 . \sin 56 . \sin 64 . \sin 45$$

(where 14 denotes the angle between the lines 1 and 4).

$$(\gamma)\quad \begin{vmatrix} \cos AOD, & \cos AOE, & \cos AOF, & \dfrac{1}{OA} \\ \cos BOD, & \cos BOE, & \cos BOF, & \dfrac{1}{OB} \\ \cos COD, & \cos COE, & \cos COF, & \dfrac{1}{OC} \\ \dfrac{1}{OD}, & \dfrac{1}{OE}, & \dfrac{1}{OF}, & 0 \end{vmatrix}$$

$$\equiv \frac{\text{area } ABC}{OA . OB . OC} \cdot \frac{\text{area } DEF}{OD . OE . OF} \text{ (in points).}$$

$$(\delta)\quad \begin{vmatrix} \cos AD, & \cos AE, & \cos AF, & \dfrac{1}{\sin OA} \\ \cos BD, & \cos BE, & \cos BF, & \dfrac{1}{\sin OB} \\ \cos CD, & \cos CE, & \cos CF, & \dfrac{1}{\sin OC} \\ \dfrac{1}{\sin OD}, & \dfrac{1}{\sin OE}, & \dfrac{1}{\sin OF}, & 0 \end{vmatrix}$$

$$\equiv \frac{\sin BC \,.\, \sin CA \,.\, \sin AB}{\sin OA \,.\, \sin OB \,.\, \sin OC} \cdot \frac{\sin EF \,.\, \sin FD \,.\, \sin DE}{\sin OD \,.\, \sin OE \,.\, \sin OF}.$$

$$(\epsilon)\quad \begin{vmatrix} AL, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv \text{area } ABC \times \text{projector of } LMN,$$

where ABC are points, and LMN lines, and AL is the perpendicular from the point A on the line L.

The interpretation of the identity

$$\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \equiv J(L, M, AB) \equiv J(LM, A, B)$$

is $$\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \begin{aligned} &\equiv \text{projector of } L, M, AB \times \text{distance } AB, \\ &\equiv \text{area of } LM, A, B \times \sin LM, \end{aligned}$$

where AL is the perpendicular as before.

Circles.

9. *To find the equation to the circle whose diameter is the line joining the intersection of A and B to the intersection of C and D.*

The circle is the locus of the intersection of two lines at right angles to one another drawn through the two points. Now if $A = kB$ and $C = k'D$ are at right angles to one another, we must have

$$\psi (A - kB,\ C - k'D),$$

or $$\psi AC - k \,.\, \psi BC - k'\psi AD + kk'\psi BD = 0,$$

and then, eliminating k, k' by the equations $A=kB$, $C=k'D$, we have for the equation to the circle

$$BD.\psi AC - AD.\psi BC - BC.\psi AD + AC.\psi BD = 0,$$

that is to say

$$\begin{vmatrix} \psi AC, & \psi AD, & A \\ \psi BC, & \psi BD, & B \\ C, & D, & 0 \end{vmatrix} = 0.$$

10. *The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.*

For, denote the equation just found by $(AB,\ CD)=0$; then it is easy to verify the identity

$$(AB,\ CD)+(AC,\ DB)+(AD,\ BC)\equiv 0,$$

which expresses the property in question.

11. If α, β, γ, δ are the *perpendiculars* from a current point on the lines A, B, C, D, the equation to the circle may evidently be written

$$\begin{vmatrix} \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha \\ \cos(\beta\gamma), & \cos(\beta\delta), & \beta \\ \gamma, & \delta, & 0 \end{vmatrix} = 0.$$

The translation of these results into the ordinary systems of coordinates requires the determination of the Absolute, to which I now proceed.

IV. *Determination of the absolute.*

12. As the title of this section might be productive of mixed feelings, I will say at once that "to determine the Absolute" means "to find the form of ϕ." It is not pretended that the method here used is simpler than that generally given; but only that it is more suggestive and of wider application.

Trilinears.

The trilinear coordinates of a point are three quantities proportional to the perpendicular distances of the point from three given lines, $X=0$, $Y=0$, $Z=0$. We may therefore write, if x, y, z are these coordinates,

$$x,\ y,\ z = \frac{X}{\sqrt{\phi X}},\ \frac{Y}{\sqrt{\phi Y}},\ \frac{Z}{\sqrt{\phi Z}},$$

and so

$$\begin{aligned}\phi\,(lx+my+nz)&\\ &= l^2\phi x + m^2\phi y + n^2\phi z + 2mn\psi yz + 2nl\psi zx + 2lm\psi xy\\ &= l^2+m^2+n^2+2mn\cos(YZ)+2nl\cos(ZX)\\ &\qquad\qquad +2lm\cos(XY).\end{aligned}$$

It only remains to determine *which* cosine is meant in each case. To do this we make the convention that x, y, z shall be all positive for a point inside the triangle. We can then see geometrically, that a line perpendicular to BC (or X) through B (or ZX), is represented by

$$z + x\cos B = 0,$$

therefore

$$\psi\,(x+z+x\cos B) \equiv \psi\,(x, z) + \cos B = 0,$$
$$\text{or}\ \cos(XZ) = -\cos B.$$

Hence, by symmetry, we have,

$$\begin{aligned}\phi\,(lx+my+nz) &= l^2+m^2+n^2\\ &\qquad -2mn\cos A - 2nl\cos B - 2lm\cos C,\end{aligned}$$

where A, B, C are the internal angles of the triangle.

One may now safely see, *a priori*, that the angle between two straight lines *ought* to be the angle through which one of them must be turned in order that its positive side may coincide with the positive side of the other. And by aid of this definition the result we have just found may be extended from Trilinear to Multilinear systems. We shall always have

$$\phi\,(lx+my+nz+\ldots) \equiv l^2+m^2+n^2\ldots - 2lm\cos(xy) - \ldots$$

Areals.

The Areal coordinates of a point are quantities proportional to the triangles it determines with the sides of a certain fixed triangle. Let α, β, γ be the sides of this triangle; then we have

$$x,\ y,\ z = \frac{\alpha X}{\sqrt{\phi X}},\ \frac{\beta Y}{\sqrt{\phi Y}},\ \frac{\gamma Z}{\sqrt{\phi Z}},$$

and so

$$\phi\,(lx + my + nz)$$
$$= \alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 - 2\beta\gamma mn \cos A - 2\gamma\alpha nl \cos B - 2\alpha\beta lm \cos C,$$

the signs being determined as before.

Interpretation of Constants.

13. Let P_1, P_2, P_3 be the three perpendiculars of the triangle of reference, and ϖ_1, ϖ_2, ϖ_3 the perpendiculars from the angular points on the line (lmn). Then by the formula in Art. 7, we have, in Trilinears,

$$\varpi_1 : \varpi_2 : \varpi_3 = lP_1 : mP_2 : nP_3,$$

since, in this system, the coordinates of the angular points are as

$$P_1,\ 0,\ 0;\quad 0,\ P_2,\ 0;\quad 0,\ 0,\ P_3.$$

Consequently,

$$l : m : n = \frac{\varpi_1}{P_1} : \frac{\varpi_2}{P_2} : \frac{\varpi_3}{P_3}.$$

In the Areal system, the coordinates of the angular points are as 1, 0, 0; 0, 1, 0; 0, 0, 1; so that in this case we have

$$l : m : n = \varpi_1 : \varpi_2 : \varpi_3.$$

It is of the greatest importance *to notice* that there are *two different* systems of Tangential Coordinates, corresponding to Trilinears and Areals respectively. In the former, the coordinates of a line are proportional to $\frac{\varpi_1}{P_1}$, $\frac{\varpi_2}{P_2}$, $\frac{\varpi_3}{P_3}$; in the latter they are proportional to ϖ_1, ϖ_2, ϖ_3. If we keep this

distinction in mind, we may say generally that the *coefficients* in the *equation* of a point are proportional to the *coordinates* of the point; and that the *coefficients* in the *equation* of a line are proportional to the *coordinates* of the line. And then we may get rid of coordinates altogether, and consider nothing but equations.

Line at Infinity.

14. Form the reciprocal of ϕ by the ordinary method; it is, in Trilinears,

$$(x \sin A + y \sin B + z \sin C)^2,$$

and in Areals,

$$\beta^2\gamma^2 \sin^2 A\, (x + y + z)^2.$$

It must be remembered that we are not here finding the *equation* of the line at infinity, but the value of the function $A\infty^2$ of the point (x, y, z). In the case of Trilinears, if we suppose x, y, z to be actually *equal* to the perpendiculars from a point on these sides, then the quantity

$$x \sin A + y \sin B + z \sin C$$

is four times the projector of the triangle of reference.

Quadrilaterals.

15. Consider four straight lines $A, B, C, D = 0$. Since the equation of *any* line can be expressed in terms of the equations of three others, there must be an identical relation

$$lA + mB + nC + sD \equiv 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

and because this is independent of the values of x, y, z, we must have

$$\begin{aligned} lA_x + mB_x + nC_x + sD_x &= 0, \\ lA_y + mB_y + nC_y + sD_y &= 0, \\ lA_z + mB_z + nC_z + sD_z &= 0, \end{aligned}$$

where A_x denotes the coefficient of x in A, &c. Eliminating therefore l, m, n, s from these four equations, we have

$$\begin{vmatrix} A, & B, & C, & D \\ A_x, & B_x, & C_x, & D_x \\ A_y, & B_y, & C_y, & D_y \\ A_z, & B_z, & C_z, & D_z \end{vmatrix} \equiv 0.$$

But, by definition of a Jacobian, this is equivalent to

$$A\,.\,J(BCD) - B\,.\,J(CDA) + C\,.\,J(DAB) - D\,.\,J(ABC) \equiv 0,$$

and this therefore is the identical relation between four given lines.

Now it is often convenient to use a set of four coordinates, x, y, z, w, connected by the identical relation

$$x + y + z + w = 0\,;$$

we have then only to put

$$x, y, z, w = A\,.\,J(BCD),\quad B\,.\,J(CAD),\quad C\,.\,J(ABD),\quad D\,.\,J(CBA),$$

and we shall have such a system. We then find that

$$\phi\,(lx + my + nz + sw)$$
$$= l^2\phi A\,.\,J(BCD)^2 + \ldots + 2lm\,.\,\psi AB\,.\,J(BCD)\,.\,J(CAD) + \ldots$$

Now let $\alpha, \beta, \gamma, \delta$ be the projectors of the triangles BCD, CAD, ABD, CBA; and divide the result just found by $\phi A\,.\,\phi B\,.\,\phi C\,.\,\phi D$; then we may write

$$\phi\,(lx + my + nz + sw) = \alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 + \delta^2 s^2 - 2\alpha\beta lm \cos(xy) - \ldots\ldots$$

We have here in fact made x proportional to the perpendicular from a point on A, multiplied by the projector of BCD. And this might be taken as the definition of the system of coordinates.

If for D we write ∞, the identity becomes

$$\infty\,.\,J(ABC) \equiv A\,.\,J(BC\infty) + B\,.\,J(CA\infty) + C\,.\,J(AB\infty).$$

Hence we find, in Trilinears, for instance,

$$\infty\,.\,\frac{J(ABC)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}} \equiv x\sin A + y\sin B + z\sin C \equiv P(ABC),$$

which agrees with the result before obtained, since

$$\frac{J(ABC)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}}$$

is really the *ratio* of the projector of ABC to the projector of the triangle of reference.

Equations.

16. The *condition* that a point A shall lie on a line L is $AL=0$. Now if we consider the coefficients of the point as coordinates, then $AL=0$ is the *equation* of the *line* L; if we consider the coefficients of the line as coordinates, $AL=0$ is the equation of the *point* A. It is not difficult to see how this notion may be generalized. Consider the form

$$\xi x+\eta y+\zeta z,$$

which may represent either a point or a line, according as $(\xi\eta\zeta)$ or (xyz) are regarded as variable. I shall denote this form by an asterisk (*), and use it exclusively to represent *equations*. For instance, the equation $J(AB^*)=0$ is the equation to the *point* or *line* AB, according as A and B are lines or points. I give one or two examples of equations found by this method.

(α) *Locus of a point subtended by four given points in a given anharmonic ratio.*

We want a point P such that $\dfrac{\sin APB \,.\, \sin CPD}{\sin APC \,.\, \sin DPB}=k.$ The locus is then immediately written down; viz. since

$$\sin APB=\frac{P\infty \,.\, J(ABP)}{\sqrt{\phi(AP)\,.\,\phi(BP)}},$$

it is

$$J(AB^*)\,.\,J(CD^*)=kJ(AC^*)\,.\,J(DB^*),$$

which is clearly a conic through the points $ABCD$.

(β) *Envelop of a line cut by four given lines in a given anharmonic ratio.*

By the formula, Art. 7, the envelop is (in Tangential Coordinates),

$$J(LM^*)\,.\,J(NR^*)=k\,.\,J(LN^*)\,.\,J(RM^*),$$

a conic touching the four given lines.

(γ) *Envelop of a line, the product of whose distances from two fixed points is constant.*

Ans. $$(A^*)\,(B^*) = k^2 A\infty \,.\, B\infty \,.\, \phi\,(^*),$$

a conic inscribed in the quadrilateral formed by joining A and B to the two points of ϕ.

(δ) *Envelop of a line of constant length resting on two given lines.*

Ans. $$\phi\,(^*)\,.\,J\,(LM^*)^2 = k^2\,.\,J\,(L\infty\,^*)^2\,.\,J\,(M\infty\,^*).$$

Let α be the point LM, β the point $L\infty$, γ the point $M\infty$; then if we remember that any point on the line $\alpha\rho$, *i.e.* at infinity, has its equation of the form $l\alpha + m\beta = 0$, this equation may be written

$$\alpha^2\,(a\beta^2 + 2b\beta\gamma + c\gamma^2) = k^2\beta^2\gamma^2,$$

or $$\frac{a}{\gamma^2} + \frac{2b}{\beta\gamma} + \frac{c}{\beta^2} = \frac{k^2}{\alpha^2};$$

the envelop is therefore the tangential inverse, in respect of the triangle $\alpha\beta\gamma$, of a conic in respect of which α is the pole of $\beta\gamma$. It is obvious from the figure that each of the lines L, M, touches at two cusps, so that the curve is of the sixth order. The equation shews that the line at infinity ($\beta\gamma$) is also a double tangent, and when L, M are at right angles, $b = 0$, and the line at infinity touches at two cusps.

(ϵ) *Envelop of a line cut by three given lines in given ratios.*

Ans. $$\lambda J\,(MN^*)\,.\,J\,(L\infty\,^*) + \mu J\,(NL^*)\,.\,J\,(M\infty\,^*) + \nu J\,(LM^*)\,J\,(N\infty\,^*) = 0,$$

a parabola touching the three given lines.

The utility of the method in questions of this sort is still more evident in the case of curves and surfaces of the second order.

V. *Planes and Points in Space.*

Expressions Considered.

17. In Geometry of Three Dimensions the absolute is an imaginary circle in which the plane at infinity cuts any sphere. The condition that a plane $A = 0$ may touch this circle, is of the second order in the coefficients of A, and may be written $\phi(A) = 0$. This being so, we have, as before,

$$\phi(\lambda A + \mu B) \equiv \lambda^2\phi(A) + 2\lambda\mu \,.\, \psi(A, B) + \mu^2 \,.\, \phi(B),$$

where the function $\psi(A, B)$ is of the first order in the coefficients of each of the two planes, and vanishes only when they are at right angles.

The condition that a point $a = 0$ may be at infinity is of the first order in the coordinates or coefficients of the point, and may be written $a\infty = 0$.

The condition that four planes $A, B, C, D = 0$ may meet in a point, is that the determinant formed with their coordinates or coefficients, that is to say, their Jacobian, shall vanish. This I express by the equation

$$J(ABCD) = 0.$$

The condition that three planes $A, B, C = 0$ may meet at infinity, or, which is the same thing, may be parallel to the same line, is accordingly

$$J(ABC\infty) = 0.$$

18. The form of the absolute ϕ is found exactly as in plane geometry; I will therefore anticipate a formula, and give it here. We have generally,

$$\phi(lx + my + nz + sw) \equiv l^2\phi(x) + \ldots + 2mn\psi(y, z) + \ldots .$$

Now by means of the formula

$$\cos(A, B) = \frac{\psi(A, B)}{\sqrt{\phi(A) \,.\, \phi(B)}},$$

this becomes, in quadriplanar coordinates,

$$\phi\,(lx+my+nz+sw) \equiv l^2+m^2+n^2+s^2-2mn\cos(y,z)-\ldots,$$

and in tetrahedral coordinates

$$\phi\,(lx+my+nz+sw) \equiv l^2\alpha^2+\ldots-2mn\,.\,\beta\gamma\,.\cos(y,z)-\ldots,$$

in both of which $\cos(y, z)$ means the cosine of the internal angle between the planes $y=0$, $z=0$ of the fundamental tetrahedron, and α, β, γ, δ are the areas of the faces. The equation to the plane at infinity is then found to be, in the quadriplanar system,

$$Px+Qy+Rz+Sw=0,$$

where $$P^2=\begin{vmatrix} 1, & -\cos(y,z), & -\cos(y,w) \\ -\cos(y,z), & 1, & -\cos(z,w) \\ -\cos(y,w), & -\cos(z,w), & 1 \end{vmatrix}.$$

It is convenient to call P the *sine* of the solid angle yzw (M. Paul Serret). If we write A, B, C, D for the solid angles of the tetrahedron, then

$$\frac{\sin A}{\alpha}=\frac{\sin B}{\beta}=\frac{\sin C}{\gamma}=\frac{\sin D}{\delta},$$

and the equation to the plane at infinity, in tetrahedral coordinates, is

$$\beta\gamma\delta\sin A\,(x+y+z+w)=0.$$

Fundamental Propositions.

19. The following propositions may be proved by Cartesian coordinates.

A. If a plane touch the imaginary circle at infinity,

(*a*) Every area measured on the plane is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of every angle *on* the plane, the sine is zero and the cosine unity.

(*d*) Of every angle made with the plane, the sine and cosine are infinite.

B. If a straight line meet the imaginary circle at infinity,

(*a*) Every length measured on the line is zero.

(*b*) Every perpendicular distance from it is infinite.

(*c*) Of the angle between any two planes through it, the sine is zero and the cosine unity.

(*d*) Of every angle made with it, the sine and cosine are infinite.

C. If a point be at infinity,

(*a*) Its perpendicular distance from any plane or straight line not passing through it is infinite.

(*b*) The volume of the tetrahedron which it forms with any other three points not in the same plane with it is infinite.

(*c*) The area of the triangle which it forms with any other two points, not such that the plane of the triangle touches the absolute (see prop. *A*, *a*), is infinite.

(*d*) Its distance from any other point, not at infinity, is infinite.

Formulæ of Adaptation.

20. These are to be deduced from the propositions just stated precisely in the same way as the corresponding formulæ in Plane Geometry were proved. I postpone for the present the consideration of formulæ relating to straight lines.

The volume contained by four points a, b, c, d is

$$\frac{J(abcd)}{a\infty \, . \, b\infty \, . \, c\infty \, . \, d\infty}.$$

If the points are given as the intersections of four planes, A, B, C, D, the expression becomes

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty) \, . \, J(CDA\infty) \, . \, J(DAB\infty) \, . \, J(ABC\infty)}.$$

The area contained by three points a, b, c is

$$\frac{\sqrt{\phi}\,(abc)}{a\infty \, . \, b\infty \, . \, c\infty},$$

where $\phi\ (abc)$ means that we are to write down the equation of the plane through a, b, c, and then form the condition that it may touch the imaginary circle at infinity. If the three points are given as intersections of the plane A with the planes B, C, D, the formula becomes

$$\frac{\sqrt{\phi A} \, . \{J\,(ABCD)\}^2}{J(ACD\infty) \, . \, J\,(ADB\infty) \, . \, J\,(ABC\infty)},$$

which is the area determined on the plane A by the planes B, C, D.

The perpendicular from the point a on the plane A is $\dfrac{aA}{\sqrt{\phi A \, . \, a\infty}}$. Here aA is used for the result of substituting the coordinates of a in the equation of A, or *vice versâ*.

If θ be the angle between two planes A and B, then

$$\cos\theta = \frac{\psi\,(A,\,B)}{\sqrt{\phi A \, . \, \phi B}}, \quad \sin^2\theta = \frac{\phi A \, . \, \phi B - \{\psi\,(A,\,B)\}^2}{\phi A \, . \, \phi B}.$$

It will be proved that the sine of the solid angle contained by three planes is

$$\frac{J\,(ABC\infty)}{\sqrt{\phi A \, . \, \phi B \, . \, \phi C}}.$$

Theorems.

21. I use Professor Sylvester's umbral notation, which, whenever determinants have to be employed, is not only convenient, but essential.

In this notation, the determinant

$$\begin{vmatrix} \psi\,(AD), & \psi\,(BD), & \psi\,(CD) \\ \psi\,(AE), & \psi\,(BE), & \psi\,(CE) \\ \psi\,(AF), & \psi\,(BF), & \psi\,(CF) \end{vmatrix}$$

is written $$\psi \left| \begin{matrix} ABC \\ DEF \end{matrix} \right. .$$

It will be easy now to interpret the notation in other cases.

For instance, the definition of sin (y, z, w), in Art. 18, may be written

$$\sin^2 (y, z, w) \equiv \cos \left| \begin{matrix} yzw \\ yzw \end{matrix} \right. .$$

I consider now the series of determinants

$$\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right., \quad \psi \left| \begin{matrix} ABC \\ DEF \end{matrix} \right., \quad \psi \left| \begin{matrix} AB \\ CD \end{matrix} \right. .$$

Suppose we wanted to find the condition that it might be possible to draw a plane

$$\lambda A + \mu B + \nu C + \sigma D = 0,$$

which should be at once perpendicular to each of the planes E, F, G, H. We should write down four equations like

$$\lambda\psi (A, E) + \mu\psi (B, E) + \nu\psi (C, E) + \sigma\psi (D, E) = 0,$$

and then eliminate $\lambda\mu\nu\sigma$ between these equations. Thus, we should arrive at the condition $\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right. = 0$. First, suppose that A, B, C, D do not meet in a point. Then the equation of any plane whatever can be put into the form

$$\lambda A + \mu B + \nu C + \sigma D = 0.$$

Now, the plane at infinity is perpendicular to all other planes. In this case, therefore, the condition $\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right. = 0$ is satisfied. Next let A, B, C, D meet in a point. Then there is an *identity*

$$\lambda A + \mu B + \nu C + \sigma D \equiv 0,$$

which gives rise to four other identities like

$$\lambda\psi (AE) + \mu\psi (BE) + \nu\psi (CE) + \sigma\psi (DE) \equiv 0,$$

so that in this case also $\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right. = 0$.

Hence, we have always, identically,

$$\psi \left| \begin{matrix} ABCD \\ EFGH \end{matrix} \right. \equiv 0 \quad \text{.......................} \quad (1).$$

In the same way we see, that $\psi \left| \begin{matrix} ABC \\ DEF \end{matrix} \right. = 0$, expresses the condition that a plane $\lambda A + \mu B + \nu C = 0$ may be so drawn as to

be perpendicular at once to each of the planes D, E, F. Now unless three planes are parallel to the same line, the only plane which is perpendicular to all three of them is the plane at infinity. If therefore the condition is satisfied, either A, B, C meet at infinity, or D, E, F meet at infinity. Thus we have

$$\psi\left|\begin{matrix}ABC\\DEF\end{matrix}\right. \equiv J(ABC\infty)\,.\,J(DEF\infty)\,\text{.........}(2).$$

The condition $\psi\left|\begin{matrix}AB\\CD\end{matrix}\right. = 0$ will be satisfied if we can draw a plane through the line (A, B) perpendicular to the two planes C, D. It is easy to see that this can only be the case when the line (A, B) is perpendicular to the line (C, D). This result may be expressed in the form

$$\psi\left|\begin{matrix}AB\\CD\end{matrix}\right. \equiv \psi(AB,\ CD)\,\text{..............}(3).$$

To interpret the theorems (1) and (2), I observe that to multiply any single row or column of a determinant by a certain quantity, is to multiply the whole determinant by that quantity. Thus we have

$$\cos\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right. = \frac{\psi\left|\begin{matrix}ABCD\\EFGH\end{matrix}\right.}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C\,.\,\phi D\,.\,\phi E\,.\,\phi F\,.\,\phi G\,.\,\phi H}} = 0,$$

$$\text{and}\quad \cos\left|\begin{matrix}ABC\\DEF\end{matrix}\right. = \frac{\psi\left|\begin{matrix}ABC\\DEF\end{matrix}\right.}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C\,.\,\phi D\,.\,\phi E\,.\,\phi F}}$$

$$= \frac{J(ABC\infty)}{\sqrt{\phi A\,.\,\phi B\,.\,\phi C}}\cdot\frac{J(DEF\infty)}{\sqrt{\phi D\,.\,\phi E\,.\,\phi F}}\,.$$

If now we make A, B, C identical with D, E, F respectively, we get

$$\frac{\{J(ABC\infty)\}^2}{\phi A\,.\,\phi B\,.\,\phi C} = \cos\left|\begin{matrix}ABC\\ABC\end{matrix}\right. = \sin^2(A,\ B,\ C) \text{ by definition; and}$$

$$\text{so}\qquad \cos\left|\begin{matrix}ABC\\DEF\end{matrix}\right. = \sin(A,\ B,\ C)\,.\,\sin(D,\ E,\ F).$$

The theorems (1) and (2) will be found to embody a great number of results.

22. I now prove certain known theorems of determinants, which are useful in this subject.

Consider five planes A, B, C, D, E, whose equations are

$$l_1x + m_1y + n_1z + s_1w = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$l_5x + m_5y + n_5z + s_5w = 0.$$

We know that, since each one can be expressed in terms of the other four, there must exist some identical relation

$$PA + QB + RC + SD + TE = 0.$$

And because this is true for all values of $x_1y_1z_1w_1$, we must have

$$Pl_1 + Ql_2 + Rl_3 + Sl_4 + Tl_5 = 0,$$
$$\&c. \qquad \&c.$$

From these five equations we can eliminate $PQRST$, and we find

$$A\,.\,J(BCDE) + B\,.\,J(CDEA) + C\,.\,J(DEAB) + D\,.\,J(EABC)$$
$$+ E\,.\,J(ABCD) \equiv 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the identical relation in question.

Now if we substitute in A the coordinates of the intersection of F, G, H, the result is clearly $J(AFGH)$. But the equation (4) is true for all values of the variables; we may therefore substitute in it the coordinates of the point (F, G, H). In this way we obtain

$$J(AFGH)\,.\,J(BCDE) + J(BFGH)\,.\,J(CDEA)$$
$$+ J(CFGH)\,.\,J(DEAB) + J(DFGH)\,.\,J(EABC)$$
$$+ J(EFGH)\,.\,J(ABCD) \equiv 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Making H identical with E, and transposing, we have

$$J(ABCE)\,.\,J(DFGE) \equiv J(AFGE)\,.\,J(BCDE)$$
$$+ J(BFGE)\,.\,J(CADE) + J(CFGE)\,.\,J(ABDE) \ldots\ldots (6).$$

Write ∞ for E, and use theorem (2); thus

$$\cos\left|\begin{matrix}ABC\\DFG\end{matrix}\right. \equiv \cos\left|\begin{matrix}AFG\\BCD\end{matrix}\right. + \cos\left|\begin{matrix}BFG\\CAD\end{matrix}\right. + \cos\left|\begin{matrix}CFG\\ABD\end{matrix}\right. \ldots\ldots (7).$$

Again, let E be the plane $w=0$ of the fundamental tetrahedron; and write (1, 2, 3, 4, 5, 6) for the traces on this plane of the planes (A, B, C, D, F, G) respectively; then we get the theorem in plane geometry

$$J(123) \,.\, J(456) \equiv J(156) \,.\, J(234) + J(164) \,.\, J(235) \\ + J(145) \,.\, J(236).$$

If we make the lines 1, 2 identical, and write ∞ for each, this becomes the theorem referred to in Art. 6.

Spheres.

23. *To find the equation to the sphere whose diameter is the line joining the point (A, B, C), to the point (D, E, F).*

The sphere may be defined as the locus of the foot of a perpendicular from the point (A, B, C) on a variable plane passing through the point (D, E, F). Now the equations to a line through (A, B, C) perpendicular to a plane L are

$$\frac{A}{\psi(A, L)} = \frac{B}{\psi(B, L)} = \frac{C}{\psi(C, L)} = \frac{1}{k}, \text{ suppose.}$$

Let now $L \equiv \lambda D + \mu E + \nu F$; then at the foot of the perpendicular,

$$\begin{aligned} \lambda\psi(A, D) + \mu\psi(A, E) + \nu\psi(A, F) - kA &= 0, \\ \lambda\psi(B, D) + \mu\psi(B, E) + \nu\psi(B, F) - kB &= 0, \\ \lambda\psi(C, D) + \mu\psi(C, E) + \nu\psi(C, F) - kC &= 0, \\ \lambda D \quad + \quad \mu E \quad + \quad \nu F \qquad\quad &= 0, \end{aligned}$$

from these equations we can eliminate λ, μ, ν, k, and we get for the equation of the sphere

$$\begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & A \\ \psi BD, & \psi BE, & \psi BF, & B \\ \psi CD, & \psi CE, & \psi CF, & C \\ D & E & F & 0 \end{vmatrix} = 0.$$

Call this $(\psi)\left|\begin{matrix}DEF0\\ABC0\end{matrix}\right.=0$; then since theorem (6) is a general theorem of determinants, we have,

$$(\psi)\left|\begin{matrix}ABC0\\DEF0\end{matrix}\right.\equiv(\psi)\left|\begin{matrix}AEF0\\BCD0\end{matrix}\right.+(\psi)\left|\begin{matrix}AFD0\\BCE0\end{matrix}\right.+(\psi)\left|\begin{matrix}ADE0\\BCF0\end{matrix}\right.;\ \dots(8),$$

these four spheres have therefore a common radical axis. Whence this geometrical theorem:

If the faces of a tetrahedron A, B, C, D are met by a straight line in the points a, b, c, d; then the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

It will be observed that the *faces* of the tetrahedron are A, D, E, $F=0$, and that the *straight line* is the intersection of the planes $B=0$, $C=0$. The relation (8) is very easily verified.

(*To be continued.*)*

* [No more on this subject was published in the *Quarterly Journal.*]

XII.

ON THE GENERAL THEORY OF ANHARMONICS*.

1. THE theory of Anharmonics on the straight line may be stated in the following symmetrical form :—

(i) There is an identical relation connecting the distances of four points, 1, 2, 3, 4, on a right line, viz.,

$$12 \, . \, 34 + 13 \, . \, 42 + 14 \, . \, 23 \equiv 0.$$

(ii) The ratios of the three terms in this identity are not altered by projection or integral linear transformation. There is a corresponding theory of four straight lines meeting in a point.

2. In applying this theory to geometry of two dimensions, we meet with this third proposition :—

(iii) If a straight line meet four fixed straight lines, so that the distances of the points which they determine on it satisfy a relation of the form

$$\lambda \, . \, 12 \, . \, 34 + \mu \, . \, 13 \, . \, 42 + \nu \, . \, 14 \, . \, 23 = 0,$$

then the envelope of the line is of the second class, touching the four given lines. There is, of course, a correlative proposition on the locus of a point subtended by four points in a given manner. The propositions (i), (ii), (iii), each including its converse and correlative propositions, may be regarded as constituting the entire theory of anharmonics in geometry of one dimension, and its application to geometry of two dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 9, pp. 3—6.]

3. I proceed to state the theory of anharmonics in geometry of two dimensions.

(i) There are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123 \,.\, 456 + 124 \,.\, 563 + 125 \,.\, 634 + 126 \,.\, 345 \equiv 0,$$

with five others obtained from this by permutation.

(ii) The ratios of the terms in these identities are not altered by projection or integral linear transformation. Under each of these propositions are included *three* correlatives, to explain which I must introduce three new definitions:

DEF. 1st. The *projector* of a plane triangle is the square of the area divided by the continued product of the sides.

DEF. 2nd. In a solid angle considered as determined by three concurrent straight lines, the sine of the angle between the first line and the plane of the other two, multiplied by the sine of the angle between the other two lines, is a symmetrical quantity in respect of the three lines, and is called the *sine* of the three lines.

DEF. 3rd. In a solid angle considered as determined by three planes, the sine of the angle between the first plane and the intersection of the other two, multiplied by the sine of the angle between the other two planes, is a symmetrical quantity in respect of the three planes, and is called the *sine* of the three planes.

4. It is convenient for several purposes to use the word "distance" as including all these notions: thus I shall speak of the *distance* of two lines in a plane, meaning the sine of the angle between them;

the distance of three points is the area of their triangle;

" " of three straight lines in a plane, the projector of their triangle;

" " of three planes, the sine of the planes;

" " of three concurrent lines in space, the sine of the lines.

And I shall have occasion afterwards to define the distance of four points and of four planes in space. By means of these definitions, the propositions (i), (ii) may be interpreted in four different ways, corresponding to the four aspects of bi-dimensional extension: the symbol 123 being always understood to mean the *distance* of the three things considered.

5. The propositions (i), (ii), including their converse and correlative propositions, constitute the entire theory of anharmonics in geometry of two dimensions. To apply this theory to geometry of three dimensions, I state the following proposition, which has only *one* correlative:

(iii) If a plane meet six fixed planes so that the distances of the lines they determine upon it satisfy a relation of the form

$$\lambda . 123 . 456 + \mu . 124 . 563 + \nu . 156 . 234 + \ldots = 0,$$

then the envelope of the plane is of the second class, touching the six given planes.

6. The theory of anharmonics in three dimensions is so entirely analogous to the two former theories, that it wants no further discussion. The distance of four points is the volume of their tetrahedron, and the distance of four planes is the cube of the volume divided by the product of the areas of the faces.

7. Two finite lines 11′, 22′, measured on the same straight line, are said to be *harmonically situate* when

$$12 . 1'2' + 12' . 1'2 = 0;$$

and when this is the case, if the two pairs of points be represented by the equations $U = 0$, $V = 0$, there is an invariant relation connecting the quadrics U, V which may be denoted by

$$\square\,(U, V) = 0;$$

the notation indicating that if we expand the discriminant of $\lambda U + \mu V$, or

$$\square\,(\lambda U + \mu V),$$

then the coefficient of $\lambda\mu$ in the expansion vanishes.

There is a similar relation between two angles having a common vertex; when this relation holds, I say that the two covertical angles are harmonically situate. I also speak of an harmonic pair of angles, or of an harmonic pair of finite lines, or lengths; meaning in this case covertical angles, or collinear lengths.

8. This being so, it is known that two lengths (I use the word *length* to denote a pair of points), anyhow placed in a plane, determine a conic passing through their ends, which conic is the locus of points which the lengths subtend harmonically, that is, in a pair of harmonic angles. I call this the harmonic conic of the two lengths. This (with the correlative propositions) completes the theory of harmonics in one dimension, and its application to two dimensions. I now come to consider harmonics in two dimensions.

9. If the harmonic conic of two lengths $11'$, $22'$ divides harmonically a third length $33'$, then the relation between the three lengths is symmetrical, and I say that the three lengths are *harmonically situate* in the plane. The following relation subsists among their distances, viz.,

$$123 \,.\, 1'2'3' + 1'23 \,.\, 12'3' + 12'3 \,.\, 1'23' + 123', 1'2'3 = 0.$$

And if each length, or point-pair, be considered as a degenerate conic, so that the equations to the three point-pairs are U, V, W $= 0$, then, when the three lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, which may be denoted by

$$\square\,(U,\ V,\ W) = 0\,;$$

$\square\,(U,\ V,\ W)$ denoting the coefficient of $\lambda\mu\nu$ in the expansion of

$$\text{Discriminant of } (\lambda U + \mu V + \nu W).$$

It is obvious that a similar relation may subsist among three angles in a plane, three pairs of lines through a point in space, or three pairs of planes through a point in space.

10. Three lengths anyhow placed in space determine a quadric surface passing through their ends, which surface is the locus of points which the lengths subtend harmonically, that is,

in a triad of harmonic angles. I call this the harmonic quadric of the three lengths.

11. If the harmonic quadric of three lengths $11'$, $22'$, $33'$ divides harmonically a fourth length $44'$, then the relation between the four lengths is symmetrical, and I say that the four lengths are *harmonically situate* in space. The following relation subsists among their distances, viz.,

$$\Sigma \, . \, 1234 \, . \, 1'2'3'4' = 0 \text{ (eight terms).}$$

And if each length, or point-pair, be considered as a degenerate quadric, so that the equations to the four point-pairs are $U=0$, $V=0$, $W=0$, $T=0$, then, when the four lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, T, which may be denoted by

$$\square\,(U,\ V,\ W,\ T)=0\,;$$

$\square\,(U, V, W, T)$ denoting the coefficient of $\lambda\mu\nu\rho$ in the expansion of

$$\text{Discriminant of } (\lambda U+\mu V+\nu W+\rho T).$$

It is obvious that a similar relation may subsist among four pairs of planes.

12. It only remains to explain the meaning of the conditions

$$\square\,(U,\ V,\ W)=0,$$
$$\square\,(U,\ V,\ W,\ T)=0,$$

when the quadrics do *not* break up into factors. Two conics U, V determine an harmonic conic F, locus of points which they subtend in a pair of harmonic angles. If a triangle self-conjugate of F can be inscribed in W, then the relation between UVW is symmetrical, and $\square\,(U, V, W)=0$. The conics may then be spoken of as three mutually harmonic conics; a similar relation may hold between three covertical cones. Thus, in fact, three quadric *surfaces* U, V, W determine an harmonic quadric F, locus of points which they subtend in three harmonic cones. If a tetrahedron self-conjugate of F can be inscribed in T, then the relation between U, V, W, T is symmetrical, and $\square\,(U, V, W, T)=0$.

XIII.

ON A GENERALIZATION OF THE THEORY OF POLARS*.

(THE present Note establishes the idea of the polar curve of a curve of given class in respect of a curve of given order, the class being less than the order; and of the polar curve of a curve of given order in respect of a curve of given class, the order being less than the class. It also deals with a certain invariant of two curves, such that the order of one is equal to the class of the other; and with certain other invariants and contravariants arising out of the theory of polars. I desire to present these ideas by themselves to the Society, because they seem likely to be useful for other purposes than that to which I propose to apply them subsequently, viz., the extension of Grassmann's Geometric Analysis.)

1. Let B_n be a curve of the nth order, and c_m a curve of the mth class. Let the equations of the curves be

$$B_n \equiv (A, B, C, D \ldots\ldots \between x\, y\, z)^n$$
$$c_m \equiv (a, b, c, d \ldots\ldots\ldots\ldots \between \xi\, \eta\, \zeta)^m$$

in point and line coordinates respectively.

In c_m write $\dfrac{\delta}{dx}, \dfrac{\delta}{dy}, \dfrac{\delta}{dz}$ in place of ξ, η, ζ respectively, and operate on B_n with the symbol thus formed. I denote the result by merely writing c_m as an operator before B_n; thus

$$c_m B_n \equiv \left(a, b, c \ldots\ldots \between \frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}\right)^m . (A, B, C, \ldots\ldots \between x, y, z)^n;$$

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 16, pp. 116—118.]

then we find

(i) If m is less than n, $c_m B_n$ is a covariant, which I call the *polar curve* of c_m in respect of B_n. It is given in the point coordinates x, y, z, and is of order $n-m$.

(ii) If m is equal to n, $c_n B_n$ is an invariant of the two curves. When this invariant vanishes, I say that the curves are *harmonic* of each other.

(iii) If m is greater than n, $c_m B_n = 0$ always.

2. It is important to shew that these new meanings of the words *polar* and *harmonic* include the old meanings. Now the mth polar of a point (x, y, z) whose tangential equation is $x\xi + y\eta + z\zeta = 0$, say the point P, is

$$\left(x\frac{\delta}{dx} + y\frac{\delta}{dy} + z\frac{\delta}{dz}\right)^m . B_n = 0;$$

which will be denoted in our present notation by $p^m B_n = 0$. But this is, according to the new definition, the polar of m times the point p; the point p taken m times being of course a particular case of a curve of the mth class. Again, the two conics

$$(a, b, c, f, g, h \between \xi\ \eta\ \zeta)^2, \qquad (A, B, C, F, G, H \between x, y, z)^2$$

have been called *harmonic* when

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh = 0,$$

the invariant being obviously the result of turning the first into an operator and applying it to the second.

3. Returning to the curves c_m, B_n, we may convert B_n into an operator by writing in it $\dfrac{\delta}{d\xi}, \dfrac{\delta}{d\eta}, \dfrac{\delta}{d\zeta}$ in place of x, y, z respectively. The result of the operation on c_m may be denoted by $B_n c_m$. As before, if $n > m$, the result is zero; if $n = m$, it is the harmonic invariant; if $n < m$, it is a curve of class $m - n$, which may be called the polar curve of B_n in respect of c_m.

4. Another definition of polar curves may be given. Suppose that the curves g_m, h_n together make up a curve which is harmonic of B_{m+n}. It is convenient to say that the curves g_m, h_n

are *complementary*. Then it is clear that *all the curves complementary to a given curve are harmonic of its polar.* This may be regarded as a generalization of the theorem: All the points conjugate to a fixed point in regard to a conic, lie on the polar of the fixed point.

5. Consider a curve of even order B_{2n}. There is an invariant of the order $\frac{1}{2}(n+1)(n+2)$ in the coefficients, which vanishes when the nth differentials are in involution; this invariant is a symmetrical determinant. Its evectant is a contravariant of the order $2n$ in the variables, and $\frac{1}{2}(n^2+3n)$ in the coefficients, which may be called b_{2n}. The curves B_{2n}, b_{2n} are so related that *if X_n be the polar of y_n in respect of B_{2n}, then the y_n is the polar of X_n in respect of b_{2n}.* That is to say, if $X_n = y_n B_{2n}$, then $X_n b_{2n} = I \cdot y_n$ where I is the invariant just defined. For example, if we consider a series of conics c_2 and their polars C_2 in regard to a given quartic curve Q_4, then there exists a curve q_4 of the fourth class, such that in respect to it the conics c_2 are the polars of C_2.

6. Two curves whose order and class are different may be made susceptible of the harmonic relation by taking each a proper number of times. Thus, curves B_6, c_8 have an invariant $(B_6)^4 (c_8)^3$, of the order 4 in the coefficients of B_6 and of the order 3 in the coefficients of c_8*. It is to be observed that the equation $B_{mn}(c_n)^n = 0$, is the most general relation of the nth order that can subsist among the coefficients of c_m.

7. The following remarks relate to theorems in Dr Henrici's paper "On certain formulæ concerning the Theory of Discriminants†."

* Thus we may have an invariant which vanishes when a curve is harmonic to itself. Let U_m be the curve, u_n its reciprocal: then $U_m^n u_n^m$ is the invariant in question, or a power of it. For a conic, it is the discriminant; for a cubic, the invariant T of the sixth order. See Salmon's *Higher Algebra*, 1st ed., note to p. 67.

If the curve U_m has no node, the class $= m(m-1)$, and the invariant is $(U_m)^{m-1} u_{m(m-1)}$. If however U_m has a node d, d^2 is part of the reciprocal, and the invariant is $d^2 u_{m(m-1)-2} (U_m)^n$. Cusps may be similarly dealt with.

† [*Proceedings of the London Mathematical Society*, Vol. II. Nos. 15, 16.]

If the polar of C_m in respect of B_n has a node, C_m is harmonic of a curve of order $\zeta m(n-m-1)^2$, which is Dr Henrici's curve $S^{(m)}$. In fact when C_m is a point taken m times, the point is *on* the curve $S^{(m)}$; this is Dr Henrici's theorem.

In general, $x^m y^n B_{m+n+1}$ denotes a straight line. If it vanishes identically, x is a node on the nth polar of y, and y is a node on the mth polar of x. In this case $x^m y^{n-1} B_{m+n-1}$ and $x^{m-1} y^n B_{m+n-1}$ are conics having nodes at y and x respectively*. In the relation

$$x^m y^n B_{m+n+1} \equiv 0$$

write $x+\delta x$ for x and $y+\delta y$ for y. This is equivalent to supposing δx and δy to be points on the tangents at x and y to the curves $S^{(m)}$ $S^{(n)}$ which are the loci of those points respectively. Then we have

$$(my\delta x + nx\delta y)\, x^{m-1} y^{n-1} B_{m+n+1} \equiv 0;$$

operate on this with y; we know that

$$n\delta y \,.\, x^m y^n B_{m+n+1} = 0,$$

it follows that

$$m\delta x \,.\, x^{m-1} y^{n+1} B_{m+n+1} = 0;$$

that is to say, the tangent at x to $S^{(m)}$ is the line-polar of y in respect of the $(m-1)$th polar of x. This is another of Dr Henrici's theorems. I have added this proof as an example of the readiness with which the operative notation lends itself to such investigations.

* Viz., these are the pairs of tangents at the two nodes. It is observable that the tangents at x to the nth polar of y, the tangent to the curve $S^{(m)}$, and the line xy, form a harmonic pencil.

XIV.

ON SYZYGETIC RELATIONS AMONG THE POWERS OF LINEAR QUANTICS*.

IN his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):

"*In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the* m^{th} *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation*

$$\lambda_1 P_1^m + \lambda_2 P_2^m + \lambda_3 P_3^m + \ldots\ldots \equiv 0\dagger."$$

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that every curve or surface touching all but one of them, touches also the remaining one. For the sake of clearness I have here stated in full only one of these four theorems.

By the use of Professor Sylvester's method of Contravariant Differentiation I have arrived at certain extensions of these theorems, which I now proceed to explain:—

Theorem I. *In order that a system of N points in a plane should all lie on a curve of order n, it is sufficient that the* p^{th}

* [From the *Proceedings of the London Mathematical Society*, Vol. III. No. 21, pp. 9—12.]

† In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.

powers of their distances from an arbitrary line should satisfy a linear homogeneous relation; the number N being given by the formula

$$N = \tfrac{1}{2}\alpha n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

Theorem II. *In order that a system of N points in space should all lie on a surface of order n, it is sufficient that the p^{th} powers of their distances from an arbitrary plane should satisfy a linear homogeneous relation; the number N being given by the formula*

$$N = \tfrac{1}{6}\alpha n(n^2 + 6n + 11) + \tfrac{1}{6}(\beta+1)(\beta+2)(\beta+3),$$

where as before

$$p = \alpha n + \beta, \quad \beta < n.$$

To render the nature of these theorems somewhat more clear, I add the following tables of the values of N for given values of p and n:—

TABLE A.—CURVES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Line	5	7	9	11	13	15	17	19	21	23	25
Conic........ ...	6	8	11	13	16	18	21	23	26	28	31
Cubic		10	12	15	19	21	24	28	30	33	37
Quartic.... . ..			15	17	20	24	29	31	34	38	43
Quintic............				21	23	26	30	35	41	43	46
Sextic					28	30	33	37	42	48	55
Septic						36	38	41	45	50	56
Octavic...........							45	47	50	54	59

TABLE B.—SURFACES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Plane	7	10	13	16	19	22	25	28	31	34	37
Quadric.	10	13	19	22	28	31	37	40	46	49	55
Cubic		20	23	29	39	42	48	58	61	67	77
Quartic			35	38	44	54	69	72	78	88	103
Quintic				56	59	65	75	90	111	114	120
Sextic					84	87	93	103	118	139	167
Septic						120	123	129	139	154	175
Octavic							165	168	174	184	199

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Plücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quartic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = \alpha\xi + \beta\eta + \gamma\zeta \; (\equiv p, \text{ say})$$

and let the equation of a curve of the n^{th} order be

$$0 = (*\between x, y, z)^n \; (\equiv B_n, \text{ say})$$

then I say that

$$(*\between\frac{\delta}{\delta\xi},\ \frac{\delta}{\delta\eta},\ \frac{\delta}{\delta\zeta})^n.\ (\alpha\xi+\beta\eta+\gamma\zeta)^n=(\overline{*}\between\alpha,\ \beta,\ \gamma)^n\ \underline{|n};$$

that is to say, *if we operate with* B_n *on the* n^{th} *power of* p, *we shall obtain the result of substituting the coordinates of* p *for* x, y, z *in* B . If, then, this result vanishes, the point p is on the curve B_n.

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4. This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

*XV.

ON SYZYGETIC RELATIONS CONNECTING THE POWERS OF LINEAR QUANTICS.

I THINK the first treatment of this subject is to be found in some very interesting articles of M. Paul Serret's; *Nouvelles Annales*, t. IV. (1865), pp. 145, 193, and 433. M. Serret's attention was confined to the *squares* of linear quantics; and in regard to these he establishes such propositions as the following:—If the squares of the characteristics of the equations of six lines satisfy a syzygetic relation, the six lines touch a conic section. That is to say, if $P_1=0$, $P_2=0$, ... $P_6=0$ are the equations of the lines, and if we have an identical relation

$$\lambda_1P_1^2+\lambda_2P_2^2+\lambda_3P_3^2+\lambda_4P_4^2+\lambda_5P_5^2+\lambda_6P_6^2\equiv 0\,;$$

or, as he finds it convenient to write

$$\Sigma_1^6\lambda P^2\equiv 0$$

where the λs are numerical coefficients, then the lines P_1 P_2 ... P_6 touch the same conic. Another of his propositions is that if eight planes P_1, P_2, ... P_8 satisfy an identical relation

$$\Sigma_1^8\lambda P^2\equiv 0,$$

then the eight planes are such that any quadric surface touching seven of them touches also the eighth. These propositions are arrived at by a somewhat circuitous path, though the steps severally are elegant. From the latter M. Serret obtains a very beautiful and immediate proof of Hesse's theorem that two tetrahedra self-conjugate to the same quadric are such that every quadric touching seven of their faces touches also the eighth. Namely, the equation of the first quadric may be written in either of the forms

$$\lambda_1P_1^2+\lambda_2P_2^2+\lambda_3P_3^2+\lambda_4P_4^2=0,\ \lambda_5P_5^2+\lambda_6P_6^2+\lambda_7P_7^2+\lambda_8P_8^2=0,$$

and these two being identical to a factor *près*, we have a syzy-

getic relation among the eight squares, from which by the second of the above propositions the theorem in question at once follows.

Having, by an application of Prof. Sylvester's most powerful method of contravariant differentiation, succeeded in extending these propositions to higher powers of linear quantics, and to curves and surfaces of any order, I found as a particular result that two quadrilaterals of the same system totally inscribed in a cubic are such that every curve of the third class touching seven of their sides touches also the eighth. Doubtful of this proposition, I communicated it to the Mathematical Society, and was subsequently informed by Mr Cotterill that the eight lines in question touch the same conic. This is equivalent to the analytic theorem, "if the cubes of eight linear quantics are syzygetic, the squares of any six of them are syzygetic." The proof of this by contravariant differentiation and the statement of a series of analogous propositions occupy the following notes.

I.

If in the tangential equation of a curve

$$c_p \equiv (\xi, \eta, \zeta)^p = 0$$

we write

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \text{ for } \xi, \eta, \zeta,$$

and operate upon

$$(lx + my + nz)^p,$$

we shall get $\underline{|p}$ multiplied by the result of substituting l, m, n for ξ, η, ζ in c_p; that is to say

$$\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^p . (lx + my + nz)^p = \underline{|p} . (l, m, n)^p.$$

For shortness, denote $(lx + my + nz)$ by Q, and let c_p mean also the differential operator

$$\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^p.$$

Then *if the operator* $\mathrm{c_p}$ *reduces* $\mathrm{Q^p}$ *to zero, the line* Q *touches the curve* $\mathrm{c_p}$, *and conversely.*

Suppose now that there are several lines $Q_{(1)}$, $Q_{(2)}$, ... and that there is an identical relation,

$$\Sigma\lambda Q^p \equiv 0,$$

connecting the p^{th} powers of the quantities Q. Let also c_p be a curve touching all but one of the lines, so that the operator c_p reduces to zero all but one of the quantities Q^p. The expression $\Sigma\lambda Q^p$ being identically zero, the result of operating upon it with c_p must be zero, or we have

$$\Sigma\lambda c_p Q^p = 0.$$

But of the terms $\lambda_1 c_p Q_{(1)}{}^p$, $\lambda_2 c_p Q_{(2)}{}^p$, ... we know that all vanish but one; it follows that this last one also vanishes, or the curve c_p touches the remaining line. We may therefore enunciate the proposition:—*If there are* n *lines* $Q_1, Q_2, \ldots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p \equiv 0,$$

then every curve c_p *of class* p *which touches* $n-1$ *of the lines will also touch the* n^{th}.

It will be sufficient merely to enunciate the obviously corresponding proposition in three dimensions:—

If there are n *planes* $Q_1, Q_2, \ldots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p = 0,$$

then every surface c_p *of class* p *which touches* $n-1$ *of the planes will also touch the* n^{th}.

In solid geometry, however, as usual, the analogy branches off into two distinct directions, and we are led to consider a somewhat different theory.

Let the number of straight lines which can be drawn through a fixed point and in a fixed plane to touch a given surface be called the *rank* of the surface (viz. this is both the class of a general plane section and the order of a general tangent cone), then that relation between the six coordinates of a line which expresses that the line touches the surface will be of a degree equal to the rank of the surface. I shall denote the expression equated to zero in this equation by a Greek letter whose suffix

indicates the rank; thus, for example, $\beta_2 = 0$ is the rank-equation* of a quadric surface.

The six coordinates being a, b, c, f, g, h, where $af + bg + ch = 0$, it is very easy to prove that if in β_n an expression of the n^{th} degree in these coordinates, we substitute for a, b, c, f, g, h respectively

$$\frac{d}{df}, \frac{d}{dg}, \frac{d}{dh}, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc},$$

we shall get an invariant symbol of operation. I shall use β_n to mean not only the function of the coordinates, but also this operator obtained by the substitution just defined. This being so, if we call the condition that the line $(abcfgh)$ shall meet a given line σ or $(lmnpqr)$, the *equation* of the line σ (namely the equation is

$$\sigma \equiv pa + qb + rc + lf + mg + nh = 0),$$

then *the condition that σ shall touch the surface β_n is*

$$\beta_n \sigma^n = 0.$$

From this it follows that

If there are n *straight lines $\sigma_{(1)}, \sigma_{(2)}, \ldots \sigma_{(n)}$, and if there is an identical relation*

$$\Sigma_1^n \, l\sigma^p = 0,$$

then every surface β_p of rank p *which touches* n − 1 *of the lines will also touch the* n*th*.

II.

At this point I digress somewhat to consider the interpretation of what I have elsewhere called the harmonic invariant of two curves or surfaces, the order of one being equal to the class of the other. First, in the case of two conics, the point-equation of the first being $B_2 = 0$, and the line-equation of the second $c_2 = 0$, the harmonic invariant is $c_2 B_2$, which is commonly called the invariant Θ. Suppose that this vanishes; then if B_2 can be written in the form

$$X^2 + Y^2 + Z^2$$

* [The expression "line-equation" would have been the more natural one, but a confusion might arise between this line-equation of a surface, and the line-equation of a plane curve.—C.]

(so that the lines XYZ form a self-conjugate triangle), since the operator c_2 reduces this to zero we see that if the conic c_2 touch two of the lines it must touch also the third. Similarly, if B_2 can be written in the form

$$X^2 + Y^2 + Z^2 + W^2$$

(so that $XYZW$ form a self-conjugate quadrilateral), if c_2 touches three of the lines it must touch also the fourth. Hence $c_2B_2 = 0$ is the condition both (1) that c_2 shall be inscribed in an infinite number of *triangles* self-conjugate to B_2, and (2) that c_2 shall be inscribed in an infinite number of *quadrilaterals* self-conjugate to B_2. These are known interpretations; the latter, given first I think by Dr Salmon under a slightly different form (*Conics*, [§ 375]), was reduced to this more simple and natural shape by Professor Cremona (*Educational Times*, Reprint)*. Next, in the case of two quadric surfaces $c_2 = 0$ and $B_2 = 0$, if $c_2B_2 = 0$, and B_2 can be expressed in either of the forms

$$\Sigma_1^4 . X^2, \ \Sigma_1^5 . X^2, \ \Sigma_1^6 . X^2,$$

we see at once that if c_2 touch all but one of the planes X it must touch also that other. Hence c_2B_2 is the condition that c_2 shall be inscribed (1) in an infinity of *tetrahedra* self-conjugate to B_2, (2) in an infinity of *pentahedra* self-conjugate to B_2, (3) in an infinity of *hexahedra* self-conjugate to B_2. The terms *conjugate hexahedron*, *conjugate pentahedron*, are introduced by M. Serret, and seem likely to be of considerable use.

Passing now to the rank-equations of the two quadrics, which I shall write $\beta_2 = 0$, $\gamma_2 = 0$, I observe that if β_2 can be thrown into the form $\Sigma_1^6\sigma^2$,
then the 6 lines σ are such that each is conjugate to all the rest; or the lines are the edges of a self-conjugate tetrahedron. If then the harmonic invariant $\gamma_2\,\beta_2$ (Dr Salmon's invariant T) vanishes, and γ_2 touches five of these lines, it will touch the sixth; or γ_2 can touch the edges of an infinite number of tetrahedra self-conjugate to β. We have not yet studied the properties relative to β_2 of a system of lines such that β_2 may be expressed in the form $\Sigma_1^p\sigma^2$ when p is greater than 6; yet it is

* [Cf. Vol. IV. p. 109; Vol. IX. pp. 62, 74.]

obvious that such systems will give new interpretations of the invariant T.

The general extension of this method of interpretation is now perfectly easy. *If* B_n *is a curve of* n^{th} *order harmonic of* c_n *a curve of* n^{th} *class, and if* B_n *can be written in the form* $\Sigma_1^p \,.\, \mathrm{X}^n$, *then if* c_n *touch* $p-1$ *of the lines* X *it will touch also the* p^{th}. Similarly for surfaces in regard to lines and planes. The converse proposition is

The curve or surface $\Sigma_1^p\,\mathrm{X}^n = 0$ *is harmonic of every curve or surface* c_n *of the* n^{th} *class which touches all the lines or planes* X.

I forbear to state the correlative propositions in which lines and planes are replaced by points.

III.

Let us return to the original question.

If there is an identical relation

$$\Sigma_1^8\,\lambda P^3 \equiv 0$$

between the cubes of the 8 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

For we know that we can find a linear differential operator which shall reduce any two of the quantics to zero; namely, let h be the point of intersection of the lines $P_{(7)}$, $P_{(8)}$, having for coordinates a, b, c, then the operator

$$a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}$$

being also denoted by h, we have $hP_{(7)} = 0$, $hP_{(8)} = 0$; and so if we operate with h on the given syzygy we obtain therefrom

$$3\Sigma_1^6\,\lambda \,.\, hP \,.\, P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}, P_{(2)} \ldots P_{(6)}$.

If there is an identical relation

$$\Sigma_1^{11}\,\lambda P^4 = 0$$

between the fourth powers of the 11 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

Let c_2 be the conic section touching the five lines $P_{(7)} \dots P_{(11)}$; then $c_2 P_{(7)}^2 = 0, \dots c_2 P_{(11)}^2 = 0$. If then we operate with c_2 on the given syzygy we shall obtain

$$12\Sigma_1^6 \lambda . c^2P^2 . P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}, \dots P_{(6)}$; provided all the $c_2 P^2$ do not vanish. If however all these vanish, all the lines touch the conic c_2, and there is again a syzygy connecting the squares of any 6 of them. Using these two demonstrations as samples, we are enabled to construct the following table.

Powers $p=$	$n=2$	3	4	5	6	7	8	9	10	11
2	6									
3	8	10								
4	11	12	15							
5	13	15	17	21						
6	16	19	20	23	28					
7	18	21	24	26	30	36				
8	21	24	29	30	33	38	45			
9	23	28	31	35	37	41	47	55		
10	26	30	34	41	42	45	50	57	66	
11	28	33	38	42	48	50	54	60	68	78

*

* [This is in effect the Table A of paper XIV., where it is explained that the number in the Table is

$$N = \frac{\alpha}{2} n(n+3) + \frac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

It may be remarked that so long as p is not greater than n, that is down to the bar in each column of the Table, the Number

$$= \tfrac{1}{2} n(n+3) + \tfrac{1}{2}(p-n+2)(p-n+1),$$

and that the several columns are then continued as follows:

Col. $n=2$.	$n=3$.	&c.
6		
8	10	
$11 = 6+5$	12	
$13 = 8+5$	15	
$16 = 11+5$	$19 = 10+9$	
$18 = 13+5$	$21 = 12+9$	
$21 = 16+5$	$24 = 15+9$	
&c.	&c.	

It is easy to see that this is in fact an equivalent construction of the table. C.]

*XVI.

[ON THE THEORY OF DISTANCES.]

[PRELIMINARY*.]

I EXPLAIN in the first place the notation employed, which is an extension of the Geometric Analysis of GRASSMANN, explained by him in the "Ausdehnungslehre" and in Crelle's Journal, and founded in part on a remark of Leibnitz.

GRASSMANN employs single large letters, as A, B, C, to represent straight lines in a plane, and single small letters, as a, b, c, to represent points. When two large letters come together, as AB, the notation is taken to mean the point of intersection of the lines A, B. So when two small letters come together, as ab, the notation is taken to mean the line joining the points a, b. The equation $ABC = 0$ means that the lines A, B, C meet in a point; the equation $abc = 0$, that the points a, b, c lie in a line; and the equation aB or $Ba = 0$, that the point a lies on the line B. No signification is given to the separated symbols ABC, abc, aB, except as equated to zero. The main principle of the application of this method is that the order of an equation in any letter contained is measured by the number of times that letter occurs; a remark which will be further explained in the sequel.

I now explain the extensions of this notation which I have found it convenient to make.

* [The *Preliminary* matter forms the substance of notes given to Prof. Henrici by the Author at the British Association Meeting in the year 1869. The paper itself (pp. 134—157), without a title, appears to have been written subsequently. The title I have given to the two communications has been taken from that of a paper Prof. Clifford read at the above meeting, an abstract of which is given below (p. 164). I have employed ι to represent $\sqrt{-1}$.]

A curve may be given by its points, or by its tangents; that is to say, we may know its equation in point-coordinates (x,y,z), the degree of the equation being the *order* of the curve; or we may know its equation in the contravariant, tangential, or line-coordinates (ξ, η, ζ), the degree of the equation being then the *class* of the curve. This being so, I denote by a large letter a quantic in (x,y,z), and I write the order of the quantic in the form of a suffix. Thus C_3 denotes a cubic in (x,y,z), and $C_3=0$ is the equation to a curve of the third order. Next, I use a small letter with a suffix to denote a quantic in (ξ, η, ζ), the order of the quantic being denoted by the suffix. Thus b_3 denotes a cubic in (ξ, η, ζ), and $b_3=0$ is the equation to a curve of the third class. When there is no suffix, the suffix 1 is to be understood; in this respect the notation coincides with that of GRASSMANN.

When two large letters come together, each is raised to the power denoted by the suffix of the other, as $A_m{}^n B_n{}^m$. The symbol then denotes a quantic which, equated to zero, gives the equation in tangential coordinates of the mn intersections of the curves A_m, B_n. Similarly $a_m{}^n b_n{}^m=0$ is the equation of the mn common tangents of the curves a_m, b_n. The reason of the indices is now apparent; such equation being of the degree n in the coefficients of the first curve, and of the degree m in those of the second.

When three large letters or three small letters come together, each is raised to a power denoted by the product of the suffixes of the other two; the symbol then denotes the resultant of the three quantics.

When a small letter comes before a large one, as $b_m C_n$, the notation is taken to mean the result of changing (ξ, η, ζ) in b_m into $(\partial_x, \partial_y, \partial_z)$ and performing the operation thus indicated on C_n. So, finally, when a large letter comes before a small one, as $B_m c_n$, the notation is taken to mean the result of changing (x, y, z) in B_m into $(\partial_\xi, \partial_\eta, \partial_\zeta)$ and performing the operation thus indicated on c_n.

In the use of these symbols to investigate the relations of geometrical magnitudes, it is to be observed that the absolute

values of (xyz) or $(\xi\eta\zeta)$ or the coefficients of a quantic are not given, but only their ratios; and consequently that the symbols defined above can have no special value but zero. If however we form a fraction such that every letter mentioned occurs an equal number of times in the numerator and denominator, this *will* have a definite numerical value, being a function of the known ratios aforesaid; and may accordingly represent a geometrical magnitude. This theory of *characteristics* is due to Prof. SYLVESTER.

It is further to be observed that the metric properties of figures in plane geometry depend upon the circular points at infinity, which I denote by i, j; and those of figures in spherical geometry upon the imaginary circle at infinity, which I denote by O_2 or o_2 according as it is given by points or tangents. The points i, j in the one case, and the circle O_2 in the other, have received the name of "the Absolute" from Prof. CAYLEY, to whom this theory is due *.

FORMULÆ FOR A PLANE CONIC.

Expressions are obtained below for the *distance* of a point from a conic given tangentially, and for the *distance* of a line from a conic given by its points. Two different geometrical definitions are obtained for each of these; their ratio is a quantity which I have called the *distance of the curve from the absolute.*

The asymptotes of the conic are denoted by P, Q; a pair of foci, viz. either the two real or the two imaginary foci, are denoted by p, q; the conic is called C_2 or c_2.

DISTANCE OF THE POINT a FROM c_2.

Let any straight line B be drawn through the point a, meeting the conic in l, m; let also the tangents from a to the conic be L, M.

* [$\overline{ab}$ = distance between points a and b, ab = line joining a and b.]

First Dist. $a, c_2 = \sin^2 LM \,.\, ap^2 \,.\, aq^2$

$$= \frac{a^2 C_2 \,.\, (aij)^2}{\overline{ai}^2 c_2 \,.\, \overline{aj}^2 c_2} \cdot \frac{\overline{ai}^2 c_2 \,.\, \overline{aj}^2 c_2}{(aij)^4 \,.\, (\overline{ij}^2 c_2)^2} = \frac{a^2 C_2}{(aij)^2 \,.\, (\overline{ij}^2 c_2)^2}.$$

Second Dist. $a, c_2 = al \,.\, am \,.\, \sin BP \,.\, \sin BQ$

$$= \frac{a^2 C_2 \,.\, (Bi \,.\, Bj)}{(aij)^2 \,.\, (B\overline{ij})^2 C_2} \cdot \frac{(B\overline{ij})^2 C_2}{(Bi \,.\, Bj) \,.\, \surd(i^2 C_2 \,.\, j^2 C_2)}$$

$$= \frac{a^2 C_2}{(aij)^2 \,.\, \surd(i^2 C_2 \,.\, j^2 C_2)}.$$

The ratio of these two is

$$\frac{\sin^2 LM \,.\, ap^2 \,.\, aq^2}{al \,.\, am \,.\, \sin BP \,.\, \sin BQ} = \frac{\surd(i^2 C_2 \,.\, j^2 C_2)}{(\overline{ij}^2 c_2)^2} = pq^2.$$

DISTANCE OF THE LINE A FROM C_2.

Let any point b be taken on the line A, the tangents from b to the conic being LM; also let A meet the conic in the points l, m.

First Dist. $A, C_2 = lm^2 \,.\, \sin^2 AP \,.\, \sin^2 AQ$

$$= \frac{A^2 c_2 \,.\, Ai \,.\, Aj}{(A\overline{ij})^2 C_2} \cdot \frac{(A\overline{ij})^2 C_2}{(Ai \,.\, Aj)^2 \,.\, i^2 C_2 \,.\, j^2 C_2}$$

$$= \frac{A^2 c_2}{Ai \,.\, Aj \,.\, i^2 C_2 \,.\, j^2 C_2}.$$

Second Dist. $A, C_2 = \sin AL \sin AM \,.\, bp \,.\, bq$

$$= \frac{A^2 c_2 \,.\, (bij)^2}{Ai \,.\, Aj \,\surd(\overline{bi}^2 c_2 \,.\, \overline{bj}^2 c_2)} \cdot \frac{\surd(\overline{bi}^2 c_2 \,.\, \overline{bj}^2 c_2)}{(bij)^2 \,.\, \overline{ij}^2 c_2}$$

$$= \frac{A^2 c_2}{Ai \,.\, Aj \,.\, \overline{ij}^2 c_2}.$$

The ratio of these two is

$$\frac{lm^2 \,.\, \sin^2 AP \,.\, \sin^2 AQ}{\sin AL \sin AM \,.\, bp \,.\, bq} = \frac{\overline{ij}^2 c_2}{i^2 C_2 \,.\, j^2 C_2} = \sin^2 PQ.$$

In the case of a sphero-conic we obtain analogous expressions for the distance of a point or line (great circle) from the conic, but the value depends on the pair of foci or cyclic planes selected; the ratio of such different values is however the same for all points and lines. Moreover, the ratio of the two distances of a point or of a line is a quantity independent of the point or line, but I have as yet obtained no geometrical definition of it. For this reason I have not treated separately the formulæ for a sphero-conic, which are of course like the preceding included in the general formulæ of curves*.

I.

All magnitudes which are concerned in plane geometry may be expressed in terms of three, which on this account are of primary importance. These are the distance of two points, the distance of a point from a line, and the sine of the angle between two lines. We obtain the most simple expressions for these magnitudes by employing rectangular Cartesian coordinates for the points, and the coordinates of Dr Booth for the lines; but it is convenient from the first to make these expressions homogeneous by the introduction of a third coordinate which may be put $=1$ or -1 in the two cases respectively. Thus if a_1, a_2, a_3 are the coordinates of the point a, $\frac{a_2}{a_3}$ and $\frac{a_1}{a_3}$ are its distances from the axes; and if A_1, A_2, A_3 are the coordinates of a line A, $-\frac{A_3}{A_1}$ and $-\frac{A_3}{A_2}$ are the intercepts it cuts off from the axes. This being so, the expressions for our primary magnitudes are

$$\text{Dist. } ab = \frac{\sqrt{\{(a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2\}}}{a_3b_3},$$

$$\text{Dist. } aB = \frac{a_1B_1 + a_2B_2 + a_3B_3}{a_3\sqrt{(B_1^2 + B_2^2)}},$$

$$\sin AB = \frac{A_1B_2 - A_2B_1}{\sqrt{(A_1^2 + A_2^2)}\,\sqrt{(B_1^2 + B_2^2)}},$$

* [Cf. however V., p. 152 *infra*, of this paper.]

and these are clearly not invariant in regard to the points and lines. Let us now ask what is the locus of the points b which are at zero distance from a. We find that it consists of the two straight lines

$$a_3b_1 + \iota\, a_3b_2 - (a_1 + \iota\, a_2)\, b_3 = 0,$$
$$a_3b_1 - \iota\, a_3b_2 - (a_1 - \iota\, a_2)\, b_3 = 0.$$

Each of these lines passes through the point a; thus we learn that *the lines of null-length are straight lines, and two of them pass through every point of the plane.*

If the point a moves about, each of the lines of null-length remains parallel to the same direction, or, which is the same thing, passes through a fixed point at an infinite distance. Let these two points be called i, j; their coordinates may be taken to be

$$i_1 : i_2 : i_3 = \frac{1}{\sqrt{2}} : \frac{\iota}{\sqrt{2}} : 0,$$

$$j_1 : j_2 : j_3 = \frac{\iota}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 0,$$

so that the line ij has coordinates $(0, 0, 1)$. Then we have

$$abi = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & \iota & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{(a_2b_3 - a_3b_2) - \iota\,(a_1b_3 - a_3b_1)\},$$

$$abj = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \iota & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{\iota\,(a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1)\},$$

$$aij = a_3, \quad bij = b_3,$$

$$abi\,.\,abj = \frac{\iota}{2}\,.\,\{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2\}$$

and therefore

$$\kappa\,.\,\text{Dist.}\ ab = \frac{\sqrt{(abi\,.\,abj)}}{aij\,.\,bij}, \quad \text{where } \kappa^2 = \frac{\iota}{2}.$$

Thus we learn that *all the lines of null-length pass through one or other of the points i, j;* and *the distance ab may be expressed in terms of the invariants of a, b, i, j* to a numerical

factor *près*. This factor κ is the same for all distances, depending only upon the absolute value given to the coordinates of i, j. To reduce κ to the value unity, we have only to multiply these coordinates throughout by $\kappa^{-\frac{1}{3}}$.

Similar expressions may now be found for the other two primary magnitudes. We have in fact

$$Ai = \frac{1}{\sqrt{2}}(A_1 + \iota A_2), \quad Aj = \frac{1}{\sqrt{2}}(\iota \,.\, A_1 + A_2),$$

$$Ai \,.\, Aj = \frac{\iota}{2} \,.\, (A_1^2 + A_2^2),$$

and thence

$$\left[\frac{1}{\kappa}\right] \text{Dist. } aB = \frac{aB}{a\overline{ij} \,.\, \sqrt{(Bi \,.\, Bj)}}$$

$$-2\,\iota \,.\, \sin AB = \frac{AB\overline{ij}}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)}}.$$

(The last expression may be simplified as follows:

We have

$$AB\overline{ij} = Ai \,.\, Bj - Aj \,.\, Bi$$

by a well-known theorem of determinants; but also

$$\begin{aligned} &Ai \,.\, Bj + Aj \,.\, Bi \\ &\quad = \tfrac{1}{2}\{(A_1 + \iota A_2)(\iota B_1 + B_2) + (\iota A_1 + A_2)(B_1 + \iota B_2)\} \\ &\quad = \iota\,(A_1 B_1 + A_2 B_2) \end{aligned}$$

and therefore

$$2\cos AB = 2\frac{A_1 B_1 + A_2 B_2}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}} = \frac{Ai \,.\, Bj + Aj \,.\, Bi}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)}}.$$

If therefore we write θ for the angle AB, we have

$$\cos\theta + \iota\sin\theta = \epsilon^{\theta\iota} = \frac{Aj \,.\, Bi}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)}}$$

$$\cos\theta - \iota\sin\theta = \epsilon^{-\theta\iota} = \frac{Ai \,.\, Bj}{\sqrt{(Ai \,.\, Aj \,.\, Bi \,.\, Bj)}},$$

and thence

$$\epsilon^{2\theta\iota} = \frac{Aj \,.\, Bi}{Ai \,.\, Bj}.$$

This last is an anharmonic ratio in which the lines AB divide the segment ij, and so is an absolute invariant.)

By analogy and for convenience of expression we shall call the sine of the angle between two lines the *distance* of the lines; we may then derive from these formulæ the following theorems:—

If a line pass through either of the points i, j, it is at an infinite distance from all other lines and points, and the distance between any two points on it is zero.

If a point lie upon the line ij, it is at an infinite distance from all other lines and points, and the distance between any two lines through it is zero.

II.

A conic may be given as a curve of the second order C_2 or as a curve of the second class c_2; but either of these expressions may be derived from the other by means of the formula

$$\begin{vmatrix} axC_2, & bxC_2 \\ ayC_2, & byC_2 \end{vmatrix} = 2\overline{ab}\;\overline{xy}\;c_2$$

and its reciprocal

$$\begin{vmatrix} AXc_2, & BXc_2 \\ AYc_2, & BYc_2 \end{vmatrix} = 2\overline{AB}\;\overline{XY}\;C_2.$$

Namely, if we make a, b identical with x, y, and A, B with X, Y, we have

$$2\overline{xy}^2c_2 = x^2C_2 \cdot y^2C_2 - (xyC_2)^2$$

and $$2\overline{XY}^2C_2 = X^2c_2 \,.\, Y^2c_2 - (XYc_2)^2.$$

The discriminant C_2c_2 may be given in a similar manner. Namely, we have

$$3\overline{xC_2}\;\overline{yC_2}\;\overline{zC_2} = 8xyz \,.\, C_2c_2.$$

The relations of a conic to the points $i\,j$ are most conveniently expressed in terms of the asymptotes and the foci. The asymptotes P, Q are tangents at the points where the conic

is met by the line ij. The equation to the pair of tangents at the points where a line X meets the conic C_2 must be of the form

$$C_2 - \lambda X^2 = 0.$$

Equating to zero the discriminant of this, we find that

$$C_2 c_2 - 3\lambda X^2 c_2 = 0,$$

whence the equation to the tangents is

$$3C_2 \, . \, X^2 c_2 - X^2 \, . \, C_2 c_2 = 0.$$

Substituting herein for X the line ij, we obtain for the asymptotes the expression

$$P \, . \, Q = 3C_2 \, . \, \overline{ij}^2 c_2 - \overline{ij}^2 \, . \, C_2 c_2.$$

Thus the product $P \, . \, Q$ is of the third order in the coefficients of the conic, whether it is given by its point- or by its line-equation.

To determine the angle between the asymptotes, we observe that the point PQ is the centre of the conic, which is the pole of the line ij, and is therefore represented by $\overline{ij}c_2$. Consequently $PQij$ must be proportional to some power of $\overline{ij}^2 c_2$. If we regard C_2 as initially given, we shall in fact have

$$(PQij)^2 = \kappa \, (\overline{ij}^2 c_2)^3,$$

each side being of the sixth order in the coefficients. If however c_2 is initially given, the formula becomes

$$(PQij)^2 = \kappa' \, (\overline{ij}^2 c_2)^3 \, . \, C_2 c_2.$$

Next we find by direct substitution

$$Pi \, . \, Qi = 3i^2 C_2 \, . \, \overline{ij}^2 c_2,$$

$$Pj \, . \, Qj = 3j^2 C_2 \, . \, \overline{ij}^2 c_2,$$

and therefore

$$\sin^2 PQ = \frac{\overline{ij}^2 c_2}{i^2 C_2 \, . \, j^2 C_2} \text{ to a factor } \textit{près}.$$

But we have also

$$\overline{ij}^2 c_2 = i^2 C_2 \, . \, j^2 C_2 - (ijC_2)^2;$$

now it is clear that $\dot{i}\dot{j}C_2$ vanishes when the points ij are conjugate in regard to the conic, or when the asymptotes are at right angles, that is, when $\sin PQ = 1$. It follows therefore that the factor in the last equation is unity, and that

$$\cos PQ = \frac{\dot{i}\dot{j}C_2}{\sqrt{(\dot{i}^2C_2 \,.\, \dot{j}^2C_2)}}.$$

We find in this way that the κ of our formula is $= -36$, so that we may write

$$(PQij)^2 = -36\,(\overline{ij}^2c_2)^3.$$

This constant might also have been determined by means of the particular case in which the conic breaks up into two straight lines, in which case these lines are themselves the asymptotes. If $C_2 = X \,.\, Y$, $-4c_2 = \overline{XY}^2$, and $-4\overline{ij}^2c_2 = \overline{XYij}^2$; by these formulæ the values just obtained may be compared with the known values of $\sin XY$, $\cos XY$.

If the conic c_2 touches the four lines joining p, q to i, j, we must have

$$\kappa p \,.\, q = c_2 + \lambda \,.\, i \,.\, j.$$

But the equation

$$0 = \text{Disct.}\,(c_2 + \lambda \,.\, i \,.\, j) = C_2c_2 + \tfrac{3\lambda}{2}\,\dot{i}\dot{j}C_2 - \tfrac{3}{4}\lambda^2\overline{ij}^2c_2$$

gives two values for λ, one belonging to $p \,.\, q$ and the other to $p' \,.\, q'$. Hence, eliminating λ, we may write

$$p \,.\, q \,.\, p' \,.\, q' = C_2c_2 \,.\, i^2 \,.\, j^2 - \tfrac{3}{2}\dot{i}\dot{j}C_2 \,.\, i \,.\, j \,.\, c_2 - \tfrac{3}{4}\overline{ij}^2c_2 \,.\, c_2^2,$$

a result of the third order in the coefficients of c_2.

If the conic touch the line ij, that is, if $\overline{ij}^2c_2 = 0$, one of the foci, say q, coincides with the point of contact, and two others, p' and q', coincide with the points i and j respectively; we have therefore in this case $qij = 0$, $p'q'i = 0$, $p'q'j = 0$. If the conic pass through the point i, which happens when $\dot{i}^2C_2 = 0$, the foci coincide two and two, p with q', suppose, and q with p'; we have then $pqi = 0$, $p'q'i = 0$. Hence the product

$$pqi \,.\, pqj \,.\, p'q'i \,.\, p'q'j$$

will vanish in three cases; (1) when $\dot{i}^2C_2 = 0$, (2) when $\dot{j}^2C_2 = 0$,

(3) when $\overline{\ddot{ij}}^2 c_2 = 0$; and it is easy to see that it cannot vanish in any other case. Consequently we must have

$$pq\dot{i} \cdot pq\dot{j} \cdot p'q'\dot{i} \cdot p'q'\dot{j} = \kappa\, (\dot{i}^2 C_2 \cdot \dot{j}^2 C_2)^x\, (\overline{\ddot{ij}}^2 c_2)^y,$$

where a comparison of dimensions gives us the equations

$$4x + y = 6,\ \ 2x + 2y = 6,\ \ x = 1,\ \ y = 2.$$

We have also by direct substitution

$$p\ddot{ij} \cdot q\ddot{ij} \cdot p'\ddot{ij} \cdot q'\ddot{ij} = \overline{\ddot{ij}}^4\, (p \cdot q \cdot p' \cdot q') = -\tfrac{3}{4}\, (\overline{\ddot{ij}}^2 c_2)^3,$$

and hence the following expression for the product of the squares of the distances pq, $p'q'$,

$$\overline{pq}^2 \cdot \overline{p'q'}^2 = \kappa' \frac{(\dot{i}^2 C_2 \cdot \dot{j}^2 C_2)^x\, (\overline{\ddot{ij}}^2 c_2)^y}{(\overline{\ddot{ij}}^2\, c_2)^6}.$$

Since $\overline{pq}^2 + \overline{p'q'}^2 = 0$, this product is the same thing as $-\overline{pq}^4$. But we get another equation between x and y by supposing the conic c_2 to break up into two points u, v, which are then themselves the foci. In that case $-4C_2 = \overline{uv}^2$, $\overline{\ddot{ij}}^2 c_2 = u\ddot{ij} \cdot v\ddot{ij}$, and the expression for the fourth power of the distance is

$$\overline{uv}^4 = \kappa^4 \frac{(uv\dot{i} \cdot uv\dot{j})^2}{(u\ddot{ij} \cdot v\ddot{ij})^4} = \frac{\dot{i}^2 C_2 \cdot \dot{j}^2 C_2}{(\overline{\ddot{ij}}^2 c_2)^4},$$

whence $x = 1$, $y = 2$, and the general formula is

$$\overline{pq}^2 = \frac{\sqrt{(\dot{i}^2 C_2 \cdot \dot{j}^2 C_2)}}{(\overline{\ddot{ij}}^2 c_2)^2}.$$

In this it is clear that c_2 is primarily given; if C_2 is given, the process of calculating it back from the coefficients of c_2 introduces the factor $C_2 c_2$, and we have

$$\overline{pq}^2 = \frac{\sqrt{(\dot{i}^2 C_2 \cdot \dot{j}^2 C_2)}\, C_2 c_2}{(\overline{\ddot{ij}}^2 c_2)^2}.$$

With the aid of this formula and the angle between the asymptotes we may now determine the axes of the conic.

If h and k be the axes, we have

$$\overline{pq}^2 = h^2 - k^2,$$

$$\cos(PQ) = \frac{h^2 + k^2}{h^2 - k^2},$$

and therefore

$$h^2+k^2=\overline{pq}^2\cos PQ=\frac{\sqrt{(i^2C_2\,.\,j^2C_2)}\,.\,C_2c_2}{(\overline{ij}^2c_2)^2}\,.\,\frac{ijC_2}{\sqrt{(i^2C_2\,.\,j^2C_2)}}$$

$$=\frac{ij\,C_2\,.\,C_2c_2}{(\overline{ij}^2c_2)^2}.$$

Again we have

$$\sin^2 PQ=1-\left(\frac{h^2+k^2}{h^2-k^2}\right)^2=-\frac{4h^2k^2}{(h^2-k^2)^2},$$

and therefore

$$-4h^2k^2=\overline{pq}^4\sin^2 PQ$$

$$=\frac{i^2\,C_2.\,j^2\,C_2.\,(C_2c_2)^2}{(\overline{ij}^2c_2)^4}\,.\,\frac{\overline{ij}^2c_2}{i^2\,C_2\,.\,j^2C_2}$$

$$=\frac{(C_2c_2)^2}{(\overline{ij}^2c_2)^3}.$$

These last formulæ are of course the well-known ones.

III.

We go on to consider the relations between a point and a conic; and in particular to determine the angle between the tangents from the point to the conic (fig. 12).

The tangents LM from x to the conic C_2 are given by the equation

$$LM=x^2C_2\,.\,C_2-(xC_2)^2,$$

we have also

$$-4\sin^2 LM=\frac{(LMij)^2}{Li\,.\,Lj\,.\,Mi\,.\,Mj}.$$

The numerator of this fraction vanishes when the intersection of L and M is on the line ij. If these tangents are distinct, the intersection is x; if they coincide, that is, if x is on the conic, or if the conic breaks up, the intersection is indeterminate. Hence $(LMij)^2$ must vanish whenever xij or x^2C_2, or C_2c_2, vanishes; and in no other case. But $(LMij)^2$ is of the fourth order in the coefficients of x and of the conic; therefore

$$(LMij)^2=\kappa x^2C_2\,.\,(xij)^2\,.\,C_2c_2.$$

The denominator may be expressed by direct operation on LM; we should find in fact

$$Li\,.\,Lj\,.\,Mi\,.\,Mj=\{x^2C_2\,.\,i^2C_2-(xiC_2)^2\}\,\{x^2C_2\,.\,j^2C_2-(xjC_2)^2\}$$
$$=\overline{xi}^2c_2\,.\,\overline{xj}^2c_2,$$

and thus finally

$$\sin^2 LM=\kappa'\,\frac{x^2C_2\,.\,(xij)^2\,.\,C_2c_2}{\overline{xi}^2c_2\,.\,\overline{xj}^2c_2}.$$

But the denominator may be put into another form which is more useful. Let us assume that the absolute values of the coefficients of p, q, p', q' are so chosen that $pi=p'i$, $qi=q'i$, $pj=q'j$, $qj=p'j$. We have by direct substitution

$$pix\,.\,qix\,.\,p'ix\,.\,q'ix=-\tfrac{3}{4}\overline{ij}^2c_2\,.\,(\overline{xi}^2c_2)^2,$$

or
$$(pix\,.\,qix)^2=-\tfrac{3}{4}\overline{ij}^2c_2\,(\overline{xi}^2c_2)^2.$$

Hence we have

$$xpi\,.\,xqi\,.\,xpj\,.\,xqj=-\tfrac{3}{4}\overline{ij}^2c_2\,.\,\overline{xi}^2c_2\,.\,\overline{xj}^2c_2,$$

and consequently, since

$$\overline{xp}^2\,.\,\overline{xq}^2=\kappa\,\frac{xpi\,.\,xqi\,.\,xpj\,.\,xqj}{(xij)^4\,(pij\,.\,qij)^2},$$

and
$$(pij\,.\,qij)^2=-\tfrac{3}{4}\,(\overline{ij}^2c_2)^3,$$

it follows that

$$xp^2\,.\,xq^2=\kappa\,\frac{\overline{xi}^2c_2\,.\,\overline{xj}^2c_2}{(xij)^4\,(\overline{ij}^2c_2)^2}=xp'^2\,.\,xq'^2.$$

Combining these two results, we find

$$xp^2\,.\,xq^2\,.\,\sin^2 LM=\kappa''\,\frac{x^2C_2\,.\,C_2c_2}{(xij)^2\,.\,(\overline{ij}^2c_2)^2},$$

a result which may be further simplified by help of the formula for the distance between the foci. Namely, we have

$$\frac{xp^2\,.\,xq^2\,.\,\sin^2 LM}{pq^2}=\frac{x^2C_2}{(xij)^2\sqrt{(i^2C_2\,.\,j^2C_2)}}\ \text{to a factor } \textit{près}.$$

We shall call this quantity the *distance** of the point from the conic; it vanishes when the point is on the conic, and is infinite if either the point or the conic has contact with the absolute.

* [Second distance; cf. p. 133.]

It is to be noted that the conic is given as a curve of the second order, in the form C_2; if it were given in the form c_2, the formula preceding would become

$$xp^2 \,.\, xq^2 \,.\, \sin^2 LM = \kappa'' \frac{x^2 C_2}{(xij)^2 \,.\, (\overline{ij}^2 c_2)^2},*$$

and this quantity might be taken as the distance of the point from a conic given as a curve of the second class. If the conic breaks up into two lines, the former expression becomes the product of the perpendicular distances of the point from the two lines; if the conic breaks up into two points, the latter expression becomes four times the squared area which they include with the given point. The former expression, however, in which the conic is given as of the second order, admits of a further interpretation, to which we now proceed.

Through the point x (fig. 13) let a line X be drawn, meeting the conic in l, m. For the product of the segments xl, xm we have the formula

$$xl \,.\, xm = \kappa^2 \frac{\sqrt{(xli \,.\, xlj \,.\, xmi \,.\, xmj)}}{(xij)^2 \,.\, lij \,.\, mij}.$$

The numerator of this expression clearly vanishes if x is on the conic, when one of the lines xl, xm becomes indeterminate; otherwise each of these lines is simply the line X. We must have therefore

$$xli \,.\, xlj \,.\, xmi \,.\, xmj = (x^2 C_2 \,.\, Xi \,.\, Xj)^2 \text{ to a factor } \textit{près}.$$

For the denominator we observe that if U is any arbitrary line,

$$lU \,.\, mU = \overline{XU}^2 C_2,$$

and taking the co-ordinates of U for the current line co-ordinates, this gives the tangential equation to l, m. From this we get

$$lij \,.\, mij = \overline{Xij}^2 C_2,$$

and our expression is therefore transformed into

$$xl \,.\, xm = \frac{x^2 C_2 \,.\, Xi \,.\, Xj}{(xij)^2 \,.\, \overline{Xij}^2 C_2}.$$

* [First distance; cf. p. 133.]

The denominator of this will clearly vanish if X is parallel to one or other of the asymptotes P, Q, or if $XPij \,.\, XQij = 0$. Let us, therefore, now seek the product of the distances of X from the asymptotes.

We have

$$-4\sin XP \,.\, \sin XQ = \frac{XPij \,.\, XQij}{(Xi \,.\, Xj)\sqrt{(Pi \,.\, Pj \,.\, Qi \,.\, Qj)}}.$$

But we find by operating with $(Xij)^2$ upon $P \,.\, Q$ that

$$XPij \,.\, XQij = 3\overline{Xij}^2 C_2 \,.\, \overline{ij}^2 c_2,$$

and moreover

$$Pi \,.\, Pj \,.\, Qi \,.\, Qj = 9 i^2 C_2 \,.\, j^2 C_2 \,.\, (\overline{ij}^2 c_2)^2.$$

Hence we have

$$-4\sin XP \,.\, \sin XQ = \frac{\overline{Xij}^2 C_2}{3Xi \,.\, Xj\sqrt{(i^2 C_2 \,.\, j^2 C_2)}}.$$

Multiplying this by $xl \,.\, xm$, the line X disappears from the result and we find

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{x^2 C_2}{(xij)^2\sqrt{(i^2 C_2 \,.\, j^2 C_2)}} \text{ to a factor.}$$

But the expression on the right is the same that we previously obtained for the distance of the point from the conic. Hence the quantity $xl \,.\, xm \,.\, \sin XP \sin XQ$ must be proportional to the quantity $\dfrac{xp^2 \,.\, xq^2}{pq^2}\sin^2 LM$. To determine the constant factor, suppose the conic to break up into a pair of points; these may be taken to be the points p, q, and the asymptotes will both coincide with the line pq (fig. 14). Here it is clear that $xl \,.\, xm \,.\, \sin XP \sin XQ = xl^2 \,.\, \sin^2 XP =$ squared distance of x from line pq; while $\dfrac{xp \,.\, xq \sin pxq}{pq} = \dfrac{\text{twice area } pxq}{pq} =$ distance of x from pq. Thus the factor is unity, and we have always

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{xp^2 \,.\, xq^2}{pq^2}\sin^2 LM.*$$

There is no difficulty in investigating the correlative formulæ. First to find the length of the chord cut off a line X

* [Second distance, p. 133.]

by the conic; let lm be the points of intersection, then we have

$$lm^2 = \left[\frac{1}{\kappa^2}\right] \frac{lmi \, . \, lmj}{(lij)^2 \, . \, (mij)^2}.$$

The numerator vanishes if the line X pass through either of the points ij, or if l, m coincide, that is to say, if X touch the conic c_2, or if the conic break up into a pair of points. Moreover, since we have

$$l \, . \, m = X^2 c_2 \, . \, c_2 - (Xc_2)^2,$$

the expression $(lmi)^2$ must be of the fourth order in the coefficients of X and of the conic, and therefore

$$(lmi)^2 = (Xi)^2 \, . \, X^2 c_2 \, . \, C_2 c_2 \text{ to a factor.}$$

Again, we have by direct substitution

$$lij \, . \, mij = X^2 c_2 \, . \, \overline{ij}^2 c_2 - (X\overline{ij}c_2)^2 = \overline{Xij}^2 C_2,$$

and thence

$$lm^2 = \frac{Xi \, . \, Xj \, . \, X^2 c_2 \, . \, C_2 c_2}{(\overline{Xij}^2 C_2)^2} \text{ to a factor.}$$

But we have already found that

$$\sin XP \, . \, \sin XQ = \frac{\overline{Xij}^2 C_2}{Xi \, . \, Xj \, \sqrt{(i^2 C_2 \, . \, j^2 C_2)}},$$

consequently

$$\text{[i]} \qquad lm^2 \, . \, \sin^2 XP \, . \, \sin^2 XQ = \frac{X^2 c_2 \, . \, C_2 c_2}{Xi \, . \, Xj \, . \, i^2 C_2 \, . \, j^2 C_2}.^*$$

Lastly, c_2 being primarily given, we have

$$\sin^2 PQ = \frac{\overline{ij}^2 c_2 \, . \, C_2 c_2}{i^2 C_2 \, . \, j^2 C_2},$$

and so

$$\text{[ii]} \qquad lm^2 \frac{\sin^2 XP \, . \, \sin^2 XQ}{\sin^2 PQ} = \frac{X^2 c_2}{Xi \, . \, Xj \, . \, \overline{ij}^2 c_2}.$$

Let now x be a variable point on the line X (fig. 15), and draw the tangents L, M from x to the conic. Then

$$\sin XL \, . \, \sin XM = \frac{XLij \, . \, XMij}{Xi \, . \, Xj \, \sqrt{(Li \, . \, Lj \, . \, Mi \, . \, Mj)}}.$$

* [First distance of line X from Conic C_2, cf. p. 133.]

In this fraction the numerator vanishes when X touches the conic, and when x is on the line ij. It must be of the first order in the coefficients of c_2 and of the second in those of x. Hence we have, to a factor *près*,

$$XLij \,.\, XMij = X^2c_2 \,.\, (xij)^2.$$

Moreover by a previous formula

$$Li \,.\, Lj \,.\, Mi \,.\, Mj = \overline{xi}^2c_2 \,.\, \overline{xj}^2c_2.$$

Thus

$$\sin XL \,.\, \sin XM = \frac{X^2c_2 \,.\, (xij)^2}{Xi \,.\, Xj\, \sqrt{(\overline{xi}^2c_2 \,.\, \overline{xj}^2c_2)}}.$$

But, also by a previous formula,

$$xp \,.\, xq = \frac{\sqrt{(\overline{xi}^2c_2 \,.\, \overline{xj}^2c_2)}}{(xij)^2 \,.\, \overline{ij}^2c_2};$$

therefore

$$\text{[iii]} \quad xp \,.\, xq \sin XL \sin XM = \frac{X^2c_2}{Xi \,.\, Xj \,.\, \overline{ij}^2c_2}$$

$$= lm^2\frac{\sin^2 XP \,.\, \sin^2 XQ}{\sin^2 PQ}.\text{*}$$

To verify this, suppose the conic to break up into two lines P, Q (fig. 16), which are themselves the asymptotes; the two foci will then coincide at the point PQ, and the tangents LM will pass through the same point. Then $xp \,.\, xq \,.\, \sin XL \,.\, \sin XM = xp^2 \sin^2 XL =$ squared perpendicular from PQ on X. And

$$\frac{lm}{\sin PQ} \,.\, \sin XP \sin XQ = lp \,.\, mp \,.\, \frac{lm}{\sin PQ} \,.\, \frac{\sin XP}{lp} \,.\, \frac{\sin XQ}{mp}$$

$$= \frac{lp \,.\, mp \,.\, \sin PQ}{lm} = \frac{2 \text{ area } lmp}{lm}$$

$$= \text{perpendicular from } PQ \text{ on } X.$$

This quantity†, of which three expressions are given in our last equation, may be called the *distance* of the line X from the conic c_2.

* [Second distance of line X from Conic c_2.]

† [i.e. iii. supra.]

IV.

We now consider a curve C_n of the n^{th} order, which may also be given as a curve $c_{n(n-1)}$ of class $n(n-1)$. To this number $n(n-1)$ there are no reductions in virtue of any singularities that C_n may have; its nodes will enter as double factors and its cusps as triple factors in $c_{n(n-1)}$. This being so, we may write

$$\text{Disct.}_\lambda\,(x+\lambda y)^n C_n = \frac{\{\underline{|n}\}^{2(n-1)}}{\underline{|n(n-1)}}\,.\,\overline{xy}^{n(n-1)}c_{n(n-1)}.$$

Conversely, if we are given a curve c_m of class m, this is also a curve $C_{m(m-1)}$ of order $m(m-1)$; but each double tangent is now a double factor and each stationary tangent a triple factor in $C_{m(m-1)}$. We may gain shortness without introducing confusion, if when C_n is primarily given, we denote $n(n-1)$ by m; and if when c_m is primarily given, we denote $m(m-1)$ by n. Thus in the latter case we shall have

$$\text{Disct.}_\lambda\,(X+\lambda Y)^m c_m = \frac{\{\underline{|m}\}^{2(m-1)}}{\underline{|n}}\,.\,\overline{XY}^n C_n.$$

The curve c_m has m^2 foci, which are the intersections of the m tangents from the point i with the m tangents from the point j. But for every tangent from i there is *one* tangent from j which meets it in a real point; thus there are m real foci. The foci may be arranged in various ways into m sets, such that no two points of the same set are collinear with i or j. By joining the points of any one set with i and j we obtain all the tangents. The real foci constitute a set; and from them we may pass to any other set by successively substituting for each pair pq their *antipoints* $p'q'$, that is to say, the remaining intersections of pi, pj with qi, qj. Now for every such substitution the following equations hold good:

$$p'q'^2 = -pq^2,$$
$$xp'\,.\,xq' = xp\,.\,xq,$$

where x is any point in the plane. Hence if we form the product Πpq^2 of the squared distances of the real foci from one

another, this product can differ only in sign from a similar product formed with any other set; and the same is true of Πxp. The number of possible sets is clearly the same as the number of terms in a determinant of the m^{th} order, viz.: $\underline{|m}$. If we then raise Πpq^2 to the power $\underline{|m}$, we must have some power of $\Pi\Pi pq^2$, the product of the squared distances of all pairs of foci from one another, excepting of course those pairs which are collinear with i or j. But there are $m^2(m-1)^2$ such pairs; thus if k is the power in question

$$\underline{|m}\,.\,m(m-1) = km^2(m-1)^2,$$

or
$$k = \frac{\underline{|m}}{m(m-1)},$$

whence
$$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2.$$

In a similar way we find

$$(\Pi xp)^m = \Pi\Pi xp.$$

Now

$$\Pi\Pi pq^2 = \Pi\Pi\,\frac{pqi\,.\,pqj}{p\dot{i}\dot{j}^2\,.\,q\dot{i}\dot{j}^2} = \frac{\Pi\Pi pqi\,.\,pqj}{(\Pi p\dot{i}\dot{j})^{4m^2(m-1)^2}} \text{ [to factor } pr\grave{e}s\text{]}.$$

To obtain the equation of the foci we may proceed as follows. The equation of the tangents from i, j to c_m is $I_m = 0$, $J_m = 0$, where
$$x^m I_m = \overline{jx}^m c_m,\quad x^m J_m = \overline{jx}^m c_m,$$

and if we then form the equation of the points of intersection of I_m, J_m (which may be written $(XIJ)_{mm} = 0$) it is of the order m in each of them and must contain the equation of the foci. But also it must contain the factor $\overline{ij}^m c_m$, and this to the degree $\overline{m-1}$; for the factor is involved as the condition that I_m should pass through a double point of J_m. Thus the equation of the foci is of the order $2m - \overline{m-1} = m+1$ in c_m and $m^2 - m(m-1) = m$ in i and j. In fact one term in it is a numerical multiple of $\overline{ij}^m c_m\,.\,(c_m)^m$.

As in the case of the conic, the product $\Pi\Pi pqi\,.\,pqj$ will vanish when $i^n C_n = 0$, or when $j^n C_n = 0$, or when $\overline{ij}^m c_m = 0$. Thus
$$\Pi\Pi pqi\,.\,pqj = \kappa\,(i^n C_n\,.\,j^n C_n)^x\,(\overline{ij}^m c_m)^y.$$

Now the left-hand side is of order $2(m-1)^2$ in the foci and besides of order $\frac{1}{2}m^2(m-1)^2$ in i and j; that is, of order $2(m+1)(m-1)^2$ in c_m, and $2m(m-1)^2+\frac{1}{2}m^2(m-1)^2=\frac{1}{2}m(m+4)(m-1)^2$ in i and j. First regard c_m as given; then C_n is of order $2(m-1)$ in the coefficient of c_m, and we have

$$2(m+1)(m-1)^2 = 4x(m-1)+y,$$

$$\tfrac{1}{2}m(m+4)(m-1)^2 = xm(m-1)+my,$$

whence $\quad x=\frac{1}{2}m(m-1),\ y=2(m-1)^2$, and consequently

$$\Pi\Pi pqi\,.\,pqj = \kappa\,(i^nC_n\,.\,j^nC_n)^{\frac{1}{2}m(m-1)}\,(\overline{ij}^mc_m)^{2(m-1)^2}.$$

Next we have

$$\Pi\Pi pij = (\overline{ij}^mc_m)^{m+1} \text{ to a factor,}$$

and therefore

$$\Pi\Pi pq^2 = \frac{\Pi\Pi pqi\,.\,pqj}{(\Pi\Pi pij)^{2(m-1)^2}} = \kappa\,\frac{(i^nC_n\,.\,j^nC_n)^{\frac{1}{2}m(m-1)}}{(ij^mc_m)^{2m(m-1)^2}},$$

but $$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2;$$

therefore $$\Pi pq^2 = \frac{(i^nC_n\,.\,j^nC_n)^{\frac{1}{2}}}{(ij^mc_m)^{2(m-1)}}.$$

We now proceed to determine Πxp, where x is any point in the plane. We have

$$\Pi\Pi xp^2 = \frac{\Pi\Pi xpi\,.\,xpj}{(xij)^{2m^2}\,\Pi\Pi(pij)^2} = \frac{(\overline{xi}^mc_m\,.\,\overline{xj}^mc_m)^m\,(ij^mc_m)^2}{(xij)^{2m^2}\,(ij^mc_m)^{2(m+1)}}$$

$$= \frac{(\overline{xi}^mc_m\,.\,\overline{xj}^mc_m)^m}{(xij)^{2m^2}\,(ij^mc_m)^{2m}},$$

and therefore

$$\Pi xp^2 = \frac{\overline{xi}^mc_m\,.\,\overline{xj}^mc_m}{(xij)^{2m}\,(\overline{ij}^mc_m)^2}.$$

The curve C_n has n asymptotes, which are the tangents at the points where it is met by the line ij. If a point x lie on one of the asymptotes, its first polar xC_n must meet C_n on the line ij. The condition for this is of the order n in xC_n, $n-1$ in C_n, and $n(n-1)$ in ij; that is, of the order n in x, $2n-1$ in C_n, and $n(n-1)$ in ij. It must also be of the form

$$Ax^nC_n + x^{n-2}B_{n-2}\,.\,(xij)^2 = 0,$$

since the n asymptotes form a curve of the n^{th} order touching C_n where it is met by $\dot{i}\dot{j}$. If A vanishes, the line $\dot{i}\dot{j}$ becomes a double factor; now this can only happen when $\bar{\dot{i}\dot{j}}^m c_m = 0$, and a comparison of dimensions shews that A differs from this only by a numerical factor. We may therefore write for the equation of the asymptotes

$$\bar{\dot{i}\dot{j}}^m c_m \,.\, C_n + B_{n-2} \,.\, \bar{\dot{i}\dot{j}}^2 \equiv \Pi P.$$

We may now find the product of the sines of the angles between them, $\Pi \sin PQ$, and of the angles they make with a line X, $\Pi \sin XP$. Namely, we have

$$\Pi \sin^2 PQ = \frac{\Pi\,(PQ\dot{i}\dot{j})^2}{(\Pi P\dot{i} \,.\, \Pi P\dot{j})^{n-1}} \text{ [to factor } \textit{près}\text{].}$$

Now

$$(\Pi PQ\dot{i}\dot{j})^2 = (\dot{i}\dot{j}^m c_m)^{2n-1},$$

and therefore

$$\Pi \sin^2 PQ = \frac{(\bar{\dot{i}\dot{j}}^m c_m)^{2n-1}}{(\bar{\dot{i}\dot{j}}^m c_m)^{2(n-1)}\,(\dot{i}^2 C_n \,.\, \dot{j}^2 C_n)^{n-1}} = \frac{\bar{\dot{i}\dot{j}}^m c_m}{(\dot{i}^2 C_n \,.\, \dot{j}^2 C_n)^{n-1}}.$$

In the next place

$$\Pi \sin^2 XP = \frac{\Pi\,(XP\dot{i}\dot{j})^2}{(X\dot{i} \,.\, X\dot{j})^n \,.\, \Pi P\dot{i} \,.\, \Pi P\dot{j}} = \frac{\{(X\dot{i}\dot{j})^n C_n\}^2}{(X\dot{i} \,.\, X\dot{j})^n \,.\, \dot{i}^n C_n \,.\, \dot{j}^n C_n}.$$

Let us now suppose a variable line X to be drawn through the fixed point x, meeting the curve C_n in the points l, m, n,... then

$$\Pi xl^2 = \frac{\Pi xl\dot{i} \,.\, \Pi xl\dot{j}}{(x\dot{i}\dot{j})^{2n}\,\Pi\,(l\dot{i}\dot{j})^2} = \frac{(x^n C_n)^2 \,.\, (X\dot{i})^n \,.\, (X\dot{j})^n}{(x\dot{i}\dot{j})^{2n} \,.\, \{(X\dot{i}\dot{j})^n C_n\}^2},$$

or

$$\Pi xl = \frac{x^n C_n \,.\, (X\dot{i} \,.\, X\dot{j})^{\frac{n}{2}}}{(x\dot{i}\dot{j})^n \,.\, (X\dot{i}\dot{j})^n C_n}.$$

Hence

$$\Pi xl \,.\, \Pi \sin XP = \frac{x^n C_n}{(x\dot{i}\dot{j})^n \,.\, \sqrt{(\dot{i}^n C_n \,.\, \dot{j}^n C_n)}};$$

this product is therefore independent of the position of the line X, and may be called the *distance* of the point x from the curve C_n.

If we draw to the curve from the point x the tangents $L, M, N, \ldots$ we shall have

$$\Pi \sin^2 LM = \frac{\Pi\,(LMij)^2}{(\Pi Li\,.\,\Pi Lj)^{m-1}} = \frac{x^n C_n\,.\,(xij)^n}{(\overline{xi}^m c_m\,.\,\overline{xj}^m c_m)^{m-1}};$$

$$\therefore\ (\Pi xp^2)^{m-1}\,.\,\Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n\,.\,(\overline{ij}^2 c_m)^{2(m-1)}},$$

and finally

$$\frac{(\Pi xp^2)^{m-1}}{\Pi pq^2}\,.\,\Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n\,.\,\sqrt{(i^n C_n\,.\,j^n C_n)}}\ \text{[to a factor]}$$

$$= \Pi xl\,.\,\Pi \sin XP \ \text{ to a factor,}$$

and by supposing the curve to break up into m points the factor is easily determined to be unity.

Considering now the line X as fixed and the point x as variable, we have

$$\Pi \sin^2 XL = \frac{\Pi\,(XLij)^2}{(Xi\,.\,Xj)^m\,\Pi Li\,.\,\Pi Lj} = \frac{(X^m c_m)^2\,(xij)^{2m}}{(Xi\,.\,Xj)^m\,.\,\overline{xi}^m c_m\,.\,\overline{xj}^m c_m};$$

$$\therefore\ \Pi \sin XL\,.\,\Pi xp = \frac{X^m c_m}{(Xi\,.\,Xj)^{\frac{m}{2}}\,.\,\overline{ij}^m c_m}.$$

But also

$$\Pi lm^2 = \frac{\Pi lmi\,.\,\Pi lmj}{(\Pi lij)^{2(n-1)}} = \frac{X^m c_n\,.\,(Xi\,.\,Xj)^{\frac{m}{2}}}{\{(Xij)^n C_n\}^{2(n-1)}},$$

and therefore

$$(\Pi \sin^2 XP)^{n-1}\,.\,\Pi lm^2 = \frac{X^m c_m}{(Xi\,.\,Xj)^{\frac{m}{2}}\,(i^n C_n\,.\,j^n C_n)^{n-1}},$$

whence

$$\frac{(\Pi \sin^2 XP)^{n-1}}{\Pi \sin^2 PQ}\,\Pi lm^2 = \frac{X^m c_m}{(Xi\,.\,Xj)^{\frac{m}{2}}\,.\,\overline{ij}^m c_m}$$

$$= \Pi \sin XL\,.\,\Pi xp, \text{ to a factor,}$$

which, as before, the special case of n lines shews to be unity. We shall call the quantity for which three expressions are here given the *distance* of the line X from the curve c_m.

V.

The elliptic geometry of two dimensions as Dr Klein calls it, or, which is the same thing, geometry on the sphere in which two opposite points are regarded as identical, differs from plane geometry in that instead of the two points ij we have the proper conic O_2 or o_2. Lines touching this conic, and points lying on it, are at an infinite distance from all other lines and points; distances measured on them are zero. Using the ordinary co-ordinates we may write

$$O_2 = x_1^2 + x_2^2 + x_3^2,$$
$$o_2 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$
$$a = a_1\xi_1 + a_2\xi_2 + a_3\xi_3,$$
$$A = A_1x_1 + A_2x_2 + A_3x_3,$$

and then

$$\sin^2 ab = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$
$$= \frac{2\overline{ab}^2 o_2}{a^2O_2 \,.\, b^2O_2} \text{ [when } O_2 \text{ given]},$$
$$\sin^2 AB = \frac{(A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2}{(A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2)}$$
$$= \frac{2\overline{AB}^2 O_2}{A^2o_2 \,.\, B^2o_2}.$$

The distance of a point a from a line B may be derived from these two as follows (fig. 17): through a draw a variable line A meeting B in b; then we have

$$\sin^2 ab = \sin^2 aAB = \frac{2\,(aB)^2 \,.\, A^2o_2}{a^2O_2 \,.\, \overline{AB}^2O_2},$$

and

$$\sin^2 AB = \frac{2\overline{AB}^2O_2}{A^2o_2 \,.\, B^2o_2};$$

$$\therefore \sin^2 ab \,.\, \sin^2 AB = \frac{4\,(aB)^2}{a^2O_2 \,.\, B^2o_2};$$

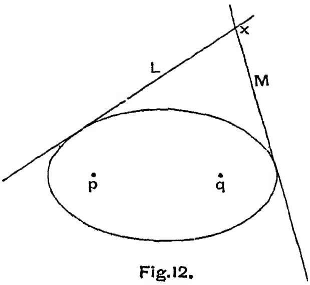


Fig. 12.

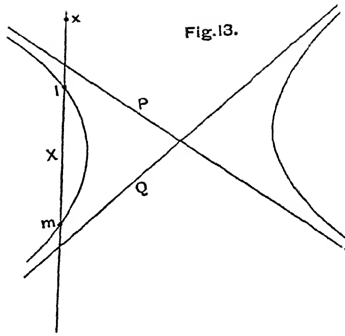


Fig. 13.

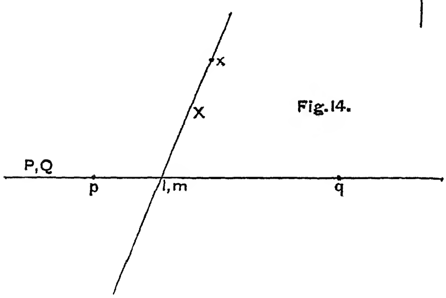
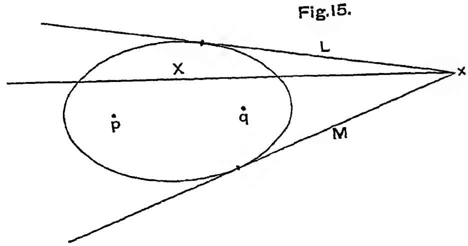


Fig. 14.

Fig. 15.



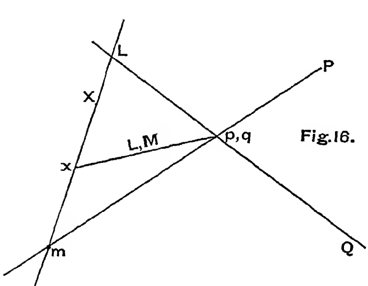
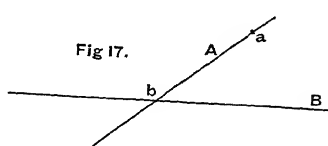


Fig. 16.

Fig 17.



a result independent of the position of the line A, which when A is at right angles to B becomes $\sin^2 aB$.

A conic C_2 or c_2 has with the absolute O_2 or o_2 four common points and four common tangents. The lines joining the common points form three pairs of common chords, P, Q; P', Q'; P'', Q''; the intersections of the common tangents form three pairs of foci, p, q; p', q'; p'', q''. The theories of the foci and the common chords are identical, and it will be sufficient to consider one of them; we shall choose for this purpose the foci.

We obtain a pair of foci by determining λ so that

$$c_2 + \lambda o_2$$

shall break up into factors; the condition for this is

$$0 = 6 \text{ Disct. } (c_2 + \lambda o_2) = C_2c_2 + 3\lambda C_2o_2 + 3\lambda^2 c_2O_2 + \lambda^3 O_2o_2,$$

and substituting here for λ, $-\dfrac{c_2}{o_2}$, we obtain the equation of the three pairs of foci:—

$$0 = C_2c_2 \,.\, o_2^3 - 3C_2o_2 \,.\, o_2^2 \,.\, c_2 + 3c_2O_2 \,.\, o_2 \,.\, c_2^2 - O_2o_2 \,.\, c_2^3.$$

If $c_2 + \lambda o_2$ break up into factors p, q, its reciprocal will be $-\frac{1}{4}\overline{pq}^2$. Thus we have

$$-\tfrac{1}{4}\overline{pq}^2 = C_2 + 2\lambda (co)_2 + \lambda^2 O_2,$$

where $(co)_2$ is the locus of points at which c_2 subtends a right angle. Consequently

$$-\tfrac{1}{4}\overline{pq}^2 o_2 = C_2o_2 + 2\lambda c_2O_2 + \lambda^2 O_2o_2 = 2\partial_\lambda \text{ Disct. } (c_2 + \lambda o_2).$$

Moreover

$$pqO_2 = c_2O_2 + \lambda O_2o_2 = \partial_\lambda^2 \text{ Disct. } (c_2 + \lambda o_2),$$

but $$p^2O_2 \,.\, q^2O_2 - (pqO_2)^2 = \tfrac{1}{3}\overline{pq}^2 o_2 \,.\, O_2o_2,$$

hence

$$\begin{aligned} 3p^2O_2 \,.\, q^2O_2 &= 3\,(c_2O_2 + \lambda O_2o_2)^2 - 4\,(C_2o_2 + 2\lambda c_2O_2 + \lambda^2 O_2o_2)\, O_2o_2 \\ &= 3\overline{c_2O_2}^2 - 4C_2o_2 \,.\, O_2o_2 - 2\lambda c_2O_2 \,.\, O_2o_2 - \lambda^2 \overline{O_2o_2}^2 \\ &= 3\,(\overline{c_2O_2}^2 - C_2o_2 . O_2o_2) - (C_2o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2o_2)\, O_2o_2. \end{aligned}$$

To simplify these formulæ, let us write

$$6\ \text{Disct.}\ (c_2 + \lambda o_2) = f = a\lambda^3 + 3b\lambda^2 + 3c\lambda + d$$

$$\tfrac{1}{3}\partial_\lambda f = x = a\lambda^2 + 2b\lambda + c$$

$$y = 3(b^2 - ac) - ax;$$

then whenever λ has a value which makes f vanish, so that $c_2 + \lambda o_2$ breaks up into the factors p, q, we must have

$$-\tfrac{1}{4}\overline{pq}^2 o_2 = x,$$

$$3p^2 O_2 \,.\, q^2 O_2 = y.$$

We shall now prove that if $x_1\ x_2\ x_3$ are the values of x, and $y_1\ y_2\ y_3$ the values of y, corresponding to the three values of λ given by $f = 0$, then

$$x_1^2 y_1 = x_2^2 y_2 = x_3^2 y_3 = -R_f, \text{ the discriminant of } f.$$

We get an equation for determining x by eliminating λ between $f = 0$ and $\frac{1}{3}\partial_\lambda f - x = 0$. Namely, the resultant of these two equations is

$$\begin{vmatrix} a & 3b & 3c & d & . \\ . & a & 3b & 3c & d \\ a & 2b & c-x & . & . \\ . & a & 2b & c-x & . \\ . & . & a & 2b & c-x \end{vmatrix} = -a^2x^3 + 3(b^2 - ac)ax^2 + aR_f,$$

$$= a(R_f + x^2 y),$$

which vanishes if

$$x^2 y = -R_f;$$

this equation is therefore true for each of the corresponding pairs of values of x and y. Substituting for these their values, we have

$$(\overline{pq}^2 o_2)^2 \,.\, p^2 O_2 \,.\, q^2 O_2 = (\overline{p'q'}^2 o_2)^2 \,.\, p'^2 O_2 \,.\, q'^2 O_2$$

$$= (\overline{p''q''}^2 o_2)^2 \,.\, p'^2 O_2 \,.\, q''^2 O_2 = -\tfrac{16}{3} R_f.$$

Now R_f is the osculant of c_2 and o_2, that is, the invariant whose vanishing is the condition that the two conics shall touch.

It follows further from the cubic equation for x that the product of its three values is $\frac{R_f}{a}$. Hence we have

$$\overline{pq}^2 o_2 . \overline{p'q'}^2 o_2 . \overline{p''q''}^2 o_2 = -64 \frac{R_f}{a},$$

and therefore

$$p^2 O_2 . q^2 O_2 . p'^2 O_2 . q'^2 O_2 . p''^2 O_2 . q''^2 O_2 = -\frac{a^2 R_f}{27};$$

whence by division

$$\sin^2 pq . \sin^2 p'q' . \sin^2 p''q'' = +\frac{12^3 [2^3]}{6^3} = +[64],*$$

since $$\sin^2 pq = \frac{[2]\,\overline{pq}^2 o_2}{p^2 O_2 . q^2 O_2} . \frac{O_2 o_2}{6} \text{ when } o_2 \text{ is given.}$$

From the same equation we learn that the sum of the reciprocals of the x is zero. Now

$$-\frac{R_f}{x^3} = \frac{y}{x} = [-2\, O_2 o_2] (\sin pq)^{-2};$$

therefore

$$(\sin pq)^{-\frac{2}{3}} + (\sin p'q')^{-\frac{2}{3}} + (\sin p''q'')^{-\frac{2}{3}} = 0.$$

$$[\text{Also } (\sin pq)^{-2} + (\sin p'q')^{-2} + (\sin p''q'')^{-2} = \tfrac{3}{4}.]*$$

We have thus two equations connecting the quantities $\sin pq$, $\sin p'q'$, $\sin p''q''$, by which when one is given the other two may be determined.

Now let x be any point of the plane (or sphere). Then we have

$$\sin^2 xp . \sin^2 xq = \frac{\overline{xp}^2 o_2 . \overline{xq}^2 o_2}{(x^2 O_2)^2 . p^2 O_2 . q^2 O_2} \left[\left(\frac{O_2 o_2}{6}\right)^2\right].$$

But the numerator of this fraction is clearly the result of operating with x^4 on the common tangents of pq and o_2, that is of $c_2 + \lambda o_2$ and o_2, a result which is clearly independent of λ. Hence we have

$$\overline{xp}^2 o_2 . \overline{xq}^2 o_2 = \overline{xp'}^2 o_2 . \overline{xq'}^2 o_2 = \overline{xp''}^2 o_2 . \overline{xq''}^2 o_2.$$

Moreover, since

$$(\overline{pq}^2 o_2)^2 p^2 O_2 . q^2 O_2 = -\tfrac{16}{3} R_f,$$

* [The introduced factor, cf. p. 152, brings the results into accordance with Prof. Cayley's equations. See Note to this paper, p. 159 (3).]

it follows that

$$\sin^4 pq = \tfrac{1}{36}\left(\frac{\overline{pq}^2 o_2 \,.\, O_2 o_2}{p^2 O_2 \;\; q^2 O_2}\right)^2 = -\tfrac{4}{27}\,\frac{R_f (O_2 o_2)^2}{(p^2 O_2 \,.\, q^2 O_2)},$$

and consequently

$$\frac{\sin xp \,.\, \sin xq}{(\sin pq)^{\frac{2}{3}}} = \frac{\sin xp' \,.\, \sin xq'}{(\sin p'q')^{\frac{2}{3}}} = \frac{\sin xp'' \,.\, \sin xq''}{(\sin p''q'')^{\frac{2}{3}}}$$

$$\left[= \frac{\sqrt{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2}}{x^2 O_2}\,\frac{1}{\sqrt[3]{16}\,\sqrt{3}}\,\frac{(O_2 o_2)^{\frac{2}{3}}}{\sqrt[6]{-R_f}}\right]$$

$$= \left[\frac{1}{\sqrt[3]{16}\,\sqrt{3}}\right]\frac{\sqrt{x^4 (co)_4}\,(O_2 o_2)^{\frac{2}{3}}}{x^2 O_2 \sqrt[6]{-R_f}}.$$

The three equations which we have just proved establish the theory of three pairs of antipoints on a sphere; viz. the three pairs of intersections of four tangents to the absolute. They take the place of the *two* equations which we have already used in regard to the two pairs of antipoints on a plane; namely,

$$pq^2 = -p'q'^2,$$

and

$$xp \,.\, xq = xp' \,.\, xq'.$$

We now proceed to use them in connection with the theory of the conic c_2.

From the point x let the tangents L, M be drawn to the conic; then

$$\sin^2 LM = [2]\,\frac{\overline{LM}^2 O_2}{L^2 o_2 \,.\, M^2 o_2} = \frac{x^2 O_2 \,.\, x^2 C_2}{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2}\ \text{[to factor } \textit{près}\text{]},$$

where pq are any set of foci. But also

$$\frac{\sin^2 xp \,.\, \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}} = \left[\frac{1}{12\sqrt[3]{4}}\right].\frac{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2 (O_2 o_2)^{\frac{4}{3}}}{(x^2 O_2)^2 \sqrt[3]{-R_f}},$$

and therefore

$$\frac{\sin^2 xp \,.\, \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}}\sin^2 LM = \left[\frac{1}{12\sqrt[3]{4}}\right]\frac{x^2 C_2 \,.\, (O_2 o_2)^{\frac{4}{3}}}{x^2 O_2 \,.\, \sqrt[3]{-R_f}}$$

$$= \left[\frac{1}{12\sqrt[3]{4}}\right].\frac{x^2 C_2 \,.\, (O_2 o_2)^2 (C_2 c_2)^{\frac{2}{3}}}{x^2 O_2 \,.\, \sqrt[3]{-R_F}}.$$

Again, if we draw through x a variable line X, meeting the conic in l, m, we shall have

$$\sin^2 xl \,.\, \sin^2 xm = \frac{\overline{xl}^2 o_2 \,.\, \overline{xm}^2 o_2}{(x^2 O_2)^2 \,.\, l^2 O_2 \,.\, m^2 O_2} = \frac{(x^2 C_2)^2 \,.\, (X^2 o_2)^2}{(x^2 O_2)^2 \,.\, X^4 (CO)_4}$$

$$= \frac{(x^2 C_2)^2 \,.\, (X^2 o_2)^2}{(x^2 O_2)^2 \,.\, \overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2},$$

and
$$\frac{\sin^2 XP \,.\, \sin^2 XQ}{(\sin^2 PQ)^{\frac{2}{3}}} = \left[\frac{1}{6\frac{4}{3}} \frac{\overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2 \,(O_2 o_2)^{\frac{4}{3}}}{(X^2 o_2)^2 \{P^2 o_2 \,.\, Q^2 o_2 \,(\overline{PQ}^2 O_2)^2\}^{\frac{1}{3}}}\right]$$

$$= \tfrac{3}{16} \frac{\overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2 \,.\, (O_2 o_2)^{\frac{2}{3}}}{(X^2 o_2)^2 \sqrt[3]{R_F}},$$

where R_F signifies the osculant of C_2 and O_2; $R_F = (C_2 c_2)^2 (O_2 o_2)^2 R_f$.

Therefore

$$\sin xl \,.\, \sin xm \, \frac{\sin XP \sin XQ}{(\sin PQ)^{\frac{2}{3}}} = \left[\frac{1}{\sqrt[6]{12^3 \,.\, 4}}\right] \frac{x^2 C_2 \,.\, (O_2 o_2)^{\frac{2}{3}}}{x^2 O_2 \,.\, \sqrt[6]{-R_F}}.$$

[The MS. here ends abruptly. In the results of the last two pages I have introduced a few additions, all, or nearly all, of which agree with results obtained by Prof. Henrici, who most kindly gave me his valuable help in revising the proof sheets.

The subject of V. has been treated from a different point of view by Prof. Cayley, who has allowed me to append the following note.

The results obtained in No. V. of the paper On the Theory of Distances may be worked out in greater detail, and in some measure in a more complete form.

Using line-coordinates, we have $x^2+y^2+z^2=0$, the conic called the absolute; a conic $(a, b, c, f, g, h \between x, y, z)^2=0$, which will be called simply the conic; and a point (x) the equation of which is $lx+my+nz=0$. The common tangents of the conic and the absolute intersect in pairs in six points $p, q : p', q' : p'', q''$, which are the foci of the conic; or if we regard the four lines simply as any four tangents of the absolute, then the six points are a system of foci; and we obtain in the first instance formulæ relating to such a system, alone or in connection with the point x: afterwards, taking them to be the foci of the conic, we further consider the two tangents L, M from the point (x) to the conic; and an arbitrary line X through the point (x).

1. The coordinates of a tangent of the absolute are (x_1, y_1, z_1), where these are any values such that $x_1^2+y_1^2+z_1^2=0$; and we consider the four tangents

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3), \quad (x_4, y_4, z_4).$$

Writing for a moment

$$\lambda,\ \mu,\ \nu = y_1z_4 - y_4z_1,\quad z_1x_4 - z_4x_1,\quad x_1y_4 - x_4y_1,$$

$$\lambda',\ \mu',\ \nu' = y_2z_3 - y_3z_2,\quad z_2x_3 - z_3x_2,\quad x_2y_3 - x_3y_2,$$

we have $\lambda x + \mu y + \nu z = 0,\quad \lambda' x + \mu' y + \nu' z = 0$

for the equations of the points p and q respectively; and the expression for the distance is given by

$$\cos^2 pq = \frac{(\lambda\lambda' + \mu\mu' + \nu\nu')^2}{(\lambda^2 + \mu^2 + \nu^2)(\lambda'^2 + \mu'^2 + \nu'^2)};$$

and if for shortness we write

$$12 = x_1x_2 + y_1y_2 + z_1z_2, \text{ \&c.},$$

then the values of λ, μ, ν give

$$\lambda^2 + \mu^2 + \nu^2 = -(14)^2,$$

$$\lambda'^2 + \mu'^2 + \nu'^2 = -(23)^2,$$

$$\lambda\lambda' + \mu\mu' + \nu\nu' = -31\,.\,24 + 12\,.\,34.$$

I write $\sqrt{23\,.\,14} = \mathrm{f},\quad \sqrt{31\,.\,24} = \mathrm{g},\quad \sqrt{12\,.\,34} = \mathrm{h},$

and I say that we have $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$. The formula becomes

$$\cos^2 pq = \frac{(\mathrm{g}^2 - \mathrm{h}^2)^2}{\mathrm{f}^4},$$

and we thence have

$$\sin^2 pq = \frac{(\mathrm{f}^2 - \mathrm{g}^2 + \mathrm{h}^2)(\mathrm{f}^2 + \mathrm{g}^2 - \mathrm{h}^2)}{\mathrm{f}^4} = \frac{-2\mathrm{fh}\,.\,-2\mathrm{fg}}{\mathrm{f}^4},\quad = \frac{4\mathrm{gh}}{\mathrm{f}^2};$$

and consequently for the three pairs of foci respectively

$$\sin^2 pq = \frac{4\mathrm{gh}}{\mathrm{f}^2};\ \sin^2 p'q' = \frac{4\mathrm{hf}}{\mathrm{g}^2};\ \sin^2 p''q'' = \frac{4\mathrm{fg}}{\mathrm{h}^2}.$$

2. The assumed relation $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$ is obtained from the equation

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix}^2 = 0,$$

which is identically true if w_1, w_2, w_3, w_4 are each $=0$; attending to the equations $x_1^2 + y_1^2 + z_1^2 = 0$, &c., this is

$$\begin{vmatrix} \cdot\ , & (12)^2, & (13)^2, & (14)^2 \\ (21)^2, & \cdot\ , & (23)^2, & (24)^2 \\ (31)^2, & (32)^2, & \cdot\ , & (34)^2 \\ (41)^2, & (42)^2, & (43)^2, & \cdot \end{vmatrix} = 0,$$

which is in fact the rationalised form of $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$.

* Results are marked with an asterisk.

3. The foregoing values give

$$\left.\begin{aligned}&\sin^2 pq\,.\,\sin^2 p'q'\,.\,\sin^2 p''q''=64,\\ &\sin^{-\frac{2}{3}} pq+\sin^{-\frac{2}{3}} p'q'+\sin^{-\frac{2}{3}} p''q''=\frac{\mathrm{f}+\mathrm{g}+\mathrm{h}}{(4\mathrm{fgh})^{\frac{2}{3}}}\,,\ =0\\ &\sin^{-2} pq+\sin^{-2} p'q'+\sin^{-2} p''q''=\frac{\mathrm{f}^3+\mathrm{g}^3+\mathrm{h}^3}{4\mathrm{fgh}}\,,\ =\tfrac{3}{4}.\end{aligned}\right\}\quad *$$

4. Considering now, in connection with the foci, the point (x) determined by the equation $lx+my+nz=0$, we have

$$\cos^2 xp=\frac{(l\lambda+m\mu+n\nu)^2}{(l^2+m^2+n^2)\,(\lambda^2+\mu^2+\nu^2)},$$

λ, μ, ν as before, and therefore

$$\lambda^2+\mu^2+\nu^2=-(14)^2.$$

Moreover

$$(l\lambda+m\mu+n\nu)^2=\begin{vmatrix} l, & m, & n\\ x_1, & y_1, & z_1\\ x_4, & y_4, & z_4\end{vmatrix}^2,\ =\begin{vmatrix} l^2+m^2+n^2, & 01, & 04\\ 01 & ,\ . & ,\ 14\\ 04 & ,\ 14, & .\end{vmatrix},$$

(if for shortness $\quad 01=lx_1+my_1+nz_1$, &c.)

$$=-(l^2+m^2+n^2)\,(14)^2+2\,.\,01\,.\,04\,.\,14.$$

The formula thus is

$$\cos^2 xp=\frac{-(l^2+m^2+n^2)\,(14)^2+2\,.\,01\,.\,04\,.\,14}{-(l^2+m^2+n^2)\,(14)^2};$$

or passing to $\sin^2 xp$, and then writing down the analogous value of $\sin^2 xq$, we have

$$\sin^2 xp=\frac{2\,.\,01\,.\,04}{14\,.\,l^2+m^2+n^2};\quad \sin^2 xq=\frac{2\,.\,02\,.\,03}{23\,.\,l^2+m^2+n^2};$$

and in like manner for the other two pairs of foci

$$\sin^2 xp'=\frac{2\,.\,02\,.\,04}{24\,.\,l^2+m^2+n^2};\quad \sin^2 xq'=\frac{2\,.\,03\,.\,01}{31\,.\,l^2+m^2+n^2},$$

$$\sin^2 xp''=\frac{2\,.\,03\,.\,04}{34\,.\,l^2+m^2+n^2};\quad \sin^2 xq''=\frac{2\,.\,01\,.\,02}{12\,.\,l^2+m^2+n^2}.$$

5. These formulæ give

$$\sin^2 xp\,\sin^2 xq\,\sin^{-\frac{4}{3}} pq=\sin^2 xp'\,\sin^2 xq'\,\sin^{-\frac{4}{3}} p'q'=\sin^2 xp''\,\sin^2 xq''\,\sin^{-\frac{4}{3}} p''q''$$

$$=4\,.\,01\,.\,02\,.\,03\,.\,04\,.\,(4\mathrm{fgh})^{-\frac{2}{3}}\,.\,(l^2+m^2+n^2)^{-2};$$

or as this may also be written

$$=4^{\frac{1}{3}}\,(01\,.\,02\,.\,03\,.\,04)\,(12\,.\,13\,.\,14\,.\,23\,.\,24\,.\,34)^{-\frac{1}{3}}\,(l^2+m^2+n^2)^{-2},\qquad *$$

where it will be recollected that 01 denotes $lx_1+my_1+nz_1$, &c., and 12 denotes $x_1x_2+y_1y_2+z_1z_2$, &c.

6. Taking now (x_1, y_1, z_1), &c., as the common tangents of the absolute and the conic, or say as the roots of the equations

$$x^2+y^2+z^2=0, \quad (a, b, c, f, g, h \between x, y, z)^2=0,$$

the expression on the right hand side, quà symmetrical function, homogeneous of the degree zero in the roots, and also homogeneous of the degree zero in the coefficients l, m, n, will be expressible as an absolute invariant of the two quadric functions and of the linear function $lx+my+nz$: and I say that the value is

$$= -4^{-\frac{2}{3}} . \square . \Omega^{-\frac{1}{3}} (l^2+m^2+n^2)^{-2},$$

$\square$ being the Resultant of the three functions, and Ω the Tactinvariant of the two quadric functions, as presently appearing. It is to be observed that $l^2+m^2+n^2$ is in fact the Reciprocant of $lx+my+nz$ and $x^2+y^2+z^2$, viz. the Reciprocant of $(a, \;.\between x, y, z)^2$ and $lx+my+nz$ is

$$(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch \between l, m, n)^2,$$

and for the quadric function $x^2+y^2+z^2$ this becomes $=l^2+m^2+n^2$.

7. Considering the three functions

$$x^2+y^2+z^2,$$
$$(a, b, c, f, g, h \between x, y, z)^2,$$
$$lx+my+nz,$$

it will be sufficient as regards the resultant to write down those terms which are independent of f. g, h; these are at once obtained by writing f, g, h each $=0$, and the resultant then presents itself as the norm of $l\sqrt{b-c}+m\sqrt{c-a}+n\sqrt{a-b}$; and we thus obtain (attending only to the terms in question)

$$\begin{aligned}\square = & \, l^4 (b-c)^2+m^4 (c-a)^2+n^4 (a-b)^2 \\ & -2m^2n^2 (c-a)(a-b)-2n^2l^2 (a-b)(b-c)-2l^2m^2 (b-c)(c-a).\end{aligned}$$

The Resultant is at once expressed in terms of the roots (x_1, y_1, z_1), &c., by the formula

$$\square = C\,(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)(lx_3+my_3+nz_3)(lx_4+my_4+nz_4),$$

or according to the foregoing notation

$$\square = C . 01 . 02 . 03 . 04,$$

where, and in what follows, C *is written to denote an essentially indeterminate constant,* having (it may be) different values in different equations.

8. Moreover writing as usual

$$\begin{aligned}K &= abc-af^2-bg^2-ch^2+2fgh,\\ \Theta &= bc-f^2+ca-g^2+ab-h^2,\\ \Theta' &= a+b+c,\\ K' &= 1,\end{aligned}$$

the Tactinvariant is taken to be

$$\Omega = 27K^2K'^2 + 4K\Theta'^3 + 4K'\Theta^3 - 18KK'\Theta\Theta' - \Theta^2\Theta'^2$$

(which for f, g, h each $=0$, reduces itself to

$$\Omega = -(b-c)^2\,(c-a)^2\,(a-b)^2).$$

The Tactinvariant vanishes if, and only if, a pair of roots (x_1, y_1, z_1), (x_2, y_2, z_2) become identical, say $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$. But we have $(y_1z_2 - y_2z_1)^2 = (y_1^2 + z_1^2)\,(y_2^2 + z_2^2) - (y_1y_2 + z_1z_2)^2, = 0$, if $x_1x_2 + y_1y_2 + z_1z_2 = 0$, that is if $12 = 0$; and similarly $(z_1x_2 - z_2x_1)^2 = 0$, and $(x_1y_2 - x_2y_1)^2 = 0$, if $12 = 0$. And we are thus led to the equation

$$\Omega = C.12.13.14.23.24.34.$$

9. The combination $\square^3\Omega^{-1}$ contains the roots homogeneously in the degree zero, and it will therefore have a determinate value, which is in fact found by the process which I present as a verification. The result is

$$\square^3\Omega^{-1} = -64\,(01.02.03.04)^3\,(12.13.14.23.24.34)^{-1}.$$

In verification, take the function $(a,\ldots\between x, y, z)^2$ to be $= x^2 + \omega y^2 + \omega^2 z^2$, ω an imaginary cube root of unity: the roots may be taken to be $(1, \omega^2, \omega)$, $(1, -\omega^2, \omega)$, $(1, \omega^2, -\omega)$, $(1, -\omega^2, -\omega)$. Attending only to the terms in l, we have

$$\square = -3l^4;\ \Omega = 27;\ 01.02.03.04 = l^4;\ 12.13.14.23.24.14$$

(is a product of factors such as $1 - \omega + \omega^2, = -2\omega$, and is) $= -64$; or the equation becomes $(-3l^4)^3\,(27)^{-1} = -64l'^2\,(64)^{-1}$, which is right. We have thus

$$-4\square\,\Omega^{-\frac{1}{3}} = (01.02.03.04)\,(12.13.14.23.24.34)^{-\frac{1}{3}},$$

and hence the foregoing value of $\sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq$: say

$$\sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq = -4^{-\frac{2}{3}}\square\,\Omega^{-\frac{1}{3}}\,(l^2 + m^2 + n^2)^{-2}. \qquad *$$

10. From the point (x) we draw to the conic tangents L, M: taking their coordinates to be (x_1, y_1, z_1), (x_2, y_2, z_2), these are the roots of

$$lx + my + nz = 0,$$
$$(a,\ldots\between x, y, z)^2 = 0,$$

and we have

$$\sin^2 LM = \frac{(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2)}.$$

11. We have $x_1^2 + y_1^2 + z_1^2 = 0$, or $x_2^2 + y_2^2 + z_2^2 = 0$, as the condition in order that the resultant $\square$ may vanish, and consequently

$$\square = C\,(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2).$$

It is easy to see that the function in the numerator will vanish if $l^2 + m^2 + n^2 = 0$, or if the Reciprocant $(bc - f^2,\ldots\between l, m, n)^2$ of the function $(a,\ldots\between x, y, z)^2$ and $lx + my + nz$ is $= 0$: or calling this reciprocant F we have

$$(l^2 + m^2 + n^2)F = C\{(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2\}.$$

The values of C in these two equations have a determinate ratio, and we find

$$\frac{(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)\,(x_2^2 + y_2^2 + z_2^2)} = \frac{-4F.(l^2 + m^2 + n^2)}{\square}.$$

In verification, assume $(a,\ldots\between x, y, z)^2 = x^2+\omega y^2+\omega^2 z^2$ as before, $lx+my+nz = x-z$; the roots (x_1, y_1, z_1) and (x_2, y_2, z_2) may be taken to be (1, 1, 1), (1, −1, 1): we have $\square = \{l^2(b-c)-n^2(a-b)\}^2, = (2\omega-1-\omega^2)^2, = 9\omega^2$; $F = bcl^2+abn^2, = b(a+c), = \omega(1+\omega^2), = -\omega^2$, and the equation is

$$\frac{8}{9} = \frac{-4\,.-\omega^2\,.\,2}{9\omega^2}.$$

Hence we have

$$\sin^2 LM = -4\,.\,F\,\square^{-1}\,.\,(l^2+m^2+n^2),$$

and consequently also

$$\sin^2 LM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq = 4^{\frac{1}{3}}\Omega^{-\frac{1}{3}}F\,.\,(l^2+m^2+n^2)^{-1},$$

which is Clifford's formula, p. 156.

12. We take through the point (x) an arbitrary line X, coordinates (α, β, γ): these coordinates satisfy therefore the equation $\alpha l+\beta m+\gamma n = 0$.

We have

$$\sin^2 XL\,.\,\sin^2 XM$$
$$= \frac{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2)-(\alpha x_1+\beta y_1+\gamma z_1)^2}{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2)}$$
$$\cdot\frac{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2)-(\alpha x_2+\beta y_2+\gamma z_2)^2}{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2)}.$$

13. To reduce this expression write for shortness

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= V,\\ x_1^2+y_1^2+z_1^2 &= \rho_1,\\ x_2^2+y_2^2+z_2^2 &= \rho_2,\\ \alpha x_1+\beta y_1+\gamma z_1 &= \sigma_1,\\ \alpha x_2+\beta y_2+\gamma z_2 &= \sigma_2,\\ x_1x_2+y_1y_2+z_1z_2 &= \tau.\end{aligned}$$

The expression is

$$\frac{V\rho_1-\sigma_1^2}{V\rho_1}\cdot\frac{V\rho_2-\sigma_2^2}{V\rho_2},$$

where the numerator is

$$= V(V\rho_1\rho_2-\sigma_1^2\rho_2-\sigma_2^2\rho_1)+\sigma_1^2\sigma_2^2.$$

But from the equations $lx_1+my_1+nz_1=0$, $lx_2+my_2+nz_2=0$, $l\alpha+m\beta+n\gamma=0$, we have

$$\begin{vmatrix}\alpha, & \beta, & \gamma\\ x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\end{vmatrix} = 0,$$

or squaring and reducing

$$\begin{vmatrix}V, & \sigma_1, & \sigma_2\\ \sigma_1, & \rho_1, & \tau\\ \sigma_2, & \tau, & \rho_2\end{vmatrix} = 0,$$

that is

$$V(\rho_1\rho_2-\tau^2)+2\sigma_1\sigma_2\tau-\sigma_1^2\rho_2-\sigma_2^2\rho_1 = 0,$$

and by reason hereof the foregoing numerator becomes

$$V(V\tau^2 - 2\sigma_1\sigma_2\tau) + \sigma_1{}^2\sigma_2{}^2, = (V\tau - \sigma_1\sigma_2)^2.$$

We thus have

$$\sin^2 XL \, . \, \sin^2 XM = \frac{(V\tau - \sigma_1\sigma_2)^2}{V^2\rho_1\rho_2},$$

$$= \frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(\alpha^2+\beta^2+\gamma^2)^2 \, . \, (x_1{}^2+y_1{}^2+z_1{}^2)(x_2{}^2+y_2{}^2+z_2{}^2)}.$$

14. The numerator-function

$$(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)$$

is $\quad = (\beta z_1 - \gamma y_1)(\beta z_2 - \gamma y_2) + (\gamma x_1 - \alpha z_1)(\gamma x_2 - \alpha z_2) + (\alpha y_1 - \beta x_1)(\alpha y_2 - \beta x_2),$

which vanishes if $\alpha : \beta : \gamma = x_1 : y_1 : z_1$ or $= x_2 : y_2 : z_2$, that is if $(a,\ldots\between\alpha,\ \beta,\ \gamma)^2 = 0$. Moreover observing that $l : m : n = \beta z_1 - \gamma y_1 : \gamma x_1 - \alpha z_1 : \alpha y_1 - \beta x_1 = \beta z_2 - \gamma y_2 : \gamma x_2 - \alpha z_2 : \alpha y_2 - \beta x_2$, it also vanishes if $l^2+m^2+n^2=0$: and we hence have

$$C\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}$$
$$= (l^2+m^2+n^2) \, . \, (a,\ldots\between\alpha,\ \beta,\ \gamma)^2$$

(viz. this equation is true when $l\alpha+m\beta+n\gamma=0$: it is a particular case of a more general formula where α, β, γ are arbitrary, and there are on the right hand side terms containing the factor $l\alpha+m\beta+n\gamma$). And we have as before

$$C(x_1{}^2+y_1{}^2+z_1{}^2)(x_2{}^2+y_2{}^2+z_2{}^2) = \square \, .$$

Squaring each side of the first equation, and dividing by the two sides of the second equation, we obtain a determinate result which is

$$\frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(x_1{}^2+y_1{}^2+z_1{}^2)(x_2{}^2+y_2{}^2+z_2{}^2)}$$
$$= \frac{(l^2+m^2+n^2)^2\{(a,\ldots\between\alpha,\beta,\gamma)^2\}^2}{\square};$$

viz. if to verify we assume as before $lx+my+nz = x-z$; and $(a,\ldots\between x,\ y,\ z)^2 = x^2+\omega y^2+\omega^2 z^2$: consequently

$$\gamma = \alpha \text{ and } (a,\ldots\between\alpha,\ \beta,\ \gamma)^2 = \alpha^2+\omega\beta^2+\omega^2\alpha^2, \ = -\omega(\alpha^2-\beta^2):$$

also $\qquad (x_1,\ y_1,\ z_1),\ (x_2,\ y_2,\ z_2) = (1,\ 1,\ 1),\ (1,\ -1,\ 1),\ \square = 9\omega^2,$

then the equation becomes

$$\frac{\{2\alpha^2+\beta^2-(2\alpha+\beta)(2\alpha-\beta)\}^2}{9} = \frac{4\{\omega(\beta^2-\alpha^2)\}^2}{9\omega^2},$$

which is right.

15. We hence have

$$\sin^2 XL \sin^2 XM = (l^2+m^2+n^2)^2\{(a,\ .\ \between\alpha,\ \beta,\ \gamma)^2\}^2 \, . \, \square^{-1} \, . \, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

and thence also

$$\sin^2 XL \sin^2 XM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq$$
$$= -4^{-\frac{2}{3}} \, . \, \{(a,\ldots\between\alpha,\beta,\gamma)^2\}^2 . \Omega^{-\frac{1}{3}} \, . \, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

which is the reciprocal of Clifford's formula, p. 157.]

ON THE THEORY OF DISTANCES*.

THIS communication relates to the following two theorems on the foci and asymptotes of curves.

Theorem i. $L, M, N, \ldots$ are the m tangents from a point a to a curve C_m of the m^{th} class; B is any line through a, meeting the curve in $m(m-1)$ points; $l, m, n, \ldots P, Q, R, \ldots$ are the $m(m-1)$ asymptotes of the curve, and $p, q, r, \ldots$ are a set of m foci.

$$\frac{\sin^2 LM \,.\, \sin^2 LN \,.\, \sin^2 MN \ldots (\overline{ap}^2 \,.\, \overline{aq}^2 \,.\, \overline{ar}^2 \ldots)^{m-1}}{al \,.\, am \,.\, an \ldots \sin BP \,.\, \sin BQ \,.\, \sin BR \ldots} = \overline{pq}^2 \,.\, \overline{qr}^2 \,.\, \overline{pr}^2 \ldots$$

Theorem ii. $l, m, n, \ldots$ are the n intersections of a line A with a curve C_n of the n^{th} order; b is any point on A from which are drawn the $n(n-1)$ tangents; $L, M, N, \ldots p, q, r, \ldots$ are a set of $n(n-1)$ foci, and $P, Q, R, \ldots$ are the n asymptotes.

$$\frac{\overline{lm}^2 \,.\, \overline{ln}^2 \,.\, \overline{mn}^2 \ldots (\sin^2 AP \,.\, \sin^2 AQ \,.\, \sin^2 AR \ldots)^{n-1}}{\sin AL \,.\, \sin AM \,.\, \sin AN \ldots lp \,.\, lq \,.\, lr \ldots} = \sin^2 PQ \,.\, \sin^2 QR \,.\, \sin^2 PR \ldots$$

The numerator and denominator of the fraction on the left-hand side of the Equation in Theorem i. are quantities either of which I call the distance of the point a from the curve C_m. The corresponding quantities in Theorem ii. I call the Distance of the line A from the curve C_n. The reason of this is in the similarity of the analytical expressions for the distance of two geometrical forms in all cases, viz. the distance vanishes when the two forms have contact, and is infinite when either of them has contact with the "absolute." The "absolute" in plane geometry (so called by Professor Cayley) is the two circular points at infinity.

I also consider the modifications undergone by these theorems in the case of spherical curves. The method of investigation employed is an extension of the "geometric analysis" of Grassmann, itself a development of a remark of Leibnitz.

* [Notices and Abstracts... from *Report of the thirty-ninth meeting of the British Association for the Advancement of Science,* held at Exeter, August, 1869, p. 9.]

XVII.

ON A CASE OF EVAPORATION IN THE ORDER OF A RESULTANT*.

A PARTICULAR case of the following Theorem was required in the course of my proof that every rational equation has a root†; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

Theorem. Let it be required to eliminate x between two equations homogeneous in x and certain other variables y, z, ..., in which equations, however, x only occurs in virtue of the occurrence of a quantity

$$w = x^\alpha y^\beta z^\gamma \ldots\ldots,$$

where $$\alpha + \beta + \gamma + \ldots\ldots = \mu;$$

let also m, n be the orders of the equations, and h, k the remainders after division of m, n respectively by μ; then the order of the resultant is

$$= \frac{mn - hk}{\mu}.$$

Demonstration. Suppose that p, q are the quotients of the division of m, n respectively by μ; that is to say, let

$$m = p\mu + h, \quad n = q\mu + k,$$

* [From the *Proceedings of the London Mathematical Society*, Vol. III. Nos. 25, 26, pp. 80—82.]

† [See p. 22, supra.]

then the two equations may be written

$$a_m + a_{m-\mu} \,.\, w + a_{m-2\mu} \,.\, w^2 + \ldots\ldots + a_h w^p = 0,$$
$$b_n + b_{n-\mu} \,.\, w + b_{n-2\mu} \,.\, w^2 + \ldots\ldots + b_k w^q = 0,$$

where the suffixes of the several coefficients indicate their orders in the variables $y, z, \ldots\ldots$ Instead of directly eliminating x from these equations, we may eliminate w; and the result of this may be written down at once by Professor Sylvester's dialytic method. It is in fact

$$\begin{vmatrix} a_m & a_{m-\mu} \ldots\ldots\ldots & & \\ . & a_m \quad \ldots\ldots\ldots & & \\ . & . \quad \ldots\ldots\ldots & & \\ & & \ldots\ldots\ldots\ldots & \\ & & & \ldots b_k \ldots \\ & & & \ldots b_{k+\mu}\ b_k \end{vmatrix}$$

where in the principal diagonal of the determinant the constituent a_m occurs q times, and the constituent b_k occurs p times. The order of the resulting term $(a_m)^q \,.\, (b_k)^p$ is $mq + kp$, and this therefore (since the determinant must be homogeneous) is the order of the resultant itself. If we had written the b coefficients before the a coefficients, we should have obtained $np + hq$ as the value of the same quantity. These two values are identical, since by hypothesis

$$q\,(m-h) = p\,(n-k) = pq\mu,$$

and therefore $qm + pk = pn + qh = r$, suppose.

Now
$$\begin{aligned} mn &= (p\mu + h)\,(q\mu + k) \\ &= pq\mu^2 + pk\mu + hq\mu + hk \\ &= \mu r + hk; \end{aligned}$$
$$\therefore\ r = \frac{mn - hk}{\mu},$$

as was to be proved.

The following extension is brought to light by a different method of proving the original theorem.

Let it be required to eliminate $k-1$ variables $x, y, \ldots$ from k equations, homogeneous in these, and certain other vari-

ables, in which equations, however, $x, y, \ldots$ only occur in virtue of the occurrence of $k-1$ quantities $u, v, \ldots$ all of the same order μ; let also $m_1, m_2, \ldots m_k$ be the orders of the equations, and

$$m_i = p_i\, \mu + h_i, \quad h_i < \mu;$$

then the order of the resultant is

$$\Pi p\, (\Sigma \frac{h}{p} + \mu).$$

For the equations may be written

$$a_h\, (u, v, \ldots)^p + \theta a_{h+\mu}\, (u, v, \ldots)^{p-1} + \ldots + \theta^p a_{h+p\mu} = 0,$$

where the h, p are to be affected successively with the suffixes 1, 2, ... k, and θ may be considered $=1$. Now these equations may be regarded as having coefficients of the constant order h, but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1 p_2 p_3 \ldots p_k$, or $\frac{h_1}{p_1} \Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2} \Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu \Pi p$. Hence the entire order of the resultant is

$$\Pi p\, (\Sigma \frac{h}{p} + \mu),$$

as stated above.

XVIII.

ON A THEOREM RELATING TO POLYHEDRA, ANALOGOUS TO MR COTTERILL'S THEOREM ON PLANE POLYGONS*.

MR COTTERILL'S theorem, presented last year to the Society, is as follows: For every plane polygon of n vertices there is a curve of class $n-3$ touching all the diagonals; the number of diagonals is such as to exactly determine this curve and no more; and when the curve touches the line at infinity, the area of the polygon is zero.

The proof of this depends essentially upon the fact that if we join the vertices of the polygon to any point in its plane, the area of the polygon is equal to the sum of the triangles so formed, taken of course with their proper signs according to the rule of Möbius.

The analogous theorem in space should therefore apply in the first instance to those solids whose volume can be expressed as the sum of tetrahedra, having one vertex at an arbitrary point of space, and the other three at three vertices of the figure; that is to say, it should apply to solids having *triangular faces.*

For such solids I find accordingly that the analogy is very complete and exact. It is convenient to define a plane containing three vertices but not being a face, as a diagonal plane; and

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 51, pp. 178—185. Mr Cotterill's paper is given, in part, in Vol. IV. No. 49.]

a line joining two vertices but not being an edge, as a diagonal line. This being so, the theorems which I shall prove are the following:

For every polyhedron of n *summits having only triangular faces* (Δ-*faced* n-*acron* CAYLEY) *there is a surface of class* n − 4 *touching all the diagonal planes.*

This surface contains all the diagonal lines.

The diagonal planes and lines are so situated, however, that the conditions of touching the planes and containing the lines are precisely sufficient to determine a surface of class n − 4.

When this surface touches the plane at infinity, the volume of the solid is zero.

To apply these propositions to polyhedra having other than triangular faces, we must consider such polygonal faces as *singularities.* Each of them, in fact, may by a small deformation of the polyhedron be resolved into a certain number of triangles; and we may thus regard a quadrangular face, for example, as the special case of two adjacent triangular faces being in one plane. Thus the quadrangular face *abcd* [fig. 18] may be regarded as produced by coplanarity of the triangles *abd, cbd.* The effect of this is also to unite together the two diagonal planes *abc, adc,* and to make the diagonal line *ac* lie in the face. Thus the surface of class $n-4$ must touch the face *abcd*; but it does not in general contain the lines *ac, bd.* It touches the face at their point of intersection. And, in general, it is not necessary to consider the diagonals of a polygonal face as diagonals of the polyhedron, and they do not in fact lie upon the surface d_{n-4}. But *a polygonal face with* m *vertices is a multiple tangent plane of order* m − 3, *and the curve of contact is Mr Cotterill's curve appertaining to the polygon.*

It is interesting to consider from this point of view the correlative propositions. Just as we have regarded a solid with a given number of summits, or *polyacron,* as having normally or in the most general case only triangular faces, while polygonal faces present themselves as singularities, and polyacra possessing them as degenerate forms; so we must regard a *polyhedron,* or

solid with a given number of faces, as having normally or in general only three-edged summits (tripleural summits, CAYLEY), while summits having a greater number of edges will present themselves as singularities, and polyhedra possessing them as degenerate. Every solid with plane faces, except the tetrahedron, must have singularities of one kind or the other; just as only loci of the second order are general at the same time of their order and of their class.

The proof of these results is as follows. Let $a, b, c, \dots l, m, n$ be the summits of a Δ-faced polyacron, and p any point in space; let also $X=0$ be the equation to the plane at infinity, and the result of substituting in X the coordinates of any point, as a, be denoted by aX. Now if fgh is a face, and the summits f, g, h, looked at from p, go round the face clockwise, then the expression $\dfrac{(pfgh)}{pX.fX.gX.hX}$ represents the volume of the tetrahedron $pfgh$ according to the rule of Möbius. (Here $(pfgh)$ means the determinant formed with the coordinates of p, f, g, h.) Hence, if V be the volume of the whole solid, we have

$$\Sigma\frac{(pfgh)}{pX.fX.gX.hX}=V,$$

the summation being extended over all the faces, and the summits of each so mentioned that every edge occurs twice in two different orders; that is to say, if we have mentioned $(pfgh)$, we must not mention $(pfgk)$, but $(pgfk)$ or $(pfkg)$ or $(pkgf)$. To render this equation homogeneous in all the quantities mentioned, I call to mind that the volume of a tetrahedron is not given absolutely by the formula $\dfrac{(pfgh)}{pX.fX.gX.hX}$, but only to a factor *près*, depending on the unit of volume employed. If we take as this unit of volume the volume of the fundamental tetrahedron, whose vertices may be denoted by 1, 2, 3, 4, then our equation becomes

$$\Sigma\frac{(pfgh)}{pX.fX.gX.hX}=V.\frac{(1234)}{1X.2X.3X.4X}\ \dots\dots\dots (1).$$

Here V is a ratio, depending on the positions of the points

$a, b, c, \ldots$ relatively to the plane X, but absolutely independent of the position of p. If, then, we make the equation integral, by multiplying both sides into $pX.1X.2X.3X.4X.\Pi.fX$, we see that the expression

$$\Sigma\,(pfgh)\frac{\Pi.fX}{fX.gX.hX}$$

must be divisible by pX; because its equivalent on the other side is so divisible, and the equation is an identity so far as p is concerned. The result of the division is of the order $n-4$ in X; or, which is the same thing, *if* X *be regarded as a variable plane, the equation*

$$\Sigma\,(pfgh)\frac{\Pi.fX}{pX.fX.gX.hX}=0\ldots\ldots\ldots\ldots\ldots(2)$$

represents a surface of class n − 4.

Two things are now clear from our previous equation and from the form of this one.

1°. If the equation is satisfied when X is the plane at infinity, then $V=0$; or, *if the surface* (2) *touch the plane at infinity, the volume of the solid is zero.*

2°. The equation (2) is satisfied if $lX=0$, $mX=0$, $nX=0$, where l, m, n are any three vertices not in the same face. Therefore *the surface* (2) *touches all the diagonal planes.*

The investigation, so far, is a mere reproduction of that of Mr Cotterill, with the addition of an extra letter to apply it to three dimensions instead of two. I shall take the liberty of calling the surface thus arrived at the *index-surface* of the polyacron, and shall denote it by the symbol v_{n-4}.

The index-surface contains all the diagonal lines. For let ab be a diagonal line, and c any other summit of the solid; then abc is a diagonal plane. For if it were a face, ab would be an edge, contrary to the supposition. Consequently $n-2$ diagonal planes can be drawn through every diagonal line; now all these are touched by the index-surface. But if more than $n-4$ tangent planes can be drawn through a straight line to a surface of class $n-4$, the line must lie altogether in the surface.

Through an edge of the solid, on the other hand, two faces and $n-4$ diagonal planes can be drawn; which latter are, of course, the tangent planes from it to the surface.

There are certain diagonal planes which it is convenient to consider separately. They are those which contain three edges of the solid; and I shall call them *single planes.* A diagonal plane may, of course, contain three diagonal lines, or two, or one, or none; but if it contains any diagonal line, the condition of touching it is already involved in the condition of containing that line. So that the facts we know about the index-surface may be summed up in saying that it passes through all the diagonal lines and touches all the single planes.

I now go on to prove that in general these conditions are precisely sufficient to determine a surface of class $n-4$. In order to do this, it will be necessary to make use of the researches of Prof. Cayley upon the Δ-faced polyacra, contained in the 1st volume of the 3rd series of the *Manchester Memoirs,* p. 248; particularly of the following passage:—

"An n-acron has n summits, $3n-6$ edges, $2n-4$ faces; and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.
2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.
3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

$$n = c + d + e + f + g + h + \&c.,$$

$$6n - 12 = 3c + 4d + 5e + 6f + 7g + 8h + \&c.;$$

and therefore $\quad 12 = 3c + 2d + e + 0f - g - 2h - \&c.,$

or $\quad 3c + 2d + e = 12 + g + 2h + \&c.;$

whence, if $c = 0$ and $d = 0$, then $e = 12$ at least."

Upon this theorem Prof. Cayley founds a method of deriving all polyacra with $n+1$ summits from those with n summits. If we remove from a polyacron a tripleural summit, as a in the figure [fig. 19], we may derive from it a new polyacron with one summit less by regarding the diagonal plane bcd as a face of the new solid. Conversely, we may add one summit to any polyacron by crowning any one of its faces with a tripleural summit, and then regarding this face as a diagonal plane This process is called by Prof. Cayley the First Process. In a similar manner, the skew quadrilateral $bced$ [fig. 20] formed by two adjacent faces may be crowned by a tetrapleural summit a, with the edges ab, ac, ae, ad, the faces bcd, cde becoming diagonal planes of the new solid; this is called the Second Process. Again, the skew pentagon $bcfed$ [fig. 21] formed by three adjacent faces may be crowned by a pentipleural summit a, with the edges ab, ac, af, ae, ad, the faces bcd, cde, cef becoming diagonal planes; this is called the Third Process. And it appears from the theorem quoted above, that every $(n+1)$-acron can be made out of an n-acron by one or other of these processes, according as it belongs to the first, second, or third case of the theorem.

I shall now show, then, that if the conditions of containing the diagonal lines and touching the single planes are precisely sufficient to determine the index-surface of an n-acron, then the same thing will be true for any $(n+1)$-acron derived from it by either of these processes. This will prove that the theorem is true for all Δ-faced polyacra, provided we can show that it is true for all pentacra. Now there is only one pentacron, the figure formed of two tetrahedra with a common face, $abcde$ [fig. 22]. This figure has the diagonal line ae and the single plane bcd; and the index-surface is the point v, which is precisely determined as the intersection of these.

In determining the number of conditions involved in passing through a system of lines, we must remember that every intersection of two lines diminishes the number by one, except where three or more lines are in one plane. We have only to deal with the case of three lines in one plane; the number of conditions is then reduced by two for the three intersections.

First Process.—Let D be the number of diagonal lines of the n-acron; then when we pass to the $(n+1)$-acron, the following is the increase in the number of conditions:

The D-lines are on a surface of class $n-3$ instead of $n-4$; this makes an increase	$+D$
There are $n-3$ new diagonals joining the new summit a to all the old summits except b, c, d	$+(n-2)(n-3)$
One or other of these, however, meets each of the old ones at least once; which is all that need be counted, because if any old diagonal meets two new ones a triangle is formed	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-3)(n-4)$ intersections......	$-\frac{1}{2}(n-3)(n-4)$
There is a new single plane bcd	$+1$
The total increase is therefore	$\frac{1}{2}(n-1)(n-2)$,

which is the difference between the number of conditions required to determine a surface of class $n-3$ and the number required for class $n-4$.

Second Process.—Let D be the number of old diagonal lines less be; then we have the following increase:

The D lines on surface of higher class.........	$+D$
There are $n-4$ new diagonals..................	$+(n-2)(n-4)$
One or other of these, however, meets each of the D lines at least once; and, as before, this is all that need be counted...	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-4)(n-5)$ intersections......	$-\frac{1}{2}(n-4)(n-5)$
The edge cd becomes a diagonal line	$+(n-2)$
If, however, we join this edge to the $n-4$ summits different from $bcde$, there is a reduction 1 in the case of each; for either the plane was a single plane, or it contained one or two diagonals.........	$-(n-4)$
The diagonal be is on surface of higher class	$+1$
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Third Process.—Let D be the number of old diagonal lines, less df, fb, be; then we have the following increase:—

The D lines on surface of higher class.........	$+D$
There are $n-5$ new diagonals..................	$+(n-2)(n-5)$
Intersections of these with D lines	$-D$
New diagonals meet in a point	$-\frac{1}{2}(n-5)(n-6)$
The edges cd, ce become diagonal lines	$+2(n-2)$
If we join these to the $n-5$ summits different from $bcdef$, there is a reduction 1 for each plane	$-2(n-5)$
The diagonals df, fb, be on surface of higher class	$+3$
Their intersections with cd, ce, and of these with one another	-3
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Passing now to the consideration of polygonal faces, I remark first that, by direct application of Mr Cotterill's theorem, we have an expression for the volume of the pyramid standing on a plane polygon. For let p be the vertex of the pyramid, q any point in the plane of the polygon; then we have

$$\Sigma' \frac{(pqab)}{pX \,.\, qX \,.\, aX \,.\, bX} = U \frac{(1234)}{1X \,.\, 2X \,.\, 3X \,.\, 4X},$$

in which U is the ratio of the volume of the pyramid to that of the fundamental tetrahedron, and Σ' refers to a summation going round the m sides of the polygon in order. From this it appears that the expression

$$\frac{1}{qX} \,.\, \Sigma' \{(pqab) \,.\, \Pi' \,.\, cX\}$$

(in which $\Pi' \,.\, cX$ means a product involving all the vertices of the polygon except a and b) is integral, independent of q and of the order $m-3$ in X. If we equate it to zero, we in fact obtain the equation of Mr Cotterill's curve belonging to the polygon.

Now if this polygon form a face P of a polyacron, we obtain, as before, the following expression for the volume of the solid:—

$$\Sigma' . \frac{(pqab)}{pX . qX . aX . bX} + \Sigma . \frac{(pfgh)}{pX . fX . gX . hX}$$
$$= V . \frac{(1234)}{1X . 2X . 3X . 4X} .$$

Thus the equation of the index-surface may be written

$$\frac{\Pi . fX}{pX . qX} . \Sigma' \{(pqab) . \Pi' . cX\} + \frac{1}{pX} . \Sigma \{(pfgh) \, \Pi . aX\} = 0.$$

Here $\Pi . aX$ must in every case contain $m-2$ factors at least belonging to points on the plane P; or it vanishes in the order $m-2$ when the coordinates of P are substituted in it. The term Σ' vanishes as we have seen in the order $m-3$ in the same case. Thus P is a multiple tangent plane of order $m-3$, and the curve of contact is determined by the term Σ'; that is to say, it is Mr Cotterill's curve belonging to the polygon.

Fig. 18.

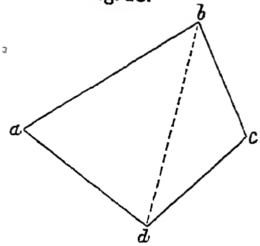


Fig. 19.

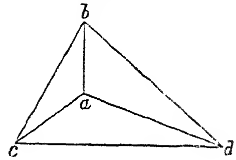


Fig. 20.

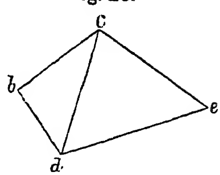


Fig. 21.

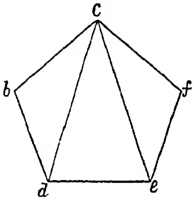
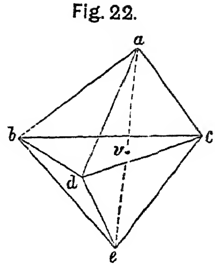


Fig. 22.



XIX.

GEOMETRY ON AN ELLIPSOID*.

THE metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four coplanar points o_1, o_2, o_3, o_4, which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2, and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y, which is the representative of x; the points i, j will then represent the generators through a. If we take the points i, j to be the absolute of the plane L, then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points o; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are concyclic.

2. We know, then, that the antipoints of the o lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real;

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 54, pp. 215—217.]

I call these u_1, u_2, u_3, u_4, and we may now take the points u as our absolute instead of the points o.

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y, so that conjugate directions at x correspond to rectangular directions at y. Next, the tangent plane at x meets the plane at infinity in a line, say β. Through the points o can be drawn two conics to touch β, say at p, q. The lines xp, xq are tangents to the lines of curvature at x, since the points p, q are conjugates both of the section of the ellipsoid at infinity and of the imaginary circle. But now let o_1o_2 and o_3o_4 meet β in r, s respectively. Then the involution made by rs and the points where β meets the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions through y corresponding to xp, xq make equal angles with the circles yo_1o_2, yo_3o_4. Hence—

The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.

Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by the two points c_1, c_2, where it meets the ellipsoid. The sections drawn through this line will be represented by the series of coaxal circles through c_1, c_2. The sections through δ, the polar of γ, will therefore be represented by the series of coaxal circles through d_1, d_2, the antipoints of c_1, c_2. Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints, as is otherwise obvious.

I denote further the principal sections and the plane at infinity by $XYZU$, which notation will serve also for the circles which represent them. Now, in general, a section of E_2 passing through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points $o_1\,o_2$, $o_3\,o_4$; $o_1\,o_3$, $o_4\,o_2$; $o_1\,o_4$, $o_2\,o_3$, correspond in this way to the circles XYZ. I want now to find the interpretation on the plane of

the rectangularity of the lines γ and δ. The planes joining them to the point o_1o_2, o_3o_4 are harmonic of the lines o_1o_2, o_3o_4. Hence the circles c_1c_2X, d_1d_2X are harmonic of the circles coaxal with them and passing through o_1o_2, o_3o_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_2YZ}{d_1d_2YZ}=\frac{c_1c_2ZX}{d_1d_2ZX}=\frac{c_1c_2XY}{d_1d_2XY}.$$

If now for d_1, d_2 we may substitute c_1, c'_1 where c'_1 is indefinitely near to c_1 *in any direction*, c_1c_2 represents the normal at c_1.

5. A circle P, orthotomic of U, represents a diametral section. Let the pole of this section be called p; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q, of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q. These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P, orthotomic of U, being given, there can always be found a point q^0, such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting point a. The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0.

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U. Next, if we draw through P and Q two new circles, one of which has o_1 o_2 for harmonics, and the other o_3 o_4, these must be harmonics of

P and Q. But a circle orthotomic of U and having $o_1\, o_2$ for harmonics, must have them for inverse points, and therefore have its centre on $o_1\, o_2$. Hence the line joining the centres of P and Q is cut harmonically by the lines o_1o_2, o_3o_4. Similarly, it is cut harmonically by o_1o_3, o_4o_2, and by o_1o_4, o_2o_3. Hence the centres of P and Q are polar opposites in regard to the quadrangle $o_1\, o_2\, o_3\, o_4$. They are therefore conjugate points in regard to the circle U.

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4. There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have $XYZU$ for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P; the point of contact in pairs will represent the axes of the corresponding section.

XX.

PRELIMINARY SKETCH OF BIQUATERNIONS*.

I.

THE *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In

* [From the *Proceedings of the London Mathematical Society,* Vol. IV. Nos. 64, 65, pp. 381—395.]

studying the motions of a particle or the composition of couples, the only construction required is that of the "force-polygon," and the theory involved is that of the addition of vectors; but in the static or kinematic of solids we require in addition the construction of the "link-polygon," and there is involved the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; *i.e.* a vector may move anywise parallel to itself, but a rotor *only* in its own line.

The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, or of two rotation-velocities whose axes do not intersect, gives rise to a system of forces on the one hand, and the most general velocity of a rigid body on the other. These still more complex quantities have been studied, and the theory of their addition or composition completely worked out, by Dr Ball.

A system of forces may be reduced in one way to a single force P, and a couple G whose plane is perpendicular to the line of action of the force, or *central axis*. Dr Ball speaks of the system of forces as a *wrench* about a certain *screw*; the axis of the screw being the central axis, and the pitch being the ratio $\frac{G}{P}$ of the couple to the force. Similarly the velocity of a

rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of the screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation-velocity or couple) is magnitude associated with direction, and as a rotor (rotation-velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench), is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor;* the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be [fig. 23] AB and AC, as they may both be made to start from any arbitrary point A. Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC, and then magnifying or diminishing it until it is of the same length as AC. The ratio of two vectors then is the combination of an ordinary numerical ratio with a *rotation;* or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis;

but is completely specified when its angular magnitude and the direction of its axis are given.

This quaternion* $\frac{AC}{AB} = q$, then, is an operation which, being performed on AB, converts it into AC, so that $q \, . \, AB = AC$. The axis of the quaternion is perpendicular to the plane BAC; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. But a quaternion can *only* operate upon a vector which is perpendicular to its axis. If AF be any vector not in the plane BAC, the expression $q \, . \, AF$ is absolutely unmeaning. A meaning is indeed subsequently given to an analogous expression *in which the signification of* AF *is different.* But it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression $q \, . \, AF$ has no meaning at all.

Let us now consider what is the operation necessary to convert one *rotor* into another. There is one straight line which meets at right angles the axes of any two rotors, and part of which constitutes the shortest distance between them. Let AC [fig. 24] be the shortest distance between the rotors AB and CD. Then AB may be converted into CD by a process consisting of three steps. First, turn AB about the axis AC into the position AB', parallel to CD. Then slide it along this axis into the position CD'. Lastly, magnify or diminish it in the ratio of CD' to CD. The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is

$$\frac{AC}{\text{circ. meas. of } BAB'}.$$

The ratio of two rotors, then, is the combination of an ordinary

* Professor Cayley, by a very convenient notation, distinguishes $\frac{|AC}{AB|}$ and $\frac{AC|}{|AB}$; viz., $AB\frac{|AC}{AB|}=1$, but $\frac{AC|}{|AB}AB=1$. It should, I think, be a convention that $\frac{X}{Y}$ is *always* to mean $\frac{X|}{|Y}$, viz., the operation which converts Y into X, or which, coming after the operation Y, is equivalent to the operation X.

numerical ratio with a *twist*. This twist is associated with a perfectly definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD, so that $t = \frac{CD}{AB}$, and $t \, . \, AB = CD$; then if EF be any other rotor which meets AC at right angles, the expression $t \, . \, EF$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC, turning it about AC as axis through an angle equal to BAB', and altering its length in the ratio $AB : CD$. But if EF be a rotor not meeting AC, or meeting it at any other than a right angle, the expression $t \, . \, EF$ will have no meaning whatever.

We have now defined the ratio of two rotors and shown that like a quaternion it has a restricted range of operation. The question naturally arises, What now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ, and $t\beta$ is another rotor, say δ. If therefore we assume the distributive law, we have

$$t\,(m\alpha + n\beta) = m\gamma + n\delta,$$

or

$$t = \frac{m\gamma + n\delta}{m\alpha + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $m\alpha + n\beta$ and $m\gamma + n\delta$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ.

The solution of this problem in the general case in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor B. Let B' be a motor having the same rotor part as B, and the same pitch as A; and let $B = B' + \beta$, where β is a vector parallel to the axis of B. Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t, and we may write

$$\frac{B}{A} = t + \frac{\beta}{A},$$

where it now only remains to find an operation which will convert a motor A into a vector β.

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω, applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis, and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, the symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = \alpha$, a vector, and the ratio $\frac{\beta}{\alpha}$ is a quaternion q so that $q\alpha = \beta$, we may write successively

$$\beta = q\alpha = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then $$\frac{B}{A} = t + q\omega,$$

or *the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.*

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive Greek letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α. The ratio $\frac{\beta}{\alpha}$ will then be a quaternion q, which is also the ratio $\frac{\omega\beta}{\omega\alpha}$ *. The general expression for a motor is then $\alpha + \omega\beta$. Let it now be required to find the ratio of two motors $\alpha + \omega\beta$, $\gamma + \omega\delta$; or the value of the expression $$\frac{\gamma + \omega\delta}{\alpha + \omega\beta}.$$

First, let $\frac{\gamma}{\alpha} = q$; then $q(\alpha + \omega\beta) = \gamma + q\omega\beta = \gamma + \omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors α, β, γ will not be coplanar, and cannot

* It follows from this that $\omega q = q\omega$, or ω is commutative with quaternions.

therefore be operated on by the same quaternion q. If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit rotors through the origin, $\frac{\delta - q\beta}{\alpha}$ will be a perfectly definite quaternion r. The equation

$$r\alpha = \delta - q\beta$$

is, like the equation $q(\alpha + \omega\beta) = \gamma + \omega q\beta$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol ω) we add ω times the first equation to the second and assume the distributive law, we obtain

$$(q + \omega r)(\alpha + \omega\beta) = \gamma + \omega\delta.$$

In this way the ratio $\frac{\gamma + \omega\delta}{\alpha + \omega\beta}$ is expressed in the form $q + \omega r$, which expression may conveniently be called a *biquaternion**. The final equation, however, is not susceptible of interpretation in the same sense as the equation $q\alpha = \gamma$. The expression $q + \omega r$ does not denote the sum of geometrical operations which can be applied to the motor $\alpha + \omega\beta$ as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table:—

Geometrical Form	Quantity	Example	Ratio
Sense on st. line	Vector on st. line	Addition or Subtraction	Signed Ratio
Direction in plane	Vector in plane	Complex quantity	Complex Ratio
Direction in space	Vector in space	Translation, Couple	Quaternion
Axis	Rotor	Rotation-Velocity, Force	Twist
Screw	Motor	Twist-Velocity, System of Forces	Biquaternion

* Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that all

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z. It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal.* An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point-equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The

scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

points which correspond to real systems of values are called real points; those which correspond to imaginary systems, imaginary points: the study of these latter, which does not strictly belong to that of three-dimensional space, is undertaken only for the sake of the former.

Loci which correspond to linear equations between the coordinates may at present be called *planes*, and their intersections *lines;* this is a purely projective definition, and these loci are not necessarily *flat* planes and *straight* lines in the metrical sense. Points, lines, and planes are included in the name *elements.*

The *metric* geometry of space* is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other algebraic forms. The word *power* will be explained as much as is wanted in the sequel; meanwhile it may be said that these fixed forms (called all together *the absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute; and that the power of an element of the absolute in regard to any arbitrary element is infinite. In other words, we *require* in general equations of the absolute in point-, line-, and plane-coordinates respectively.

A unicursal space the points of which may be represented uniquely by value-systems of the coordinates x, y, z, without the exception of any point-equations or loci-values, is called a *linear* space. This is merely a projective definition, and leaves the absolute, therefore the whole of metric geometry, undetermined.

There is a particular determination of the absolute in a linear space which is of the utmost importance. It is that in which the points of the absolute are those of a certain quadric surface, while the lines and planes of the absolute are those which touch this surface; or in which the three equations of

* This theory of metric geometry is due to Prof. Cayley: "Sixth Memoir on Quantics," *Phil. Trans.*, 1859.

the absolute are of the second degree. There are three cases* to be considered, as being the only ones of which observed space can form a part:—

(1) *Elliptic* geometry; all the elements of the absolute are imaginary.

(2) *Hyperbolic* geometry; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal.*

(3) *Parabolic* geometry; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z, and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j, the quantity

$$\frac{ab \,.\, ij}{\sqrt{(ai \,.\, aj \,.\, bi \,.\, bj)}} \equiv \overline{ab},$$

(which is a function of anharmonic ratios, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of either in regard to the other. The *distance* of these two points is an angle θ such that

$$\sin \theta = \overline{ab}.$$

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute,

* On this division see Dr Klein, "Ueber die so-genannte Nicht-Euklidische Geometrie," *Math. Annalen*, Bd. 4. The second case is the geometry of Lobatschewsky and Bolyai.

the power of the planes A, B in regard to one another is the quantity

$$\frac{AB \,.\, IJ}{\sqrt{(AI \,.\, AJ \,.\, BI \,.\, BJ)}} = \overline{AB},$$

and the angle between them is an angle ϕ such that

$$\sin\phi = \overline{AB}.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines or planes are conjugate in regard to the absolute, they are at right angles. Thus all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, such that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn *one* line perpendicular to a given plane; namely, the line joining the point to the pole of the plane. If, however, the point *is* the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) *In general, two lines can be drawn so that each meets two given lines at right angles, and these are polars of one another.* One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus *there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the*

absolute. I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versâ*. Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal*. If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore

* This motion is described in another connection by Drs Klein and Lie, *Math. Annalen*, Bd. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A' the polar motor; then we have $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$. Now $A + A'$ and $A - A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k. Then every rotor through the origin will be denoted by an expression of the form $ix + jy + kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix + jy + kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2 + y^2 + z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w + ix + jy + kz \equiv w + \rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If α be a rotor through the origin and q a quaternion, the product $q\alpha$ can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r. In this general case

the equation $q\alpha = r$ can only be interpreted by giving to α its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor α and the quaternion q, the final result will be the same as if it had been originally operated upon by the quaternion r. If, however, the axes of q and α are at right angles, the scalar part of r will be wanting, and we may write the equation $q\alpha = \rho$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor α produces the rotor ρ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q + r)\, s = qs + rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 *are equivalent to one another; and all rectangular twists of pitch* -1 *are equivalent to one another.*

The rectangular twist of pitch 1 shall be denoted by the symbol ω; the expression $\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude, obtained from it by a left-handed twist. During the operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore

$$\omega^2 = 1.$$

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore

$$\alpha + \omega\beta.$$

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now
$$\xi = \frac{1+\omega}{2}, \quad \eta = \frac{1-\omega}{2};$$

then
$$\xi^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \xi,$$

$$\eta^2 = \frac{1-2\omega+\omega^2}{4} = \frac{2-2\omega}{4} = \eta,$$

$$\xi\eta = \frac{1-\omega^2}{4} = 0.$$

Any motor $\alpha + \omega\beta$ can also be expressed in the form $\xi\gamma + \eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma + \eta\delta$ by ξ, the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma + \eta\delta$ into a motor $\xi\alpha + \eta\beta$. For if we perform the operation

$$\left(\xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta}\right)(\xi\gamma + \eta\delta),$$

remembering the laws of multiplication of ξ, η, we obtain the result $\xi\alpha + \eta\beta$. If then $\frac{\alpha}{\gamma} = q$, $\frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega \, . \, \frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion.*

The motor $\xi\alpha + \eta\beta$ will be a *rotor* if

$$S(\alpha+\beta)(\alpha-\beta)=0,$$

or if

$$T\alpha = T\beta;$$

and it is easy to see from this that the biquaternion $\xi q + \eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

1. *Position-Rotor of a Point.*—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being x_1, x_2, x_3, x_4, the equation to the absolute is $\Sigma x^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by

$$\dot{i}_1 \frac{x_1}{x_4} + \dot{i}_2 \frac{x_2}{x_4} + \dot{i}_3 \frac{x_3}{x_4}, \text{ or } \Sigma \dot{i}_k \frac{x_k}{x_4} \; (k = 1, 2, 3),$$

where $\dot{i}_1, \dot{i}_2, \dot{i}_3$ are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if

$$\rho = \dot{i}_1 \frac{x_1}{x_4} + \dot{i}_2 \frac{x_2}{x_4} + \dot{i}_3 \frac{x_3}{x_4},$$

$$(T\rho)^2 = \frac{x_1^2 + x_2^2 + x_3^2}{x_4^2} = \tan^2 \widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π. If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\breve{\rho} \equiv \breve{\alpha}$ (mod. $\breve{\pi}_\alpha$). To obviate this indetermination, there is required a one-valued unicursal function having the period π; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line.*—Let OM [fig. 25] be the perpendicular from the origin O upon the straight line MP;

and let ON be a line perpendicular to OM in the plane MOP. Then from the triangle MOP we have

$$\frac{\tan OM}{\tan OP} = \cos MOP;$$

or if $\quad OM = \alpha, \;\; OP = \rho, \;\; ON = \beta, \;\; T\alpha = T\rho \cos MOP;$

so that α is the component of ρ in the direction OM, and we have $\rho = \alpha + \beta x$, where x is some scalar.

By varying x, then, we get all the points in the line MP. But if α_1 is *any* particular value of ρ, the equation may just as well be written $\quad \rho = \alpha_1 + \beta x,$

where now α_1 is not necessarily perpendicular to β.

This form may be reduced to the preceding as follows:

To find the perpendicular from O, put $\delta T\rho = 0$; this gives

$$S\alpha_1\beta + \beta^2 x = 0,$$

and the equation becomes

$$\rho = \alpha_1 - \beta S \frac{\alpha_1}{\beta} - \beta x,$$

where $\alpha_1 - \beta S \frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. *Rotor along Straight Line whose Equation is given.*

Let OR [fig. 26] be the rotor through the origin which has right parallelism with MP. Then $\angle NOR = OM$. Let OK be perpendicular to ON and OM, and of such length that

$$\frac{\tan OK}{\tan ON} = \tan NOR.$$

Then, if $\quad \gamma = OK, \;\; OR = \beta + \gamma.$

Now $\frac{T\gamma}{T\beta} = T\alpha$, and $U\gamma = U\alpha\beta$, since γ is perpendicular to α and β. Hence $\gamma = \alpha\beta$; and if R be a rotor along MP, m a scalar,

$$\text{right vector of } R = \xi R = m\xi(\beta + \gamma) = m\xi(\beta + \alpha\beta),$$

so $\quad \text{left vector of } R = \eta R = m\eta(\beta - \gamma) = m\eta(\beta - \alpha\beta);$

therefore $\quad R = m(\beta + \omega\alpha\beta).$

Now if R have the same length as β, we have

$$\beta^2 = R^2 = m^2 (\beta^2 + \overline{\alpha\beta}^2) = m^2\beta^2 (1 - \alpha^2);$$

therefore
$$R = \frac{\beta + \omega\gamma\beta}{\sqrt{(1-\alpha^2)}}.$$

Conversely, equation to axis of rotor $\gamma + \omega\delta$ is

$$\rho = \frac{\delta}{\gamma} + \gamma x.$$

This finds the rotor in the case in which $\rho = \alpha + \beta x$, where $S\alpha\beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha - \beta S\frac{\alpha}{\beta} + \beta x,$$

whence
$$R = \frac{\beta + \omega\left(\alpha - \beta S\frac{\alpha}{\beta}\right)\beta}{\sqrt{\left(1 - \alpha^2 - \beta^2 S^2\frac{\alpha}{\beta} + 2S\alpha\beta S\frac{\alpha}{\beta}\right)}}$$

$$= \frac{\beta + \omega V\alpha\beta}{\sqrt{\left(1 + S\alpha\beta S\frac{\alpha}{\beta} - \alpha^2\right)}}.$$

4. *Rotor* ab *joining Points whose Position-Rotors are* α, β.

The equation of this rotor is

$$\rho = \alpha + (\beta - \alpha)x,$$

whence
$$mR = \beta - \alpha + \omega V\alpha\beta.$$

Now if a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 are the coordinates of the points, we have

$$(TR)^2 = \tan^2 ab = \frac{\Sigma(a_h b_k - a_k b_h)^2}{(\Sigma a_h b_h)^2} = -\frac{(\alpha-\beta)^2 + (V\alpha\beta)^2}{(1 - S\alpha\beta)^2},$$

therefore
$$R = \frac{\beta - \alpha + \omega V\alpha\beta}{1 - S\alpha\beta}.$$

COR.—If ρ [fig. 27] be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V\rho d\rho}{1 - \rho^2}.$$

5. *Rotor parallel to β through Point whose Position-Rotor is α.*

The general equation to a line through the point α is $\rho = \alpha + \lambda x$, where λ is any rotor through the origin. A rotor along this line is $\lambda + \omega V\alpha\lambda$; if this is right parallel to β, we have

$$\xi(\lambda + V\alpha\lambda) = \xi\beta, \qquad (\xi\omega = \xi)$$

or

$$\lambda + V\alpha\lambda = \beta.$$

Operating by $S\alpha$, we have, since $S.\alpha V\alpha\lambda = 0$,

$$S\alpha\lambda = S\alpha\beta,$$

whence, by addition, $\quad \lambda + \alpha\lambda = \beta + S\alpha\beta,$

and $\quad \lambda = (1+\alpha)^{-1}(\beta + S\alpha\beta) = \beta - (1+\alpha)^{-1}V\alpha\beta.$

The rotor required is

$$\lambda + \omega V\alpha\lambda, \text{ or } \lambda + \omega(\beta - \lambda).$$

This becomes, then,

$$\beta - (1+\alpha)^{-1}V\alpha\beta + \omega(1+\alpha)^{-1}V\alpha\beta = \beta - 2\eta(1+\alpha)^{-1}V\alpha\beta.$$

Instead of operating by $S\alpha$ on the equation

$$\lambda + V\alpha\lambda = \beta,$$

we might have operated with $V\alpha$, and got

$$V\alpha\lambda + \alpha V\alpha\lambda = V\alpha\beta, \text{ since } V.\alpha V\alpha\lambda = \alpha V\alpha\lambda,$$

therefore $\quad V\alpha\lambda = (1+\alpha)^{-1}V\alpha\beta,$

and $\quad \lambda = \beta - V\alpha\lambda = \beta - (1+\alpha)^{-1}V\alpha\beta.$

Similarly, we have for the rotor *left* parallel to β,

$$\lambda = \beta + (1-\alpha)^{-1}V\alpha\beta,$$

and the rotor is

$$\lambda + \omega(\lambda - \beta) = \beta + (1-\alpha)^{-1}V\alpha\beta + \omega(1-\alpha)^{-1}V\alpha\beta$$
$$= \beta + 2\xi(1-\alpha)^{-1}V\alpha\beta.$$

XXI.

GRAPHIC REPRESENTATION OF THE HARMONIC COMPONENTS OF A PERIODIC MOTION*.

FOURIER'S theorem asserts that any motion having the period P may be decomposed into simple harmonic motions having periods P, $\frac{1}{2}P$, $\frac{1}{3}P$, &c.; and assigns the amplitudes and phases of these motions by means of definite integrals. In fact, if $\phi(x+2\pi)=\phi(x)$ for all values of x,

$$\text{then} \quad \phi(x)=\tfrac{1}{2}b_0+b_1\cos x+b_2\cos 2x+\ldots+b_m\cos mx+\ldots$$
$$+a_1\sin x+a_2\sin 2x+\ldots+a_m\sin mx+\ldots,$$

where
$$\pi b_m=\int_{-\pi}^{+\pi}\phi(\alpha)\cos m\alpha\,d\alpha,$$

$$\pi a_m=\int_{-\pi}^{+\pi}\phi(\alpha)\sin m\alpha\,d\alpha;$$

and this is made applicable to the general case of periodic motion by putting $\frac{x}{2\pi}=\frac{t}{P}$, where t is the time elapsed since the era of reckoning.

The terms $b_m\cos mx+a_m\sin mx$ constitute a simple harmonic motion of period $\frac{P}{m}$; the object of the present communication is to represent this motion by a graphical construction.

If a right circular cylinder be made to revolve uniformly about its axis, while a pencil in contact with its surface has

* [From the *Proceedings of the London Mathematical Society*, Vol. v. No. 67, pp. 11—14.]

a rectilinear motion parallel to the axis, the pencil will trace out upon the cylinder a curve representing its motion. In particular, if this motion is simple harmonic and of a period equal to that of the revolution of the cylinder, the curve traced out will be an ellipse. The amplitude of the motion will be $r \cot\theta$, where r is the radius of the cylinder and θ the inclination of its axis to the plane of the ellipse; the phase at epoch is determined by the orientation of the major axis. This ellipse may thus be regarded as a graphical representation of the simple harmonic motion.

Now let the pencil have any arbitrary motion whose period is P. For convenience let us suppose that the axis of the cylinder is vertical. Then, if the cylinder be made to turn once round in the time P, a curve C_1 will be traced on it, representing the arbitrary periodic motion. Next let the cylinder turn round twice in the period P; a curve C_2 will be traced on it. And generally when the cylinder turns round m times in the time P, a curve C_m will be traced on it, going m times round the cylinder. All these curves C will be closed curves, because the motion is periodic.

At this point I call to mind Mr Hayward's extension of the meaning of "area," whereby it is made to have direction as well as magnitude*. Any closed contour, not necessarily plane, being given, the area of its projection on a plane is found to be a maximum when the plane has a certain aspect. The magnitude of this maximum area, considered as having this particular aspect, is called the area of the contour; and the area of the projection on any other plane is proportional to the cosine of the angle which it makes with the maximum plane. If, therefore, we know the area of the projection of a contour on any three planes at right angles, we can find the area of the projection on any other plane.

Now I say that it is possible to draw on the cylinder an ellipse which shall have the same area in magnitude and direction as the curve C_1. For the projection of this curve on a

* [*Proceedings of the London Mathematical Society*, Vol. IV. No. 59, pp. 289—291.]

plane perpendicular to the axis (*i.e.* a horizontal plane) is merely the circular section of the cylinder, which is the same as the projection of any ellipse traced on it. If therefore we cut the cylinder by a plane parallel to the maximum plane of the contour C_1, the elliptic section E_1 will have the same area as that contour in magnitude and direction.

The contour C_2 goes twice round the cylinder; therefore its projection on a plane perpendicular to the axis is the circle described twice in the same direction, and its area is twice that of the projection of any ellipse. If therefore we cut the cylinder by a plane parallel to the maximum plane of C_2, we shall obtain an ellipse E_2 whose area is half that of the contour C_2 and parallel to it. Similarly, the contour C_m goes m times round the cylinder, and a plane parallel to its maximum plane will determine an ellipse E_m whose area is $\frac{1}{m}$th of that of C_m.

Now first let a circle be drawn on the cylinder whose height is the mean height of C_1. On this circle is the middle point of all the component oscillations.

Next, while the cylinder goes round once in the period P, let the pencil follow the ellipse E_1; it will then have a simple harmonic motion of period P, which is, in fact, the first or fundamental component. Then, while the cylinder goes round twice in the time P let the pencil follow the ellipse E_2: the resulting simple harmonic motion of period $\frac{P}{2}$ is the second component. Generally, while the cylinder goes round m times in the time P, let the pencil follow the ellipse E_m; this simple harmonic motion is the mth component.

The demonstration of this result is very simple. The values of a_m and b_m may be written as follows:

$$\pi b_m = \int_{-\pi}^{+\pi} \phi x \cos m x d\alpha = \frac{1}{m}\int_{\alpha=-\pi}^{\alpha=\pi} \phi x d(\sin m x),$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi x \sin m x dx = -\frac{1}{m}\int_{\alpha=-\pi}^{\alpha=\pi} \phi \alpha d(\cos m x).$$

Suppose now that ϕx is set up vertically at P [fig. 28], when $FCA = mx$, then $d \cos mx$ is the element of CA, and $d \sin m\alpha$ is the element of CB; so that the differentials under the integral signs are respectively elements of the areas projected on vertical planes through AA' and BB'. If in these integrals we write $b_m \cos \alpha + a_m \sin \alpha$ in place of $\phi\alpha$, we get the areas of the corresponding projections of the ellipse E_m; these are πb_m and πa_m respectively. Thus the area of C_m projected on three planes at right angles is m times that of the ellipse E_m; or the areas of the two curves have the same aspect and are in the ratio $m : 1$; which was to be proved.

XXII.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS*.

THE following communication is an attempt to apply Jacobi's geometrical representation of the addition-theorem in elliptic functions to the theory of their transformation. For this purpose I use the said representation in the following form.

Consider two circles, one of which is wholly within the other, but which are not concentric; as in the figure [fig. 29]. The points of the outer circle may be uniquely represented by a parameter x, such that if 0 and ∞ are the points represented by these values respectively, and $0t$ is the tangent at the former, x is proportional to the ratio of the sines of the angles which $0x$ makes with the lines $0t$ and 0∞. Let the angle $x0t=\phi$; then, if we make $x=i\tan\phi$ ($i=\sqrt{-1}$), the values 1, -1 of the parameter will belong to the circular points at infinity. Let then k^{-1}, $-k^{-1}$ be the values belonging to the imaginary points of intersection of the two circles. Through the points 0, x let tangents be drawn to the inner circle, meeting the outer circle in c, ξ; these being so chosen that, when x moves continuously to 0, ξ will move continuously to c. Then Jacobi's theorem is that, if

$$x=\operatorname{sn}(u,k),\quad \xi=\operatorname{sn}(r,k),\quad c=\operatorname{sn}(\gamma,k),\quad \text{then } r=u+\gamma.$$

The extension to any two conics, made by Prof. Cayley, may be put into the same form. The representation of each point

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 90, 91, pp. 29—38.]

of a conic by a parameter is determinate when we know the parameters of any three points. Now the four intersections of two conics U, V may be divided into pairs in three ways, and will so determine three involutions upon the conic U. Let one of these be chosen, and let the parameters 0 and ∞ be assigned to its united points. Then, if the value 1 be assigned to one of the intersections, -1 will belong to another of them; and the remaining two will have parameters equal in magnitude but of contrary signs. Call these $\pm k^{-1}$, and draw the tangents as before; then Jacobi's theorem remains true, except that we must write $\pm \operatorname{sn}^{-1} \xi = \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c$; the sign of $\operatorname{sn}^{-1} \xi$ depending on the reality of the common tangents.

The proof of it depends on the symmetrical (2, 2) correspondence, considered by Euler, in relation to the addition of elliptic integrals. Given that the relation between the points x and ξ is that the line $x\xi$ touches the conic V, it is clear that to one position of x correspond two positions of ξ; and to one position of ξ, two positions of x; or, the points have a (2, 2) correspondence. Hence the equation connecting the quantities x, ξ must be of the second order in each; and it must obviously be symmetrical. Let the equation be

$$(ax^2 + 2bx + c)\,\xi^2 + 2\,(bx^2 + 2b'x + c')\,\xi + (cx^2 + 2c'x + c'') = 0;$$

or, which is the same thing,

$$(a\xi^2 + 2b\xi + c)\,x^2 + 2\,(b\xi^2 + 2b'\xi + c')\,x + (c\xi^2 + 2c'\xi + c'') = 0.$$

The values of x which make the two corresponding values of ξ coincide are given by the equation

$$X = (ax^2 + 2bx + c)\,(cx^2 + 2c'x + c'') - (bx^2 + 2b'x + c')^2 = 0,$$

and similarly the values of ξ which make the two corresponding values of x coincide are given by $\Xi = 0$, where Ξ is the same function of ξ that X is of x. Now, by differentiating the original equation, we easily find $dx : \sqrt{X} = d\xi : \sqrt{\Xi}$.

The roots of the equation $X = 0$ are clearly the parameters of the points of intersection of the two conics; for these are the only points on U from which two coincident tangents can be drawn to V. If, then, the parameters of these points have

been made equal to ± 1, $\pm k^{-1}$, X must be proportional to $(1-x^2)(1-k^2x^2)$, and the differential equation becomes

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}};$$

whence, since, when $x=0$, $\xi=c$,

$$\pm \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c = \operatorname{sn}^{-1} \xi,$$

which is the theorem in question.

If we change V into $U+\sigma V$, and allow σ to vary, this varying conic will always have the same intersections with U, and therefore k will be constant; but c will depend upon the value of σ. It is clear that c^2 is given uniquely when σ is given, but, when c is given, there are two values of σ. When $c=0$, the two conics must touch at 0, and therefore must coincide; thus both values of σ vanish. When $c=\infty$, the conic $U+\sigma V$ becomes a pair of lines intersecting on the line $0\,\infty$; let α and β be the values of σ which belong to these, and which are, of course, roots of the equation $\square(U+\sigma V)=0$.

Then we must have $c^2 = \dfrac{m\sigma^2}{(\sigma-\alpha)(\sigma-\beta)}$,

where m is an undetermined constant.

Suppose, now, that a polygon is inscribed in U by the following process; a tangent is drawn from x to a conic σ_1, meeting U again in x_1; then from x_1 to a conic σ_2, meeting U in x_2, and so on; finally, let x_{n-1} be joined to x. If $c_1, c_2 \dots c_{n-1}$ be the constants belonging to these conics respectively, we shall have

$$\begin{aligned}
\pm \operatorname{sn}^{-1} x_1 &= \operatorname{sn}^{-1} x \quad + \operatorname{sn}^{-1} c_1, \\
\pm \operatorname{sn}^{-1} x_2 &= \operatorname{sn}^{-1} x_1 \quad + \operatorname{sn}^{-1} c_2; \\
&\&\text{c.} \qquad\qquad \&\text{c.} \\
\pm \operatorname{sn}^{-1} x_{n-1} &= \operatorname{sn}^{-1} x_{n-2} + \operatorname{sn}^{-1} c_{n-1};
\end{aligned}$$

whence, by addition, with proper changes of sign,

$$\begin{aligned}
\pm \operatorname{sn}^{-1} x_{n-1} &= \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_1 \pm \operatorname{sn}^{-1} c_2 \pm \dots \pm \operatorname{sn}^{-1} c_{n-1} \\
&= \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_n, \text{ suppose.}
\end{aligned}$$

From this equation it appears that the last side of the polygon will always touch the same conic of the series $U+\sigma V$, wherever the starting-point x is taken. Or, if a polygon be inscribed in U, and move so that all but one of its sides touch conics of the series $U+\sigma V$, then the remaining side will also touch a conic of the series. This theorem of Poncelet's is proved in this way by Jacobi. It is to be noticed that the signs of the quantities c depend upon which tangent is drawn from the corresponding point x; the two tangents belong in a definite way to the two tangents from 0. The final value of c_n being thus determined, one of the two conics belonging to it is singled out by the sign given to $\operatorname{sn}^{-1} x_{n-1}$. In fact, the whole series of conics $U+\sigma V$ is divided into three parts by the three line-pairs it contains. For two conics in the same part of the series the signs of $\operatorname{sn}^{-1}\xi$ are certainly the same; for conics in different parts they may be different.

And it appears that the conics mentioned in the theorem may belong to any part of the series if the signs be properly chosen in each of the equations. It is remarked by Jacobi that with these restrictions the position of x_{n-1} does not depend upon the order in which the conics are taken.

Now suppose two such polygons to be drawn with the same system of conics, x being continuously moved to ξ, and at the same time x_1 to ξ_1, &c. We shall then have the equations

$$\pm \operatorname{sn}^{-1} x_1 = \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c_1,$$
$$\pm \operatorname{sn}^{-1} \xi_1 = \operatorname{sn}^{-1} \xi + \operatorname{sn}^{-1} c_1,$$

the signs being the same in both. Consequently

$$\pm (\operatorname{sn}^{-1} x_1 - \operatorname{sn}^{-1} \xi_1) = \operatorname{sn}^{-1} x - \operatorname{sn}^{-1} \xi,$$

or the lines $x_1\xi_1$, $x\xi$ touch the same conic of the series $U+\sigma V$.

Proceeding in this way, we may show that

$$\pm (\operatorname{sn}^{-1} x_r - \operatorname{sn}^{-1} \xi_r) = \operatorname{sn}^{-1} x - \operatorname{sn}^{-1} \xi;$$

whence it appears that the lines joining corresponding vertices of the two polygons all touch the same conic of the series when the n conics touched by the sides belong to the same part; but if they belong to different parts, each joining line touches one

of two conics which harmonically divide the pairs of lines $U+\alpha V$, $U+\beta V$.

Attending now only to the first case, it will be convenient to re-state the two theorems together, as follows:—

If a polygon be inscribed in a conic U *so that all its sides but one touch conics of the series* $U+\sigma V$, *the remaining side will also touch a conic of the series.*—(Poncelet's Theorem.)

When all these conics can pass continuously into one another without breaking up into two straight lines, the lines joining corresponding vertices of two such polygons will all touch a conic of the series.

Let us now consider the particular case in which all the sides of the moving polygon touch the same conic. Here the second theorem is true without restriction; the lines joining corresponding vertices of two such polygons will always touch one conic passing through the intersections of the other two. In this case also *the vertices of the variable polygon determine upon the conic* U *an involution of the* n^{th} *order*; that is to say, if the parameters of the vertices of one polygon be determined by an equation $p_n=0$ of the n^{th} order, and those of another polygon by an equation $q_n=0$, then the vertices of any third polygon will be determined by an equation $p_n-yq_n=0$, where y is a variable quantity, which we may call the parameter of the polygon. The relation between y, the parameter of the polygon, and x, the parameter of any one of its vertices, is $y=p_n:q_n$, where p_n, q_n are rational integral functions of the n^{th} order in x.

Suppose, then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V. Let $x, x_1 \dots x_{n-1}$ be the vertices of such a polygon; then, if $x=\operatorname{sn} u$, we must have $x_1=\operatorname{sn}(u+\gamma)$, $x_2=\operatorname{sn}(u+2\gamma)$, $x_3=\operatorname{sn}(u+3\gamma) \dots x_{n-1}=\operatorname{sn}\{u+(n-1)\gamma\}$, and consequently $x=\operatorname{sn}(u+n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $U-\sigma V$, which can be inscribed in n-gons inscribed in U is equal to the number of periods whose n^{th} parts are not congruent, that is, for n a prime number it is $n+1$.

Now let another polygon be drawn having a vertex at ξ, and let η be its parameter. Then the lines $x\xi$, $x_1\xi_1$ ... &c. will all touch a conic W. Let this conic be held fixed, and the two polygons moved so that the lines joining corresponding vertices always touch W. Then to every value of y will belong two values of η, and *vice versâ*, and this (2, 2) correspondence is symmetrical. Hence *a symmetrical* (2, 2) *correspondence between individual vertices implies a symmetrical* (2, 2) *correspondence between the polygons.*

Now, the parameter of every polygon is determined when we know the parameters of three polygons. Let the parameters of the polygons which have a vertex at 0, 1, and ∞ be made equal to 0, 1, and ∞ respectively; this amounts to saying that $p_n=0$ has a root 0, $q_n=0$ has a root ∞, and $p_n-q_n=0$ has a root 1. It is clear, then, from the symmetry of the figure, that y must be an odd function of x, so that $p_n+q_n=0$ will have a root -1. This amounts to saying that p_n is x multiplied by a rational integral function of x^2, and q_n (which is really only of the order $n-1$) is another rational integral function of x^2. This being so, let $\pm\lambda^{-1}$ be the parameter of those polygons which have vertices at the remaining two points of intersection of the conics. Then the quantities y and η are connected by a symmetrical (2, 2) correspondence such that the values of y which give equal values for η are ± 1, $\pm\lambda^{-1}$. Therefore, if $y=\operatorname{sn}(u', \lambda)$, we must have $\eta=\operatorname{sn}(u'+\delta, \lambda)$, where δ is a constant.

We have arranged that y is divisible by x or $\operatorname{sn} u$, by making 0 the parameter of that polygon which has a vertex at the point 0. It appears thus that y must also be divisible by $x_1, x_2 \ldots x_{n-1}$, since, when any one of these is zero, y vanishes. We may write, therefore, $y=mxx_1 \ldots x_{n-1}$, where m is a constant, since y is only infinite when one or other of the x is infinite. The products x_1x_{n-1}, x_2x_{n-2}, &c., are given rationally as ratios of quadratic functions of x by the original equation of (2, 2) correspondence. To determine m, we have

$$y=m\operatorname{sn} u\operatorname{sn}(u+\gamma)\operatorname{sn}(u+2\gamma)\ldots\operatorname{sn}\{u+(n-1)\gamma\}\,(n\gamma=aK+biK')\,;$$

but, since $y=1$, when $x=\operatorname{sn} u=1$, or $u=K$, we have also

$$1=m \operatorname{sn} K \operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\},$$

and therefore $y=\dfrac{\operatorname{sn} u \operatorname{sn}(u+\gamma) \ldots \operatorname{sn}\{u+(n-1)\gamma\}}{\operatorname{sn} K \operatorname{sn}(K+\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}$;

the denominator may, of course, be simplified, as in Jacobi's formula, but for present purposes it may be left as it stands. Now λ^{-1} is the value of y when $x=k^{-1}$, or when $u=K+iK'$,

$$\frac{\lambda}{k}=\frac{\operatorname{sn} K \operatorname{sn}(K+\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}(K+iK') \operatorname{sn}(K+iK'+\gamma) \ldots \operatorname{sn}\{K+iK'+(n-1)\gamma\}},$$

which, again, may be easily reduced.

If we now assume that $u=Mu'$, where $y=\operatorname{sn}(u', \lambda)$ and M is a constant, we may determine M by observing that it is the value of $x : y$ when both of them are zero. Namely we have

$$M=\frac{\operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}\gamma \operatorname{sn} 2\gamma \ldots \operatorname{sn}(n-1)\gamma}.$$

And thus, with the exception of the assumption just made, the theory of transformation is established by means of that of the in-and-circumscribed polygon.

Let us now endeavour to generalize the theory of the in-and-circumscribed polygon. In the first place, we may observe that every involution of the third order gives rise to an elliptic transformation; for two triangles inscribed in a conic are always circumscribed to the same conic*. For convenience, we will now consider the involution as determined on the conic V; namely, the triangles form groups of three tangents which are in involution. And we may now generalize our theorem as follows:—

If a complete n-*gram move with its sides touching a conic so*

* Every substitution $y=\dfrac{U}{V}$, where U, V are cubic functions of x, has an elliptic differential which it transforms. The cubic forms U, V are first polars of two points in regard to a single quartic form F (Gundelfinger, *Math. Annalen*). Let $X=0$ give the four points whose first polars in regard to F have a square factor; then $dx : \sqrt{X}$ is the elliptic differential required. We have $X=jF+iH$, where H is the Hessian of F, and i, j the quadrinvariant and cubinvariant. [Cf. however p. 221, infra.]

that they form groups of n *tangents in involution, the locus of the* $\frac{1}{2}$ n (n − 1) *vertices is a curve of order* n − 1.

For, consider the number of points which the curve has in common with any one tangent of the conic. It determines uniquely in the involution the group of n tangents to which it belongs, and can have no other point on the locus of intersections except those $n-1$ in which it meets the other $n-1$ tangents of this group. Therefore &c.

If, in a curve of order n − 1, *it is possible to inscribe one complete* n*-gram whose sides all touch a conic, then it is possible to inscribe a singly infinite number, and the sides determine upon the conic an involution of the* n*th* *order.*

Let $A, B, C, \ldots N=0$ be the equations of the sides of the complete n-gram, then the equation of the curve may be written

$$\frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}+\ldots+\frac{\nu}{N}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ldots(1),$$

where $\alpha, \beta, \gamma \ldots \nu$ are constants. But since A is a tangent to the conic, its equation may be written in the form $X+aY+a^2Z=0$, where a is the parameter of the tangent and X, Y, Z three fixed lines. The condition for three tangents to meet in a point is

$$0=\begin{vmatrix} 1, & x, & x^2 \\ 1, & y, & y^2 \\ 1, & z, & z^2 \end{vmatrix}=(y-z)(z-x)(x-y) \ldots\ldots\ldots\ldots(2),$$

where x, y, z are their parameters. Hence the condition that the tangents whose parameters are x, y shall meet on the curve (1) is

$$\frac{\alpha}{(a-x)(a-y)}+\frac{\beta}{(b-x)(b-y)}+\ldots+\frac{\nu}{(n-x)(n-y)}=0 \ldots(3);$$

or, which is the same thing,

$$\Sigma\frac{\alpha}{a-x}=\Sigma\frac{\alpha}{a-y},$$

and therefore, if tangents xy and also xz meet on the curve, it follows that yz will meet on the curve. Starting, then, from any arbitrary tangent x, we can find the $n-1$ points in which

this meets the curve, and draw from them $n-1$ other tangents; the intersection of any two of these will lie on the curve, by what we have just proved. That is to say, we can inscribe a complete n-gram which shall have for one side any arbitrary tangent of the conic.

Now, suppose that x is given, then, regarding the equation (3) as determining $n-1$ values of y, we can find the product of the roots. Namely, it is

$$=\Sigma\frac{\alpha\,.\,\Pi a}{a\,(a-x)}\;:\;\Sigma\frac{\alpha}{a-x},\quad \Pi a = a\,.\,b\,.\,c\,\ldots\,n.$$

If we multiply this product by x, we obtain the product of the parameters of all the tangents forming a complete n-gram; let this be called λ; then, observing that

$$\frac{\alpha x}{a\,(a-x)}=\frac{\alpha}{a}-\frac{\alpha}{a-x},$$

we shall find $$\left(1-\frac{\lambda}{\Pi a}\right)\Sigma\frac{\alpha}{a-x}=\Sigma\frac{\alpha}{a}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now this is an equation of the n^{th} order in x, the roots of which are the parameters of the sides of a complete n-gram; and λ is the product of these roots. Since λ is linearly involved, the equation shows that these groups of n tangents form an involution of the n^{th} order, and that λ is proportional to the parameter of such a group in the involution when the groups containing the tangents 0, ∞ are made to have the parameters 0, ∞ respectively. It appears also that the sum of the roots, sum of their products in pairs, &c., are each given as linear functions of λ, and might each be used as parameters of the involution.

We shall now endeavour to find an expression for $ABC\ldots N$.

Let $1-\lambda\,(\Pi a)^{-1} : \Sigma\alpha\, a^{-1}$ be called θ, and let $\Pi_a\,(x-a)$ mean the product $(x-a)\,(x-b)\ldots(x-n)$, then the equation for x may be put into the form

$$\Pi_a\,(x-a)+\theta\Pi_a\,(x-a)\,.\,\Sigma\alpha\,(x-a)^{-1}=0\;\;\ldots\ldots\ldots(5),$$

where the first term is of order n, and the second of order

$n-1$ in x. By a slight change of notation, let the n roots be called $x_1, x_2, \ldots x_n$, and let $\Pi_x (y-x)$ mean

$$(y-x_1)(y-x_2)\ldots(y-x_n).$$

Then we have

$$(x-x_1)(x-x_2)\ldots(x-x_n) = \Pi_a (x-a) + \theta\Pi_a (x-a)\,\Sigma\alpha\,(x-a)^{-1};$$

and therefore

$$\Pi_x (y-x) = \Pi_a (y-a) + \theta\Pi_a (y-a)\,\Sigma\alpha\,(y-a)^{-1}.$$

Multiplying together two such equations, we obtain

$$\begin{aligned}\Pi_x (y-x)\,.\,\Pi_x (z-x) = \\ &\Pi_a (y-a)\,.\,\Pi_a (z-a) \\ &+ \theta\Pi_a (y-a)\,.\,\Pi_a (z-a)\,.\,\{\Sigma\alpha\,(y-a)^{-1} + \Sigma\alpha\,(z-a)^{-1}\} \\ &+ \theta^2\Pi_a (y-a)\,.\,\Pi_a (z-a)\,.\,\Sigma\alpha\,(y-a)^{-1}\,.\,\Sigma\alpha\,(z-a)^{-1}.\end{aligned}$$

Now, if we examine this equation, we shall find that the left-hand member is $A'B'C'\ldots N'$, where $A'=0$, $B'=0\ldots$ are the tangents which make up the n-gram belonging to the parameter θ; the first term on the right is $ABC\ldots N$, the n-gram for $\theta=0$, and the last term is $\theta^2 Z_1 Z_2 \ldots Z_n$, the n-gram for $\theta=\infty$, which has the line $Z=0$ in it. Thus we may write the equation

$$\Pi A' = \Pi A + \theta P_n + \theta^2 \Pi Z \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and it only remains to find the nature of the curve P_n of the n^{th} order. We may see from its equation,

$$0 = \Pi_a (y-a)\,\Pi_a (z-a)\,\{\Sigma\alpha\,(y-a)^{-1} + \Sigma\alpha\,(z-a)^{-1}\},$$

that it passes through the points of contact and all the intersections of the n tangents ΠA; and then it is clear, from the symmetry, that it must pass through the points of contact and all the intersections of the tangents ΠZ. But perhaps the simplest way is to consider the envelope of the n-gram $\Pi A'$, which we know must consist of the conic K_2 once, and the locus of the nodes C_{n-1} twice; thus we shall have

$$4\Pi A\,.\,\Pi Z - P_n^2 = K_2 C_{n-1}^2$$

to a factor *près*, and this equation gives at once the intersections of P_n with K_2, and with the n-gram.

We may now state the following propositions:—

Given any two in-and-circumscribed n*-grams* ΠX *and* ΠZ, *there exists always a curve* P_n *of order* n *which passes through their* $n(n-1)$ *vertices and their* 2n *points of contact with the conic.*

The equation of any other n*-gram may be written in the form*

$$0 = \Pi X + \lambda P_n + \lambda^2 \Pi Z.$$

The relation between λ, *the parameter of the* n*-gram, and* x *the parameter of one of its sides, is*

$$(\Pi a - \lambda)\, \Sigma a\, (a - x)^{-1} = \Pi a \,.\, \Sigma a a^{-1},$$

and λ *is the product of the roots of this equation.*

I have here taken ΠX, $(X_1 X_2 \dots X_n)$ for the first n-gram, corresponding to $\lambda = 0$, instead of ΠA, corresponding to $\lambda = \Pi a$ or $\theta = 0$.

We may show, conversely, that if the envelope of

$$0 = P + \lambda Q + \lambda^2 R,$$

where $P = 0$, $Q = 0$, $R = 0$ are three curves of the n^{th} order, is $4PR - Q^2 = K_2 C^2$, C being of order $n-1$; then P and R are each an assemblage of n straight lines. For the curve P has nodes on all its intersections with C, since $4PR = Q^2 + K_2 C^2$; that is, $\frac{1}{2}n(n-1)$ nodes, so that it must consist of n straight lines.

{This point of view immediately suggests the extension of the whole theory to quadric surfaces. If the envelope of $P + \theta Q + \phi R + \theta\phi S$ is $PS - QR = K_2 C^2$, where P, Q, R, S are of order n, and C of order $n-1$, each of the surfaces P, Q, R, S will meet the quadric K_2 in two curves of the n^{th} order, and therefore will have $\frac{1}{2}n^2$ contacts with it; and similarly will meet C_{n-1} in two curves of order $\frac{1}{2}n(n-1)$, which intersect in $\frac{1}{4}n^2(n-1)$ points; these are not contacts, but nodes on the surface P. We thus get a theory of surfaces of order n, having $\frac{1}{2}n^2$ contacts with a quadric surface, and $\frac{1}{4}n^2(n-1)$ nodes on a fixed surface of order $n-1$. It appears that n must be even, and of course the variable surface is subject to other conditions.

Thus, in the case of a cone doubly tangent to the quadric, and having its vertex on a fixed plane, it has also to pass through two fixed points on the plane and four on the quadric.

The application of the identity $4PR - Q^2 = K_2 C^2$ to surfaces only reproduces the theory of the plane conic }

At any point of intersection of the two n-grams,

$$0 = \Pi X + xP_n + x^2\, \Pi Z,$$

$$0 = \Pi X + yP_n + y^2\, \Pi Z,$$

we shall have $xy : -x-y : 1 = \Pi X : P_n : \Pi Z.$

Consequently, any symmetrical (m, m) correspondence between the two n-grams expresses that they intersect on a curve of order mn, namely $(\Pi X, -P_n, \Pi Z)^m$, if the equation of the (m, m) correspondence is $(xy, x+y, 1)^m$.

If we substitute the values $xy : -x-y : 1$ for $\Pi X : P_n : \Pi Z$, in the equation of a third n-gram $0 = \Pi X + zP_n + z^2\, \Pi Z$, we shall get simply $(z-x)(z-y) = 0$. Consider, then, m different n-grams ΠA_1, $\Pi A_2 \ldots \Pi A_m$, and form the curve of order $n(m-1)$

$$\frac{\beta_1}{\Pi A_1} + \frac{\beta_2}{\Pi A_2} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If the n-grams x, y meet on this curve we shall have

$$\Sigma \frac{\beta}{(a-x)(a-y)} = 0\,;$$

or, what is the same thing,

$$\Sigma \frac{\beta}{a-x} = \Sigma \frac{\beta}{a-y},$$

where $a_1, a_2 \ldots a_m$ are the parameters of the n-grams. It follows, as before, that a singly infinite number of groups of n-grams can be totally inscribed in the curve (7); a group being totally inscribed when all the intersections of any two n-grams of the group are on the curve.

So far we have dealt only with totally inscribed n-grams; and, as this case is represented only by the triangle when the curve of inscription is a conic, it might seem that there should

be more general theorems corresponding to the inscription of polygons of a greater number of sides in a conic. But, in fact, the case of total inscription is the general case, and all others are cases of decomposition of the curve C_{n-1}. Consider, for example, a hexagon inscribed in a conic. If we produce all the sides, we shall get nine more intersections; three of these lie on a straight line, and the other six on a conic which circumscribes two triangles. These two conics and the straight line make up the curve C_5 of the fifth order. Again, let eight lines A, B, C, D, F, G, H, K (fig. 30) touch a conic, and let the twelve points marked o in the figure touch a cubic; then the octagram may be moved round the conic so as to keep these twelve points on the cubic. The points marked × will lie on a fixed straight line, and a second cubic will pass through the intersections of A, B, C, D among themselves, and of F, G, H, K among themselves. These two cubics and the straight line make up the curve C_7 of the seventh order; and it is easy to see that there is an analogous case for any even number of lines. In order that a porismatic polygon may be inscribed in a curve, it is necessary that either the order or the curve or the number of sides should be even.

XXIII.

NOTES ON THE COMMUNICATION ENTITLED "ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS."*

SOME of the following notes† would have been incorporated in the paper by the process of revision for the press, if that had not been kindly performed for me during an enforced absence from the English climate. As regards all but one of them, I am glad of the opportunity which has thus been afforded me of extension and correction. But it is a matter of great regret to me that I discovered too late the priority of M. Darboux in the principal theorem of the second part of the paper; viz., the porismatic character of a polygram circumscribed to a conic and totally inscribed in a curve of order one less than the number of sides. In one of the notes to a book which it is almost inexcusable in a geometer not to have read, marked, learned, and inwardly digested‡, M. Darboux has stated and proved the theorem, and has followed it by further investigations of the highest interest and importance. The method even of my investigation is the same as that of M. Darboux (as indeed was inevitable from the nature of the subject), namely, the representation of a point in a plane by means of the parameters of the tangents drawn from it to a fixed conic. It is not the first time

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 102, 103, pp. 225—233.]

† These remarks apply also to certain developments which I have since thought it better to communicate under separate titles.

‡ *Sur une classe remarquable de courbes et de surfaces algébriques.* Paris, Gauthier-Villars, 1873. Note II., p. 183.

that I have had the honour of following, however distantly, in the footsteps of that eminent geometer; but on other occasions it has been my good fortune to discover the fact in time.

Completion of the Geometric Proof of the Transformation-Formulæ.

In my former paper one point was assumed as given by the analytical theory of transformation, namely, that the new argument u' is equal to the old argument u divided by a constant M. Having now found a simple proof of this, I will take the liberty of re-stating in outline, for the sake of clearness, the whole demonstration; availing myself of a remark of M. Darboux.

It is proved by Jacobi's method that if $x_1\, x_2 \ldots x_n$ are parameters of the points of contact of the n sides of a polygon circumscribed to a conic U and inscribed in a conic V, and if ± 1, $\pm k^{-1}$ are the parameters of the points of contact of the common tangents of the two conics, then

$$x_1 = \operatorname{sn} u,\ x_2 = \operatorname{sn}(u+\gamma),\ x_3 = \operatorname{sn}(u+2\gamma),\ \ldots\ x_n = \operatorname{sn}\{u+(n-1)\gamma\},$$

where $n\gamma = 4K + 4iK'$, and the modulus of the elliptic function is k.

This being so, an infinite number of in-and-circumscribed polygons can be drawn.

If $p_n = 0$ be the equation in x whose roots are $x_1\, x_2 \ldots x_n$, and $q_n = 0$ an equation in x whose roots are the parameters of the sides of another such polygon; then $p_n - yq_n = 0$ will have for its roots the parameters of the sides of an in-and-circumscribed polygon, whatever value be given to y.

For the locus of the $\frac{1}{2}n(n-1)$ intersections of the tangents at the n points $p_n - yq_n = 0$, when y is made to vary, is a curve of order $n-1$, which has $2n$ points in common with the conic V, and therefore contains that conic entirely*.

The quantity y being now regarded as the parameter of a varying polygon, let p_n be chosen to represent that polygon

* This is in substance the remark of M. Darboux referred to. *Op. cit.*, p. 190.

which has the parameter of one of its sides equal to zero, and q_n to represent that which has the parameter of one of its sides infinite. Then y, which is $p_n : q_n$, will be an odd function of x, because of the symmetry of the figure and the fact that $q_n = 0$ (having one root infinite) is only of degree $n-1$.

If then p_n and q_n be affected with such constant multipliers that $y=1$ when $x=1$, we must have $y=-1$ when $x=-1$. And we may suppose that $y = \pm\lambda^{-1}$ when $x = \pm k^{-1}$.

Now the intersections of corresponding sides of two in-and-circumscribed polygons lie on a conic touching the common tangents of U and V.

For the parameters being respectively sn u, sn $(u+\gamma)$, &c., and sn v, sn $(v+\gamma)$, &c., the common difference of the arguments is $u-v$.

If we suppose the two polygons to vary subject to this condition, their parameters y and η will be connected by a (2, 2) correspondence, such that the values of y which make the two corresponding values of η coincide are ± 1, $\pm\lambda^{-1}$.

Therefore, if $y = \mathrm{sn}\,(u', \lambda)$, we must have $\eta = \mathrm{sn}\,(u'+c', \lambda)$, where c' is a constant. But the relation between corresponding sides x, ξ, of these polygons is $x = \mathrm{sn}\,(u, k)$, $\xi = \mathrm{sn}\,(u+c, k)$, since they intersect on the fixed conic V.

Hence the quantities u, u' are so related that a constant difference between two values of u implies a constant difference between the corresponding values of u'. Hence* (by Euclid's definition of proportion) a varying difference between two values of u implies a *proportional* difference between the corresponding values of u'. But $u'=0$ when $u=0$; therefore the two quantities are proportional, and $u = Mu'$ where M is constant.

Now the relation between the parameters of two consecutive sides of a polygon being of the second degree in each, the products x_2x_n, x_3x_{n-1}, &c., are given as ratios of quadratic functions of x_1. Hence the product $x_1x_2 \dots x_n$, regarded as a function of

* This is Archimedes' proof that a body which passes over equal spaces in equal times will pass over proportional spaces in unequal times.

x_1, is a rational fraction whose numerator is of order n, and whose denominator is of order $n-1$. But y is a rational fraction whose numerator and denominator are of just these orders; and y vanishes whenever one of the quantities $x_1, x_2 \dots x_n$ vanishes, and becomes infinite when one of them becomes infinite. Therefore

$$y = mx_1x_2x_3 \dots\dots x_n,$$

where m is a constant; that is to say,

$$\operatorname{sn}\left(\frac{u}{M}, \lambda\right) = m \operatorname{sn} u \operatorname{sn}(u+\gamma) \operatorname{sn}(u+2\gamma) \dots\dots \operatorname{sn}\{u+(n-1)\gamma\},$$

when $n\gamma = 4K + 4iK'$. We determine m by remarking that $y = 1$, when $x = 1$, or when $u = K$; and then λ, by remarking that $y = \lambda^{-1}$, when $x = k^{-1}$, or when $u = K + iK'$. Finally M is determined by differentiating the equation and making $u = 0$.

(Cayley's Theorem.)—Every Cubic Transformation has an Elliptic Differential which it transforms.

This theorem was given by Prof. Cayley in the *Philosophical Magazine*, Vol. 15 (Fourth Series), p. 363. I here reproduce his investigation, slightly altered to suit the generalization which follows. On the very beautiful solution of the complete question (Given the elliptic differential, to find the transformation) by Hermite (*Crelle*, vol. 60, p. 304) and Clebsch (*Theorie der binären alg. Formen*, p. 405), I hope to say something at another time*.

Let U, V be any two cubic functions of x, and consider the transformation $y = \dfrac{U}{V}$.

Suppose that

$$\text{Disct } (U - Vy) = A + By + Cy^2 + Dy^3 + Ey^4,$$

where, of course, $A = $ Disct. U, $E = $ Disct. V. And let y_1, y_2, y_3, y_4

* The foot-note in my previous paper gave an erroneous expression for X. The article there referred to (Gundelfinger, *Math. Annalen*, Vol. VII., p. 452) is a simplification of the method and results of Clebsch in regard to the typical representation of two cubics.

be the roots of the equation Disct. $(U - Vy) = 0$. Then each of the cubics $U - Vy_1$, $U - Vy_2$, $U - Vy_3$, $U - Vy_4$ has a square factor, because its discriminant vanishes. Now, if $U - Vy_1$ has a square factor $(x - a)^2$, then $x - a$ divides $U' - V'y_1$; that is, for the value $x = a$ we have at the same time

$$\begin{matrix} U - Vy_1 = 0 \\ U' - V'y_1 = 0 \end{matrix} \qquad \left\{ U',\ V' = \frac{dU}{dx},\ \frac{dV}{dx} \right\},$$

and therefore also $VU' - V'U = 0$; that is to say, those four linear factors which are squared in the cubics $U - Vy_1$, &c., occur as single factors in $VU' - V'U$. It follows that

$$A\,(U - Vy_1)\,(U - Vy_2)\,(U - Vy_3)\,(U - Vy_4) = (VU' - V'U)^2 . X,$$

where X represents the product of the single factors of the four cubics. Or, which is the same thing,

$$AU^4 + BU^3V + CU^2V^2 + DUV^3 + EV^4 = (VU' - V'U)^2 X.$$

Now, since $y = \dfrac{U}{V}$, we have $dy = \dfrac{VU' - V'U}{V^2}\,dx$. Hence, if we transform the differential $\dfrac{dy}{\sqrt{\text{Disct.}\,(U - Vy)}}$ by the substitution $y = \dfrac{U}{V}$, we get

$$\frac{VU' - V'U}{V^2} \cdot \frac{V^2}{(VU' - V'U)\sqrt{X}}\,dx \text{ or } \frac{dx}{\sqrt{X}}.$$

That is, we have
$$\frac{dy}{\sqrt{\text{Disct.}\,(U - Vy)}} = \frac{dx}{\sqrt{X}},$$

where y is connected with x by the equation $y = \dfrac{U}{V}$, which is the theorem in question.

New Stand-Point for the Algebraic Transformation-Theory.

We may generalize this result by applying an analogous treatment to transformations of any order. The problem is considered in the following form: Given a transformation $y = U : V$, it is required to find—

(1) What are the necessary and sufficient conditions to be satisfied by the functions U, V, in order that the transformation $y = U : V$ may be able to transform an elliptic differential;

(2) These conditions being supposed satisfied, what is the differential which can thus be transformed.

We will consider first the case in which U and V are of odd order, say $2m + 1$, or, to speak more correctly, the case in which $U - Vy$ is of order $2m + 1$ in x.

The necessary conditions may at once be derived from consideration of the varying in-and-circumscribed polygon the parameters of whose sides are the roots of the equation $U - Vy = 0$.

Starting with any one side of the polygon, which touches what we may call the inner conic, we find its intersections with the outer conic, and then from these draw new tangents to the inner conic. Proceeding in this way symmetrically on both sides of the original tangent, we find at last that the two tangents to the inner conic meet on the same point of the outer conic. We must clearly end with two tangents, and not with one, because the polygon has an odd number of sides.

We might, however, start with a vertex of the polygon on the outer conic, draw two tangents to the inner conic, then from their intersection with the outer conic two more tangents, and so on: at last we shall reach a pair of vertices such that the line joining them touches the inner conic. In this case we must end with two vertices on one side, not with one vertex on two sides, for the same reason as before, that we have supposed the polygon to have an odd number of sides.

Now suppose that in the first mode of construction we start with a common tangent to the two conics; then its two intersections with the outer conic will coincide, and consequently the tangents from them to the inner conic coincide also. We may, however, go on with the construction; and, after drawing $m + 1$ successive tangents, we shall have an exceptional case of an in-and-circumscribed polygon, in which the side first drawn (the common tangent of the two conics) counts singly, and each

of the m others counts doubly, so that the polygon has altogether $2m+1$ sides. But the last pair of tangents being coincident, must be regarded as intersecting on the inner conic; and therefore their point of contact must be an intersection of the two conics.

So that we cannot by the second mode of construction get a degenerate polygon of an odd number of sides different from those just considered. If we start with a vertex at a point of intersection of the two conics, the tangents drawn from this to the inner conic will of course coincide, and so therefore will the points in which they meet the outer conic again, and we may continue the process; but we only have a right to stop when the line joining the two last coincident vertices on the outer conic (*i. e.*, the tangent at that point to the outer conic) touches the inner conic; that is to say, when we come upon a common tangent of the two conics.

What actually happens may be illustrated by the case of an in-and-circumscribed pentagon. Let ab be a common tangent of the two conics, touching the inner conic at a, and the outer at b. From b draw the other tangent bc to the inner conic, meeting the outer conic again at c; then from c draw the other tangent cd to the inner conic meeting the outer conic again at d. Then, *if pentagons can be drawn inscribed to the outer conic, and circumscribed to the inner, the point* d *will be an intersection of the two conics.* And the pentagon whose sides are dc, cb, bab, bc, cd is a degenerate case of an in-and-circumscribed pentagon; the side bab being single, and the sides bc, cd, each of them double.

Observe that, if d were not an intersection of the two conics, we should still have an improper solution of the problem, to find five points on the outer conic such that the line joining every successive two shall touch the inner conic. But if we started from d as an intersection of the conics, and then found the points c, b, a as before, except that ba is now not a tangent to the outer conic at b, we should not have found a solution of that problem, but of this other—To find five tangents to the inner conic, so that the intersection of every successive two shall be upon the outer conic.

For the purpose of our present investigation, the result may be stated thus: an in-and-circumscribed polygon of an odd number of sides can only have two sides coincident, when one of its sides is a common tangent of the two conics, and all the others coincide in pairs.

Or, the equation $U - Vy = 0$ can only have coincident roots, when one represents a common tangent, and the others coincide in pairs: so that $U - Vy$ has one single factor, and m square factors. Moreover, there are four values of y, and four only, which bring this about. Now the discriminant of $U - Vy$ is of the order $4m$. Hence *we must have*

$$\text{Disct.}\ (U - Vy) = Y^m, \text{ where } Y = (1, y)^4,$$

and if $y_1 y_2 y_3 y_4$ *are the roots of the equation* $(1, y)^4 = 0$, *then each of the quantities* $U - Vy_1$, &c., *has one single factor and* m *square factors.*

These conditions are necessary; we shall shew that they are sufficient, by solving the second part of the problem.

In the first place, we have

$$(1, 0)^4 . (U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (U, V)^4,$$

and therefore, as before,

$$(U, V)^4 = (VU' - V'U)^2 X,$$

where $(VU' - V'U)^2$ is the product of all the square factors, and X of the four single factors. From this it follows immediately that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

if $y = \dfrac{U}{V}$; which is the transformation required. The result may thus be stated :—

The necessary and sufficient conditions that the substitution $y = U : V$ shall be able to transform an elliptic differential, $U - Vy$ being of order $2m + 1$ in x, are that Disct. $(U - Vy)$ shall be a perfect mth power, and that those forms $U - Vy$ which have a square factor at all shall have m square factors. This

being so, the differential $\dfrac{dy}{\sqrt[m^2]{\text{Disct. } (U - Vy)}}$ will be transformed by the given substitution into $\dfrac{dx}{\sqrt{X}}$, where X is the product of the four single factors.

It may be observed that, if those forms $U - Vy$ which have a square factor at all have m square factors, it will follow that Disct. $(U - Vy)$ is a perfect mth power; but the converse is not true.

Passing now to the case of a transformation of even order, we enquire, as before, in what cases the in-and-circumscribed polygon can have two sides coincident. If we start with a tangent to the inner conic, and from its intersections with the outer conic draw two more tangents, and so on; there cannot be an in-and-circumscribed polygon of an even number of sides, unless we come to a pair of intersections such that the line joining them touches the inner conic. Suppose then that the first side is a common tangent, so that its two intersections with the outer conic coincide, and that we draw another tangent from this point, another from its second point of intersection, and so on; we must finally come to a point on the outer conic where the tangent touches the inner conic; that is, we must come to another common tangent. In the case of a quadrilateral, for example, let ab be a common tangent, touching the inner conic at a and the outer at b. From b draw the other tangent to the inner conic, meeting the outer again at c; then cd must be also a common tangent, touching the outer conic at c and the inner at d. Thus the sides of the degenerate quadrilateral are bab, bc, cdc, cb, the sides bab, cdc counting singly, and bc double. And in general the two common tangents will count singly, and all the rest double. It is manifest that there are only two degenerate polygons of this kind, each containing two of the four common tangents.

The second construction also gives us two degenerate polygons, but of quite a different character. Starting with a point on the outer conic, we draw two tangents to the inner, and from their new intersections with the outer, two more, and so on;

we must at last come to two tangents which meet on the outer conic. If then our starting-point is a point of intersection, so that the two tangents coincide all through the process, we must come to a pair whose intersection, that is, their point of contact with the inner conic, is on the outer conic; or, which is the same thing, we must come to another intersection of the conics. To use again the quadrilateral as an illustration, the tangents to the inner conic at two points of intersection α and γ must meet on the outer conic at β, and the sides of the quadrilateral are then $\alpha\beta$, $\beta\gamma$, $\gamma\beta$, $\beta\alpha$, so that *all* of them count double. And generally, in degenerate polygons of this kind, all the sides count double. There are clearly two such degenerate polygons, each having two points of intersection for vertices.

To sum up, then, there are four degenerate polygons of even order $2m$; two of them have each two common tangents as sides, and two of them have each two points of intersection as vertices. The former have the common tangents as single sides, and all the rest double; the latter have all their sides double.

It follows that, if the substitution $y = U : V$ is capable of transforming an elliptic differential, $U - Vy$ being of order $2m$ in x, there are only four values of y which make $U - Vy$ have a square factor; two of these make it have m square factors, and the other two make it have $m-1$ square factors and two single factors. Consequently the former two are m-fold roots, and the latter two $(m-1)$-fold roots, of the equation Disct. $(U - Vy) = 0$. That is to say, we have

$$\text{Disct. } (U - Vy) = Y^{m-1} . (y - y_1)(y - y_2),$$

where $\quad Y = k(y - y_1)(y - y_2)(y - y_3)(y - y_4) = (1, y)^4$, say.

Hence, as before, $\quad (U, V)^4 = (VU' - V'U)^2 . X,$

where X is the product of the four single factors due to y_3 and y_4. From these equations it follows directly that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

which is the transformation required.

In regard to these conditions it is to be observed that in general they imply a special constitution of the quantics U, V,

as well as a special relation of them to each other. This consideration, however, does not come in until the sixth order of transformation is reached. Thus, in the case of the quartic transformation, the only condition is that U, V shall be simultaneously reducible to the canonical form; which being so, we may find linear combinations of them such that one is the Hessian of the other, thus falling back upon Hermite's very elegant form. In the quintic transformation U may be taken arbitrarily, but the involution $U - Vy$ is then completely determined. In the sextic transformation, however, U and V must each be the product of three quartics in involution (viz., the same involution in the two cases); so that a certain invariant of each must vanish. (Salmon's *Higher Algebra*, p. 210 and Appendix; see Clebsch, *Alg. Formen*, p. 298.)

*ADDITIONS TO PAPER ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

1. Completion of doctrine of in-and-circumscribed polygon so as to make it a formal proof of the transformation theory. In p. [210] the quantities u, u' are proportional because constant difference in the values of u means constant difference in the values of u', and they vanish together.

2. Porismatic representation of spherical harmonics as sums of sectorial harmonics. "Nodal curve" of order n may be expressed as a sum of n^{th} powers, $n+1$ in number in a singly infinite number of ways.

3. Polyhedra whose faces osculate a skew-cubic. Special case noticed by Frahm. Involution of one variable: curve and ruled surface. Involution of two variables, surface of order $n-2$; (2,2,2) correspondence for quadric surfaces; cases of degeneration: Δ-faced polyacra. Application of Cotterill's theorem; vanishing area and volume of the porismatic polygon and polyacron.

4. Multiplication. Sylvester's theory of derived points on cubic. Do. for quadriquadric. Scrolls. Arithmetical Theorem.

* [These mere heads of an intended paper are printed as they were found: no complete paper seems to embody them. Perhaps 1 was worked up into *XXIV., and 2 seems to be connected with XXV.]

Fig 23.

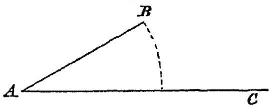
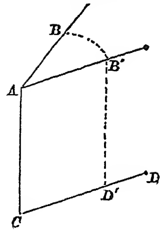


Fig. 24.



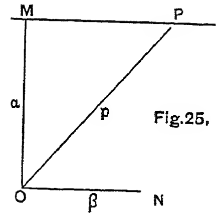


Fig.25,

Fig 26.

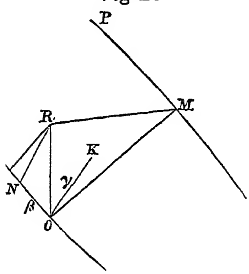


Fig. 27.

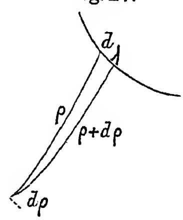


Fig. 28.

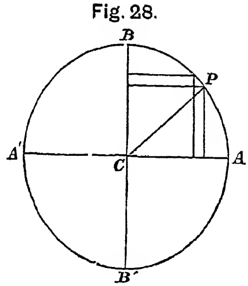


Fig. 29.

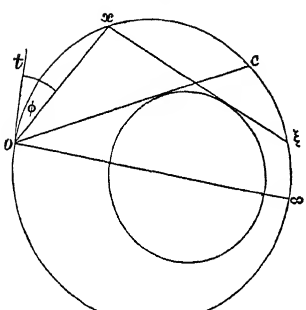
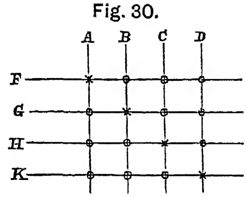


Fig. 30.



*XXIV.

ON IN-AND-CIRCUMSCRIBED POLYHEDRA.

THE extension of the theory of the in-and-circumscribed polygon, made in my paper "On the Transformation of Elliptic Functions," was suggested by a particular case of it studied by Dr. Lüroth in the *Math. Annalen,* I. [pp. 37—53]. It had been remarked by Clebsch (*Crelle,* [Bd. 59]) that not every curve of the fourth order can have its equation expressed as the sum of five fourth-powers; but that for this to be possible a certain invariant (the determinant of the six second derived forms) must vanish. Dr. Lüroth proved that in this case the proposed reduction can be effected in a singly infinite number of ways, and that the lines represented by equating the fourth-powers to zero all touch the same conic. All these pentagrams, moreover, are totally inscribed in a covariant quartic of the original curve (locus of points whose covariant cubics are equianharmonic, so that their Hessians break up into three straight lines). We thus have a varying pentagram circumscribed to the conic and totally inscribed in a quartic; and it was this pentagram which suggested the theorems in my paper above referred to.

The starting-point of the present communication is a remark by Dr. Frahm on a question allied to the foregoing (*Math. Ann.* VII. p. 635). It had been assumed by Dr. Salmon that the equations of three quadric surfaces may simultaneously be reduced to the form of the sum of five squares. Now Hesse

established a connection between the theory of three quadric surfaces and that of a plane quartic curve; namely, if u, v, $w=0$ are the three surfaces, then the equation

$$\text{Disct. } (\lambda u + \mu v + \nu w) = 0$$

is of the fourth order in λ, μ, ν, and taking these as co-ordinates of a point in a plane, the equation is that of a quartic curve which corresponds point for point with the locus of vertices of the cones $\lambda u + \mu v + \nu w = 0$. Dr. Frahm remarked that if the three quadrics could be simultaneously reduced to Salmon's canonical form, then this quartic curve is totally circumscribed to a pentagram, and is therefore not the general quartic, but the special form pointed out by Luroth; so that the problem of effecting this reduction is porismatic—it can either not be solved at all or be solved in a singly infinite number of ways. In the latter case, then, we have a singly infinite number of pentaplanes in regard to which the reduction can be effected; and each of these is totally inscribed in a curve of order 6 and deficiency 3, locus of the vertices of the cones

$$\lambda u + \mu v + \nu w = 0.$$

On considering the envelope of these pentaplanes, I found it to be a twisted cubic. The road to further generalizations was now clearly open.

I.

The equation of any osculating plane to a twisted cubic may be written

$$X + 3\theta Y + 3\theta^2 Z + \theta^3 W = 0,$$

where X, Y, Z, $W = 0$ are four fixed planes, and θ a parameter determining the particular osculating plane. The developable generated by tangent lines to the cubic is

$$(XZ - Y^2)(YW - Z^2) - 4(XW - YZ)^2 = 0 \quad \text{.........(1)},$$

and from this we see that Y passes through the tangent line of X and the point of contact of W, Z through the tangent line of W and the point of contact of X; or we may say that XY

and ZW are two tangent lines, YZ their chord of contact. The equations to the cubic itself are

$$\left\|\begin{matrix} X, & Y, & Z \\ Y, & Z, & W \end{matrix}\right\| = 0.$$

The co-ordinates of the point of intersection of three planes x, y, z are

$$\left\|\begin{matrix} 1, & 3x, & 3x^2, & x^3 \\ 1, & 3y, & 3y^2, & y^3 \\ 1, & 3z, & 3z^2, & z^3 \end{matrix}\right\| = 3xyz : -yz - zx - xy : x + y + z : -3,$$

and if we substitute these in the function $X + 3aY + 3a^2Z + a^3W$ belonging to any fourth plane, we get $3(x-a)(y-a)(z-a)$.

If a variable group of n *osculating planes of a twisted cubic form an involution of the* n$^{\text{th}}$ *order, the locus of their lines of intersection is a ruled surface* R *of order* $2(\mathrm{n}-1)$, *and the locus of their points of intersection is a curve* γ *the order of which is* $\frac{1}{2}(\mathrm{n}-1)(\mathrm{n}-2)$, *and which is a triple curve on the ruled surface.*

Consider the sections of this curve and surface by a fixed osculating plane L of the cubic. There is one group of the involution to which it belongs, and it meets the curve only where it meets the lines of intersection of the remaining $n-1$ planes of this group; that is, in $\frac{1}{2}(n-1)(n-2)$ points. This therefore is the order of the curve. All other osculating planes meet the plane L in lines which touch a fixed conic in that plane; in fact it meets the developable (1) in this conic and in its tangent line taken twice. The variable group of n planes in involution determines upon L a variable group of n tangents in involution; and the locus of their intersections is a curve of order $n-1$, by what we have already proved. Besides this curve, the plane L meets the ruled surface in $n-1$ straight lines, in which it meets the $n-1$ other planes of the group which contains it; so that the order of the whole intersection is $2(n-1)$, which is therefore the order of the surface. That the curve is a triple curve on the surface is clear from the fact that through any point of it there may be drawn three straight lines in the

surface, these being symmetrically related and not in the same plane.

By means of the in-and-circumscribed polygon determined upon the plane L we may find the equation of the ruled surface. Since the parameters of the several osculating planes of the cubic may be taken as parameters of the tangents to the conic in the plane L in which they are cut by that plane, it follows that the condition for two osculating planes x, y to belong to the same group is the same as the condition for the two tangents x, y to belong to the same group; that is to say, it is a condition of the form

$$\Sigma_i \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \text{ or } \Sigma_i \frac{\alpha_i}{x-a_i} = \Sigma_i \frac{\alpha_i}{y-a_i} \ldots\ldots\ldots (2),$$

where $a_1, a_2, \ldots a_n$ are parameters of some one group, and $\alpha_1, \alpha_2, \ldots \alpha_n$ are constants.

When this condition is fulfilled, the line of intersection of the planes x, y meets the curve γ. Now through any point on the surface R there can be drawn a line which is the intersection of two osculating planes of the cubic, and these two planes will satisfy the condition (2). If therefore the point of intersection of xyz lies on the surface R, the condition (2) must be satisfied either for yz or for zx or for xy. That is, we must have

$$\Sigma \frac{\alpha_i}{(y-a_i)(z-a_i)} \cdot \Sigma \frac{\alpha_i}{(z-a_i)(x-a_i)} \cdot \Sigma \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \ldots\ldots (3).$$

It remains to translate this equation into the ordinary point-co-ordinates. Let $A_1, A_2, \ldots A_n = 0$ be the equations to the n planes whose parameters are $a_1, a_2, \ldots a_n$; that is, let

$$A_i = X + 3a_i Y + 3a_i^2 Z + a_i^3 W;$$

then the result of substituting in A_i the co-ordinates of the point of intersection of x, y, z is $3(x-a_i)(y-a_i)(z-a_i)$. If then we multiply together terms of like suffixes in the factors of (3), we get in the product the sum of n terms

$$9\Sigma_i \frac{\alpha_i^3}{A_i^2}.$$

Next, the equation of the plane passing through the tangent line of A_i and the point of contact of A_j is

$$(iij) = X + 2a_i Y + a_i^2 Z + a_j (Y + 2a_i Z + a_i^2 W) = 0,$$

and when we substitute in this the co-ordinates of intersection of xyz, we obtain

$$(x-a_j)(y-a_i)(z-a_i) + (y-a_j)(z-a_i)(x-a_i) + (z-a_j)(x-a_i)(y-a_i).$$

Selecting then from (3) the products of two like suffixes by one unlike, we get the sum of $n(n-1)$ terms

$$\Sigma_{ij} \frac{a_i^2 a_j \,.\, (iij)}{A_i^2 \,.\, A_j}.$$

Lastly the equation of the plane passing through the points of contact of A_i, A_j, A_k is

$$(ijk) = X + (a_i + a_j + a_k) Y + (a_j a_k + a_k a_i + a_i a_j) Z + a_i a_j a_k W = 0,$$

and when we substitute in this the co-ordinates of intersection of x, y, z, we get $\frac{1}{2}\Sigma (x - a_i)(y - a_j)(z - a_k)$, where the x, y, z are to be permuted in all possible ways. Thus the products in (3) of three unlike suffixes give us the $\frac{1}{6}n(n-1)(n-2)$ terms

$$\tfrac{1}{2}\Sigma_{ijk} \frac{a_i a_j a_k \,.\, (ijk)}{A_i A_j A_k},$$

and the equation to the surface R is therefore

$$0 = 18\Sigma_i \frac{a_i^3}{A_i^2} + 2\Sigma_{ij} \frac{a_i^2 a_j \,.\, (iij)}{A_i^2 A_j} + \Sigma_{ijk} \frac{a_i a_j a_k \,.\, (ijk)}{A_i A_j A_k}.$$

The equation shews that when cleared of fractions it is as it ought to be of the order $2(n-1)$.

XXV.

ON A CANONICAL FORM OF SPHERICAL HARMONICS*.

THE canonical form in question is an expression of the general harmonic of order n as the sum of a certain number of sectorial harmonics, this number being, when n is even,

$$\frac{5n-10}{2},$$

and when n is odd,

$$\frac{5n-9}{2}.$$

Laplace's operator,

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$

may be obtained from the tangential equation of the imaginary circle $\xi^2+\eta^2+\zeta^2=0$, by substituting $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ for ξ, η, ζ. If, therefore, a form $U\equiv(x, y, z)^n$ is reduced to zero by this operation, it follows from Prof. Sylvester's theory of contravariants that the curve $U=0$ is connected by certain invariant relations with the imaginary circle. I find that U can be

* [Notices and Abstracts.. from *Report of the Forty-first Meeting of the British Association for the Advancement of Science*, held at Edinburgh, August, 1871, p. 10. A discussion followed the reading of Prof. Clifford's paper, and a result was that in the same Report, pp. 25, 26, is printed a communication by Sir W. Thomson, *On the General Canonical Form of a Spherical Harmonic of the nth order*. In this Sir W. Thomson answers the question, Can canonical forms not be found in which the nodal conic of each constituent is not resolvable into circular cones and planes ?]

expressed in the form

$$U \equiv A^n + B^n + C^n + \ldots$$

where $A = 0$, $B = 0$, ... are great circles touching the imaginary circle, the number of terms being as above. Now if $L = 0$, $M = 0$ be two such great circles meeting in a real point a, and if ϕ be a longitude and θ latitude referred to a as pole, it is easy to see that

$$L^n + M^n = l \sin^n \theta \sin n\phi + m \sin^n \theta \cos n\phi,$$

a sum of two sectorial harmonics, which is the proposed reduction.

When n is less than 5, exceptions of interest occur. For $n = 3$, if we take a, b, corresponding points on the Hessian of the *nodal curve* $U = 0$ (Thomson and Tait, *Treatise on Natural Philosophy*, § 780 [first edition]), and if we call ϕ_1, ϕ_2 the longitudes, θ_1, θ_2 the latitudes referred to these poles, we have

$$\begin{aligned} U \equiv\ & l \sin^3 \theta_1 \sin 3\phi_1 + m \sin^3 \theta_1 \cos 3\phi_1 \\ & + n \sin^3 \theta_2 \sin 3\phi_2 + s \sin^3 \theta_2 \cos 3\phi_2. \end{aligned}$$

For $n = 4$, the nodal curve is of the species first noticed by Clebsch, of which many most beautiful properties have been pointed out by Dr. Lüroth. The form U is expressible as the sum of *five* fourth powers; so that if we take a, b real points of intersection of two pairs of them, c a real point on the fifth, calling ϕ_1, ϕ_2, ϕ_3, θ_1, θ_2, θ_3 longitudes and latitudes referred to them, we have

$$\begin{aligned} U \equiv\ & l \sin^4 \theta_1 \sin 4\phi_1 + m \sin^4 \theta_1 \cos 4\phi_1 \\ & + p \sin^4 \theta_2 \sin 4\phi_2 + q \sin^4 \theta_2 \cos 4\phi_2 \\ & + r \sin^4 \theta_3\, \epsilon^{4\iota\phi_3}. \end{aligned}$$

XXVI.

ON THE FREE MOTION UNDER NO FORCES OF A RIGID SYSTEM IN AN N-FOLD HOMALOID. (Provisional Notice.)*

THE problem of the rotation under no forces of a rigid body about a fixed point in ordinary three-dimensional space is the same as the problem of free motion under no forces; for the motion about the centre of inertia takes place as if it were a fixed point. But it is also the same thing as the problem of the free motion of a rigid system on the surface of a sphere, or in elliptic space of two dimensions†. So also the problem of the free motion of a solid in elliptic space of three dimensions is the same as that of the free motion, or motion about a fixed point, in parabolic or homaloidal space of four dimensions. And, in general, the problem of free motion in elliptic space of n dimensions is identical with that of free motion, or motion about a fixed point, in parabolic space of $n+1$ dimensions.

The form of the problem which is considered in what follows is that which deals with the motion about a fixed point in parabolic space of n dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. VII., Nos. 92, 93, pp. 67—70.]

† According to Dr Klein's nomenclature, a space every point of which can be uniquely represented by a set of values of n variables is called elliptic, parabolic, or hyperbolic, when its curvature is uniform and positive, zero, or negative. The geometry of the sphere becomes elliptic when opposite points are regarded as identical.

I.

Let the co-ordinates of a point, referred to a rectangular system, be $x_1, x_2, \ldots x_n$. If this point belongs to a rigid system in motion, its velocity is given by the equations

$$\dot{x}_h = \Sigma p_{hk} x_k \quad (h, k = 1, 2, \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $p_{hk} = -p_{kh}$, $p_{hh} = 0$. The $\frac{1}{2}n(n-1)$ quantities p are of the nature of rotational velocities of the rigid system. It may be observed that if the vector from the origin to the point x be represented in terms of n unit vectors $\iota_1 \iota_2 \ldots \iota_n$, satisfying the equations $\iota_h \iota_k = -\iota_k \iota_h$, $\iota_h^2 = -1$, then the velocity of the rigid body may be represented in terms of the $\frac{1}{2}n(n-1)$ products $\iota_h \iota_k$; namely, we may write

$$\rho = \Sigma \iota_h x_h, \quad -2p = \Sigma p_{hk} \iota_h \iota_k,$$

and then the equation (1) may be put into the form

$$\dot{\rho} = Vp\rho.$$

If dm be the element of mass at the point x, its kinetic energy is

$$\tfrac{1}{2}\Sigma \dot{x}^2 dm = \tfrac{1}{2} dm \, \{\Sigma p_{hk}^2 (x_h^2 + x_k^2) + 2\Sigma p_{hl} p_{kl} x_h x_k\}.$$

Let then $$\int \dot{x}_h^2 dm = \alpha_h, \quad \int x_h x_k \, dm = \beta_{hk},$$

the integrations extending over the whole rigid system; then, if T be the kinetic energy of the system,

$$2T = \Sigma (\alpha_h + \alpha_k) p_{hk}^2 + 2\Sigma \beta_{hk} p_{hl} p_{kl} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Write now $$q_{hk} = \frac{\delta T}{\delta p_{hk}}, \quad -2q = \Sigma q_{hk} \iota_h \iota_k,$$

then q is the *momentum* of the system; it is a linear function of the velocity, or $q = \phi(p)$, and twice the kinetic energy is the scalar part of the product of velocity and momentum, $2T = Spq$. The equations of motion are $\dot{q} = f$, where f is the system of applied forces, or in the present case of no forces $\dot{q} = 0$; viz., this is equivalent to $\frac{1}{2}n(n-1)$ equations. From this we get the first integrals, q = constant, and (since $0 = Sp\dot{q} = 2\dot{T}$) T = constant.

But these equations are inconvenient, because the α and β are variable, depending upon the position of the body. We must therefore follow Euler in referring the motion to axes moving with the body, and coinciding with the principal axes at the fixed point. This will make all the β vanish; and we shall have
$$q_{hk} = (\alpha_h + \alpha_k)\, p_{hk}.$$

From equation (1) we obtain

$$\dot{q}_{hk} = \int dm\, (\ddot{x}_h x_k - x_h \ddot{x}_k) = \Sigma_l \int dm \{\dot{x}_l (p_{hl} x_k - p_{kl} x_h) + x_l (\dot{p}_{hl} x_k - \dot{p}_{kl} x_h)\}.$$

Assuming that $\int dm x_h x_k = 0$ when h and k are different, and remembering that $\dot{q}_{hk} = 0$, we may write this equation in the form
$$(\alpha_h + \alpha_k)\, \dot{p}_{hk} + (\alpha_h - \alpha_k)\, \Sigma_l p_{hl} p_{lk} = 0 \quad \text{.............. (3).}$$
Here $\dot{p}_{hk}$ means the rate of change of that component of velocity which coincides at the moment with one of the principal components; it must be distinguished from the rate of change of the principal component, which we shall call $(\dot{p})_{hk}$. In general, if the system of axes has the rotational velocities r_{hk}, we shall have
$$\dot{p}_{hk} = (\dot{p})_{hk} + \Sigma (p_{hl} r_{kl} - p_{kl} r_{hl});$$
but in the present case the r are equal to the p, because the axes move with the body, and so $\dot{p}_{hk} = (\dot{p})_{hk}$. Thus in the equations (3), which are analogous to Euler's equations for three dimensions, the symbols $\dot{p}_{hk}$ may be understood to mean the rates of change of the principal components of rotation.

If the symbol V_2 is regarded as selecting all the binary products of $\iota_1 \iota_2 \ldots \iota_n$ out of any rational integral function of them, the last equations may be written

$$\dot{q} = V_2 pq;$$

and it may be observed that we have also

$$q = \int dm V_2 \rho\dot{\rho}.$$

II.

These equations may be integrated by means of Θ-functions of $n-2, = s$, arguments, one of which is a linear function of the time. It is most convenient to use these in the form employed by Göpel for the case $s=2$. Let $u_1, u_2, \dots u_s$ be the arguments, and let $B_{11}, B_{12}, \dots B_{1s}, \dots B_{s1}, \dots B_{ss}$ be s^2 constant quantities, and let

$$U_h = (u_h + 2m_1 B_{1h} + 2m_2 B_{2h} + \dots + 2m_s B_{sh})^2 = (u_h + 2\Sigma m_k B_{kh})^2;$$

then we shall write

$$G(u_1, u_2, \dots u_s) = \Sigma_m e^{\Sigma U_h},$$

where the whole numbers m are to take independently all values from $-\infty$ to $+\infty$. It is clear that the function G is unaltered if we simultaneously increase $u_1, u_2, \dots u_s$ by equimultiples of the quantities $2B_{h1}, 2B_{h2}, \dots 2B_{hs}$, for this is only increasing m_h by an integer. Moreover, if we determine s^2 quantities A so that

$$4\Sigma_k A_{hk} B_{hk} = \pi i, \quad \Sigma_k A_{hk} B_{lk} = 0,$$

then the function $e^{-\Sigma uu} G$ is unaltered if we simultaneously increase $u_1, u_2, \dots u_s$ by equimultiples of $4A_{h1}, 4A_{h2}, \dots 4A_{hs}$. In what follows we shall write for shortness $G(u+X_h)$ instead of $G(u_1 + X_{h1}, u_2 + X_{h2}, \dots u_n + X_{hn})$, omitting always the last suffix, and mentioning only one argument u.

A linear function of the A, B with coefficients 0 or 1 will be called a *quadrant;* there are clearly 2^{2s} quadrants, if zero be included among them. Let X and Y be two quadrants, A_h the difference between those parts of them which involve the quantities A_{hk}, then the function

$$e^{-\Sigma A u} \frac{G(u+X)}{G(u+Y)} = Al_{X,Y}(u)$$

is $2s$-periodic in the arguments u, the periods being $4A$, $4B$. It is convenient to speak of the *distance* of two quadrants X and Y, meaning the number of coefficients of the A and B which must be changed from 0 to 1, or *vice versâ* to make one of them into the other. This distance may be any of the numbers

1, 2, ... $2s$; and accordingly there are $2s$ really distinct $2s$-periodic functions.

It is, however, possible to form a group of $n, = s+2$, quadrants $X_1, X_2, \dots X_n$ having such a relation to the quadrant O that if we write $Al_{hk}(u)$ for $e^{-\Sigma Au}\, G(u + X_h + X_k) : Gu$, the $\frac{1}{2}n(n-1)$ functions $Al_{hk}(u)$ satisfy the equations

$$\delta_u Al_{hk}(u) = \Sigma c_l Al_{hl}(u) \,.\, Al_{kl}(u) \;\ldots\ldots\ldots\ldots\ldots (4),$$

where the δ_u applies to any one of the arguments; but the values of the c will depend upon which argument is taken. In the case of the hyperelliptic functions, the four quadrants X may be taken to be the quantities A_1, A_2, B_1, B_2.

It appears therefore that the equations (3) may be integrated if we write $p_{hk} = \lambda_{hk} Al_{hk}(u)$, where $u_1 = at + e$.

III.

From the $\frac{1}{2}n(n-1)$ equations (3) let us pick out $n-1$, namely

$$-(\alpha_1 + \alpha_h)\,\dot{p}_{1h} = (\alpha_1 - \alpha_h)\,\Sigma p_{hk} p_{1k};$$

if we write $(\alpha_1 + \alpha_h)\,p_{1h} = (\alpha_1 - \alpha_h)\,\xi_h$, these equations become

$$-\dot{\xi}_h = \Sigma p_{hk}\,\xi_k.$$

But these are the equations for the velocity of a fixed point ξ relative to the moving axes in $n-1$ dimensions. The rest of the equations (3), if we write in them $p_{1h} = 0$, become the equations for the component rotations in $n-1$ dimensions. Thus the solution for the rotational velocities *and* the position of a point fixed in space, for $n-1$ dimensions, are obtained by diminishing the number of periods in the solution for n dimensions; it consists accordingly of the Al functions expressing the rotation-velocities, and of Rosenhain's combination of Θ-functions and exponentials, expressing the position.

XXVII.

ON THE CANONICAL FORM AND DISSECTION OF A RIEMANN'S SURFACE*.

THE object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end I assume only certain results of Puiseux, which are put together at the outset.

I.

Puiseux's Theory of an n-valued Function.

If two variables s and z are connected by an equation of the form $f(s, z), = (s, 1)^n (z, 1)^m, = 0$, each is said to be an algebraic function of the other. Regarding z as a complex quantity $x + iy$, we represent its value by the point whose co-ordinates are x, y, on a certain plane. To every point in this plane belongs one value of z, and consequently, in general, n values of s, which are the roots of the equation $f = 0$. The points of the plane may be divided into those at which the n values of s are distinct, and those at which two or more of them are equal. The latter points are finite in number, and correspond to the roots of the equation which is got by equating to zero the discriminant of f in regard to s. If the roots of this equation are distinct, there are $2(n-1)m$ such points, because the discriminant of the

* [From the *Proceedings of the London Mathematical Society*, Vol. VIII., No. 122, pp. 292—304.]

equation of the nth order in s is of degree $2(n-1)$ in the coefficients, and these coefficients are of the order m in z. But a point at which r values of s become equal corresponds to an $(r-1)$-fold root of the discriminant-equation.

Let us now consider an arbitrary point O of the plane [fig. 31], corresponding to a value z_0 of z, which is not a root of the discriminant-equation. Then the equation $f(s, z_0)=0$ will give n different values for s, which we may call $s_1, s_2 \dots s_n$. If we move along any path from the point O to another point P of the plane, the value of z will change continuously, and each of the quantities $s_1, s_2 \dots s_n$ will also change continuously. If therefore the path OP does not go through a point where two values become equal, these n quantities will be distinct all the way, and each of the n values of s at P will belong to a definite one of the values of s at O. But if the path goes through such a point, two or more of the n quantities will become equal and then diverge again, so that it will be impossible after that to distinguish them so as to say which of these belongs to a particular one of the values at the point O. We cannot always avoid this difficulty by going *round* the point, for it is found that the values at P to which the values at O correspond may depend upon the path OP, so that the correspondence is different for a path which goes to the right of the point and for a path which goes to the left of it. When this is the case, the point is called a branch-point. Suppose that, when we go from O to A, the two values p and q of s at O approach one another and become equal at A; then it is found that the value at P which represents p when we go along the path OBP may represent q when we go along OCP, and *vice versâ*. So that, if we travel along $OBPCO$, round the point A and back to O, the values p and q will change continuously into one another. If more than two values are equal at A, the corresponding values at O may be cyclically interchanged by a path going round A. We shall assume, however, that only two values become equal at each branch-point; and, moreover, that no branch-point is at an infinite distance*.

* Roots of the discriminant-equation which are not branch-points correspond to double points on the curve $f(s, z)=0$. Such points behave, in regard to

A path going along any line from O to very near A, then round A in a very small circle, and then back to O along the same line, will be called a *loop*.

If we start from O and go round any closed curve not including any branch-points, the n values of s at O will be restored in the same order. For the path may be gradually shrunk into a point without crossing any branch-points, so that no two of the n values can become confused at any point of it. The same thing is true if the closed path includes *all* the branch-points. Suppose it a large circle through O; then it may be gradually increased till it coincide with the tangent at O, then curved over on the other side, and shrunk up into a point; and during the whole process the n values will be distinct at every point of the path.

We shall now go on to shew that this n-valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of n infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. This surface is called a RIEMANN'S surface; we shall demonstrate its existence at the same time that we shew how to construct it in the most convenient form.

II.

Construction of the Riemann's Surface.—Lüroth's Theorem.

Draw loops from O [fig. 32] to all the branch-points, and let the first, A, interchange the values p and q. If we go round all the loops successively, starting with the value p at O, we must, as we have seen, come back to that value; but this may happen before we have used all the loops. Let B be the first branch-point after going round which the value p is restored. Draw a line from A to B cutting all the loops which alter p, but none of the others. Then, if we go round any of the

the function s, like two coincident branch-points belonging to the same pair of values, and they have no influence on the connection of the different values of s.

branch-points between A and B without crossing the line AB or going round any other branch-points, we shall not alter the value p.

Suppose that A interchanges pq, B interchanges ps, and that the branch-points between A and B are 1, 2, 3, 4, interchanging respectively qr, rs, hk, pl. The value q must in fact be changed into the value p through a longer or shorter series of values; the loops interchanging hk and pl are put in as examples. Now if we go round 4 by the dotted loop passing round outside A, the effect is the same as going in succession round A, 1, 2, 3, 4, 3, 2, 1, A. By the time we have gone round A, 1, 2, 3, we cannot have the value p, for that is first restored by B; and we cannot have the value l, for then 4 would restore the value p. Hence we have some value which is not altered by the loop to 4; and consequently, when we retrace our path, we shall come back to the value p.

Next, let us draw a loop to B which passes within the line AB, but goes round all the included branch-points, as in the figure. The effect of this loop will be to change q into p; for it is the same thing as going round 1, 2, 3, B, 3, 2, 1. Now the effect of 1, 2, 3, B is to change q into p, and this p is not altered in coming back because all the branch-points which alter p are outside the line AB.

Suppose then that all the branch-points of this group which alter p are connected with O by loops going round A, so that they no longer alter p; and that B is connected with O by the loop just described, so that no branch-points are contained in the triangle AOB.

Starting now from this new loop OB, with the value p, let us go round all the loops as before from left to right. We know that when all the loops have been gone round, ending with OA, the value p must be restored. If it is not restored before we have gone round OA, we must draw a line BA cutting all the loops which change the value p but none of the others. But if the value p is restored before we have gone round OA, say after going round OC; then we must draw a new loop to C, going round all the branch-points between A and C except those which change the value p. This new loop will, by our previous

reasoning, change p into q. Hence, if the value p is restored before we have gone round OA, we can make a new loop OC which changes p into q; and this comes next to OB. To those branch-points whose loops have been cut by this new loop we must draw new loops going round to the right of C, so as not to cut OC. The figure comes then into this form [fig. 33], containing

(1) Loops to the left of OA which do not change the value of p, like the dotted loop $O4$ in the previous figure;

(2) Three consecutive loops OA, OB, OC which change p into q;

(3) Loops to the right of OC which may or may not change p.

If now we start with the loop OC and proceed to the right, the value p must be restored *before* we have gone round OA; for, starting with OA and going all round, we must restore the value p in the end. Let p then be restored by OD; and draw a line CD cutting all those loops which change p, but none of the others. Replace the loops which change p by new ones going round between B and C; and replace OD by a new loop going outside all the branch-points whose loops do not alter p. The figure now consists of these elements:

(1) Two triangles AOB, COD, containing no branch-points, and such that the loops OA, OB, OC, OD interchange p and q;

(2) Loops between OB and OC which do not change p;

(3) Unknown loops between OD and OA.

About these unknown loops we may make three suppositions.

First, suppose that none of them change p. Then the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines AB, CD.

Secondly, suppose that some of these loops change p, but that, when we start with the loop OD and go round to the right, the value p is first restored by OA or OC. (It is clear that it cannot be first restored by OB, because the two loops OA, OB, taken together, make no change in any value; nor by any loop

between OB and OC, for none of them change p.) Then we must join D with A by a line cutting all the loops which change p, but no others; and B with C by a line cutting none of the loops between OB and OC. In that case the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines BC, DA.

Thirdly, suppose that the value p is restored *before* we come to OA, say at OE. Then we must proceed as before, finding a new line EF which shall have the properties of AB or CD. The figure will then consist of three triangles AOB, COD, EOF, containing no branch-points, and such that the loops OA, OB, OC, OD, OE, OF interchange p and q; loops between OB and OC, and between OD and OE, which do not change p; and unknown loops between OF and OA.

It is clear that this process must ultimately stop, and then we shall be left with a finite number of lines such that, if we start from O, follow any continuous path, and come back again, without crossing any of these lines, we shall not alter the value p. The lines are either AB, CD, EF, &c., or else they are BC, DE, &c.; in either case the loops $OA, OB, \ldots$ interchange p and q.

It follows that, if we take an infinite plane sheet and cut it through along these lines, we may consider a single value of the function s to be attached to every point of the sheet in such a way that this value varies continuously when we move about continuously in the sheet; but there will be different values on the two sides of any cut—namely, we must attach to every point P of the sheet that value of s which changes continuously into p when we go from P to O without crossing any of the cuts. There is only one such value; for if two different paths from O to P gave different values at P, it would be possible to change the value p by means of a closed curve returning to O; and this we have proved not to be the case.

When the lines cut through are AB, CD, ..., the triangles AOB, COD, ... contain no branch-points; but when the lines are BC, DE, ..., the triangles BOC, DOE do in general contain branch-points. We may, however, draw new loops to C, E, ... so as to exclude these branch-points, and the new loops will still change p into q. For no closed curve going round B and C

so as not to cut BC can change the value p, by what we have already proved; but the loop OB changes p into q, therefore OC must change q into p.

We shall assume then that the cuts are AB, CD, ..., and that the triangles AOB, COD, ... contain no branch-points.

Now let us deal with the value q at O in the same way as we have dealt with the value p. It is first to be observed that a path going round one or more of the lines AB makes no change in *any* value at O; so that, if we agree never to cross these lines, we may leave the branch-points A, B, ... entirely out of consideration.

This being so, let us take a loop which changes q into some other value, say r. There must be such a loop, if the function is more than two-valued; for otherwise the values p, q would form a two-valued algebraic function of z, and the expression $f(s, z)$ would have a factor of the second degree in s.

Starting then with this loop, we may proceed in exactly the same way as before, and draw lines $A'B'$, $C'D'$, ... such that a closed curve, starting from O and coming back to it without cutting any of these lines or any of the previously drawn lines, will not alter the value q. Moreover, we shall have drawn loops OA', OB', ..., each of which changes q into r, and such that the triangles $A'OB'$, $C'OD'$, ... contain no branch-points. And since our previous triangles AOB, COD, ... contained no branch-points, it will not have been necessary to cut through them in drawing the new lines $A'B'$, $C'D'$, ...

We shall now speak of the first set of lines AB, CD,... as the lines (pq), and of the second set as the lines (qr).

Let us take two infinite plane sheets, cut them both through along the lines (pq), but only the second one along the lines (qr). To every point of the *first* sheet we will suppose attached that value of s which is arrived at by continuous change of the value p at O; and to every point of the *second*, that value which is arrived at by continuous change of the value q at O.

In each sheet there will be a finite difference in the values on the two sides of each of the cuts (pq); but the value on one side in the upper sheet will be equal to the value on the other side in the lower sheet. At the cut AB, for example, the value

continuous with p on the side next to O is equal to the value continuous with q on the side remote from O; because a path taken round A or B from O and back again changes the value p continuously into the value q.

Thus, if we take p, q to denote values at the cut continuous with p, q at O, they will be situated as in the figure [fig. 34], which represents a section across AB perpendicular to the two sheets. If then we make the two sheets cross one another along the lines p, q, as here represented [fig. 35], then these two values will be continuously distributed on the double-sheeted surface so formed.

We may now continue the process with the value r. We must first find a loop which changes r into some other value, say t, and then proceed as before, taking care not to cross the lines qr. (We may cross the lines pq as often as we please, provided that we have not previously crossed the lines qr; for these lines can have no effect upon r unless it has been previously changed into q.) Thus we shall draw lines rt such that the value r cannot be altered by a closed curve not cutting the lines qr or rt, and having their extremities joined to O by loops which change r into t. If we take, then, a third sheet, cut it through along the lines qr and rt, and then join it crosswise to our second sheet along the lines qr; the three values pqr may be continuously distributed on this three-sheeted surface.

By proceeding in this way it is clear that we shall construct an n-sheeted surface, the sheets of which are connected chainwise by cross lines, so that the first is connected only with the second, the second with the third, and so on; but there is no direct connection except between consecutive sheets. And the n values of the function may now be attached to the points of this surface, so that one value only belongs to each point, and that in moving this point about on the surface the value belonging to it always changes continuously. Thus, if we start from a given point of the surface (on a given sheet), and travel by any path so as to come back to the same point (on the same sheet), we shall in all cases return to the former value of the function s.

The theorem that the Riemann's surface may be so con-

structed that the sheets are only connected *chainwise*—*i. e.*, so that there are no cross-lines except between consecutive sheets —is due to Dr. Lüroth.

III.

Clebsch's Theorem.

All the links between successive sheets except the last may be made to consist of one cross-line only.

First, we shall prove that, if there are two or more lines (pq), one of them may be converted into a line (qr).

The original position of the two lines (pq) and the line (qr) is drawn in fig. [36]. If we move the line qr, keeping, of course, its ends fixed, the effect is to interchange the sheets QR in the area over which it moves; so that, by passing it over the line (pq) on the right, we change this into a line (pr). The position is then as in fig. [37]. If now we pass the remaining line (pq) over this line (pr), we change it into a line (qr); thus we are left with two lines (qr) and one line (pq). [Fig. 38.]

In this way we may convert all but one of the lines (pq) into lines (qr). Then we may convert all but one of the lines (qr) into lines (rs); and so on. Then the first $n-1$ sheets will be connected chainwise by one cross-line each, and the last two by all the remaining cross-lines.

The Riemann's surface is now said to be in its canonical form.

The process of transformation may be made clearer by looking at a section of the three sheets by a plane perpendicular to them cutting the lines pq, qr, pq [figs. 39, 40, 41].

IV.

Transformation of the Riemann's Surface.

The Riemann's surface now consists of n infinite plane sheets, such that the sheet 1 is connected with 2 by a single cross-line, 2 with 3 by another cross-line, and so on; but $(n-1)$

with (n) by a number of cross-lines which we shall call $p+1$. Thus the whole number of cross-lines is $n-2+p+1=n+p-1$. If w is the number of branch-points, this is twice the number of cross-lines, or $w=2(n+p-1)$. Hence $p=\frac{1}{2}w-n+1$.

Let now this n-fold plane be inverted in regard to any point outside it, so that it becomes an n-fold sphere passing through the point. Any two successive sheets of the sphere will be connected by one cross-line, except the two outside sheets, which are connected by $p+1$ cross-lines.

To every point of this n-sheeted spherical surface will correspond one value of the function s, namely, that which belongs to the corresponding point upon the n-fold plane. As for the centre of inversion, it is to be regarded as n distinct points upon the several sheets, corresponding to the n values of s when $z=\infty$.

We shall now prove that this n-fold spherical surface can be transformed without tearing into the surface of a body with p holes in it.

First, suppose we have only two sheets, connected by a single cross-line which joins the branch-points AB. Let the figure [42] represent a section by the plane which bisects AB at right angles.

Suppose each hemisphere of the inner sheet to be moved across the plane of the great circle containing AB (indicated by the dotted line in the figure), so that the points m, n change places. In this process the two hemispheres will have to penetrate and cross each other; but this may be supposed to take place without altering the continuity of either. Each point may be supposed to move on a straight line perpendicular to the dotted plane, till it coincides with what was its reflexion in regard to that plane. The effect on the cross-line will be to change it from the form drawn in fig. [42] to that drawn in fig. [43]; instead of the two sheets crossing along the line, each of them will be doubled under it. The result is that, if we now look down on the double sphere from a point vertically over the line AB, we shall see a spherical shell with a hole in it, in the form of a slit along the line AB [fig. 44]. Conceive the spherical shell to be made of india-rubber or some more elastic substance;

then by mere stretching, without tearing, the slit may be opened out until the shell takes the form of a flat plate; that is, of a body with *no* holes in it.

Next, consider a two-sheeted spherical surface with $p+1$ cross-lines, and suppose them all arranged along the same great circle; which may obviously be done by stretching, without tearing, the surface. Let this great circle be the one represented by the dotted line in figs. [42] and [43]. Then we may apply to the inner sheet the same process as before; viz., we may interchange the two hemispheres into which the sheet is divided by the dotted plane. The effect is to convert all the cross-lines into slits or holes in a spherical shell; and we have supposed that there are $p+1$ of these. One of the slits may be stretched out in the same way as AB was before, so as to convert the spherical shell into a flat plate; but in this flat plate there will remain p holes. A double sphere with $p+1$ crossing lines is thus converted, without tearing, into the surface of a body with p holes in it.

Lastly, suppose that the inner sheet of this two-sheeted sphere is connected by one cross-line with a third inside sheet, the third sheet by one cross-line with a fourth inside it, and so on, until there are n sheets. Let the inner sheet of all be reflected in regard to the plane of the great circle through its crossing line, so that it makes with the sheet next to it a spherical shell with one hole in it. Then, without tearing, the inner sheet may be shrunk up until it merely covers over this hole. The same process may now be applied to shrink up the second sheet into the third, and so on, until we are left with only the two outside sheets connected by $p+1$ cross-lines. These, however, as we have seen, may be converted, without tearing, into the surface of a body with p holes in it. Hence the proposition follows, that *an n-sheeted Riemann's surface with w branch-points may be transformed, without tearing, into the surface of a body with p, $= \frac{1}{2}w - n + 1$, holes in it.*

V.

The Number of Irreducible Circuits.

A closed curve drawn on a surface is called a *circuit.* If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible;* otherwise it is *irreducible.* In general there is a finite number of irreducible circuits on a closed surface which are *independent,* that is, no one of which can be made by continuous motion to coincide with a path made out of the others. All other irreducible circuits can then be expressed as compounds of these independent ones. For example, on the surface of a ring (*i.e.,* of a body with one hole through it) there are two independent irreducible circuits; one *round the hole,* as *abc* [fig. 45], and one *through the hole,* as *ade.* If a circuit goes neither round the hole nor through the hole, it can be shrunk up into a point. If it cannot be so shrunk up, it must go a certain number of times round or through the hole or both, that is, it may be made up of circuits like *abc* and *ade.*

In the same way we may see that, on the surface of a body having p holes through it, there are $2p$ independent irreducible circuits; one *round* each hole, and one *through* each hole. For simplicity consider the case $p = 3$. We suppose the body in the form to which we reduced the Riemann's surface, namely, that of a flat plate, represented by figs. [46] and [47], in which A, B, C are the holes. The circuits through each hole are so drawn as to connect the hole directly with the outer rim, like the circuit which is drawn through the hole A. A circuit passing through *two* holes, as B, C [fig. 46], may be moved continuously till it consists of two circuits going through the two holes separately. Similarly, a circuit round two or more holes, as B, C [fig. 47], may be pinched at various points until it is made up of circuits round the separate holes. Such a circuit as *abcd* [fig. 46] may be moved into the form *abcd* [fig. 47], in which it consists of two circuits going through the hole A, but in opposite directions. On this account it may be called a *nugatory* circuit.

VI.

The Canonical Dissection.

Suppose now that it is desired to cut through the Riemann's surface in such a way that it shall still hang together, but that it shall no longer be possible to draw an irreducible circuit upon it. This we may do if we successively prevent the different kinds of irreducible circuits considered in the last section. To prevent the possibility of going *round* any hole, we must cut the surface along a circuit which goes *through* the hole. To prevent the passage *through* a hole, we must cut through a circuit which goes *round* a hole.

Let us make sections a_1, a_2, a_3 [figs. 48, 49] round the holes, and b_1, b_2, b_3 through the holes. Then we shall have prevented the drawing of any irreducible circuits except nugatory ones, like *abcd* in the previous figures. To prevent these also, we may cut the surface along the line c_1 which goes from p to q, that is, from a point on b_2 to a point on b_3, and along the line c_2 which goes from q to r, that is, from a point on b_3 to a point on b_1. We must not cut from r to p also, for then we should divide the surface into two separate parts. We may now open out the upper and under portions of the surface in fig. [48], until it assumes the form of fig. [49]. It then becomes obvious that all our cuts form a continuous line, which is now the boundary of the surface, and is made up of the pieces (beginning at p and going round to the right) $c_1 b_3 a_3 b_3 c_2 b_1 a_1 b_1 a_1 b_1 c_2 b_3 a_3 b_3 c_1$. Moreover, it is a matter of intuition that no irreducible contour can now be drawn on the surface.

This system of cuts is called a *canonical dissection* of the surface. In the general case it consists of p cuts a going round the holes, p cuts b going through them, and $p-1$ cuts c joining b_2 to b_3, b_3 to b_4, ... b_p to b_1, but not b_1 to b_2. The cuts c may, if we like, join the a-cuts together, or generally they may join the systems (ab) together, a *system* meaning an a-cut and a b-cut belonging to the same hole. In fact, the c-cuts are only of importance as completing the single boundary of the surface,

and so enabling us to see that no irreducible circuit is any longer possible.

It only remains to translate this result so that it may be applicable to the original form of the Riemann's surface, viz., an n-fold plane. We shall do this in the case $p=2$, which will sufficiently explain the general case. We have now two sheets connected by three cross-lines mn, pq, rs [fig. 50]. One of these must be chosen to represent the outer rim of our flat plate; the other two will then correspond to the holes in it. Let mn, pq represent the holes, and rs the outer rim; lines in the upper sheet shall be drawn in full, and lines in the lower sheet shall be dotted. Then we must first make cuts a_1, a_2, which go round the holes mn, pq; these may lie entirely in the upper sheet. Next we must make cuts b_1, b_2, which connect the holes respectively with the outer rim rs. These cuts lie partly in the upper sheet, where they intersect the cuts a, and partly in the under sheet. Lastly, we must connect the system $a_1\,b_1$ with the system $a_2\,b_2$ by the cut c; this is drawn in the figure from b_1 to b_2 in the under sheet. It is impossible to draw an irreducible circuit on the two-fold plane when it is thus dissected*.

In general, we have proved that in the n-sheeted Riemann's surface which represents the function s determined by the equation $f(s, z) = 0$, there are $p+1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one on which no irreducible contour is possible by the following process:—Cut the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these lines with the one taken to represent the rim by a cut b along a closed curve which crosses each of the two cross-lines once. Then connect the systems (ab) chainwise by $p-1$ cuts c.

* It is to be understood that a circuit is *reducible* when all parts of it can be continuously moved away to infinity without crossing any branch-point; because in this theory infinity counts as a single point.

Fig 31.

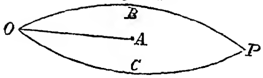


Fig. 32.

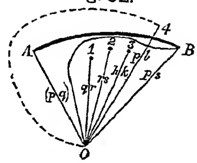


Fig. 33.

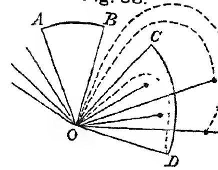


Fig 34.

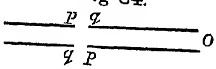


Fig. 35

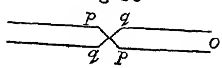


Fig. 36.

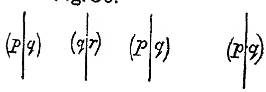


Fig. 37.

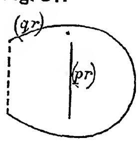


Fig. 38.

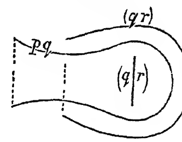


Fig 39.

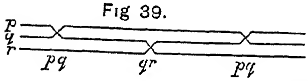


Fig. 41.

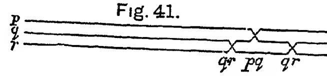


Fig. 40.

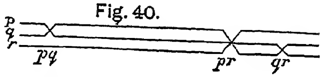


Fig 42

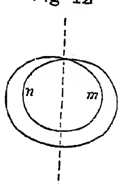


Fig 43

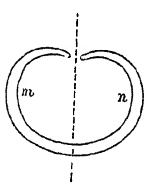


Fig 44

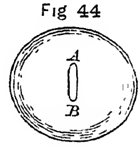


Fig. 45

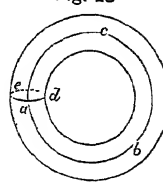


Fig 46.

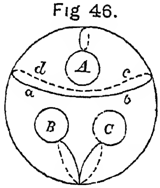


Fig. 47.

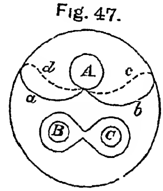


Fig. 48

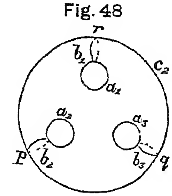


Fig. 49.

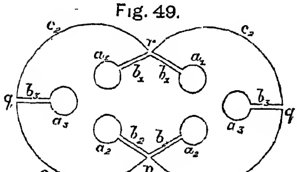
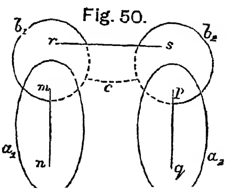


Fig. 50.



XXVIII.

REMARKS ON THE CHEMICO-ALGEBRAICAL THEORY.

(Extract from a letter to Mr Sylvester*.)

"THE new Journal [see foot-note] I look forward to with the greatest interest: it will be the only English periodical in which one will have room to print formulæ, except the *Philosophical Transactions*. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet†. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras. Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published years ago in the *Cambridge Mathematical Journal*. I have a strong impression that you will find there the analogy of covariants and invariants to compound radicals and saturated molecules.

I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$ and that I have altogether 6 sets of 2 variables each, namely

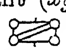
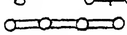
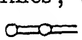
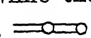
$$x_1x_2,\ y_1y_2,\ z_1z_2,\ u_1u_2,\ v_1v_2,\ w_1w_2.$$

* [From the *American Journal of Mathematics, Pure and Applied*, Vol. I. No. 2, pp. 126—128.]

† [This is xxx. of the present volume.]

Suppose the forms are

$$(xyzu),\ (yzvw),\ (xv),\ (uw)\ ;$$

viz. $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: *to obtain this invariant,* regard the *variables as alternate numbers*, and *simply multiply all the forms together.* By *alternate* numbers I mean those whose multiplication is polar $(xy = -yx)$ and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol , the quadratics by -o-. Thus the invariant $(xyzu)$ $(yzvw)$ (xv) (uw) will be represented by the figure ; whereas, $(xyzu)$ $(yzvw)$ (xu) (vw) is this form . The former is clearly the product of the two quartic covariants   got by cutting it across the dotted lines; while the latter is the product of the quadri-covariants , . A *bond* between two forms means a set of variables common to them. Of course, we may regard two or more of the forms as identical and so form invariants of a single form; thus  is the discriminant of a cubic*...Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call 'unravelment'....

The part of the theory which astonished me most is its application to *intergredient* variables when the number in a set is greater than 3,—such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get one intergredient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergredient variables are merely their products so many together. With

this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal—all the angles rounded off so that you can't see through it so clearly...." †

† [Dr Sylvester has appended several interesting notes, from which a few extracts are given here. "I think Prof. Clifford overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it. In a word he has found the universal pass-key to the *quantification of the graphs*...'All that Prof. Clifford adds' is the very pith and marrow of the matter which before was wanting." Dr Sylvester further remarks, "I will take the example of this figure [cf. * in text] to illustrate Prof. Clifford's rule for finding the *algebraical content* of the graph. Let the bonds be called $\begin{smallmatrix} x & & t & \\ y & z & u & v \end{smallmatrix}$. Then there will be four forms corresponding to the four apices or atoms, viz.

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)\ (x_1, x_2)\ (y_1, y_2)\ (z_1, z_2),$$
$$(b_1, b_2, \ldots\ldots\ldots\ldots b_8)\ (z_1, z_2)\ (t_1, t_2)\ (u_1, u_2),$$
$$(c_1, \ldots\ldots\ldots\ldots c_8)\ (t_1, t_2)\ (u_1, u_2)\ (v_1, v_2),$$
$$(d_1, \ldots\ldots\ldots\ldots d_8)\ (v_1, v_2)\ (x_1, x_2)\ (y_1, y_2),$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. [Dr Sylvester objects to the term *alternate numbers* in this connexion.] Take the polar product of these forms; the coefficient of

$$x_1 \cdot x_2 \cdot y_1 \cdot y_2 \cdot z_1 \cdot z_2 \cdot t_1 \cdot t_2 \cdot u_1 \cdot u_2 \cdot v_1 \cdot v_2$$

will be an invariant of three lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally if we make the coefficients of $x_1y_1z_2$, $y_1z_1x_2$, $z_1x_1y_2$ all alike, and again the coefficients of $x_2y_2z_1$, $y_2z_2x_1$, $z_2x_2y_1$ all alike, and identify the letters x, y, z, the form becomes a binary cubic and the invariant becomes its discriminant. We know *à priori* by my permutation-sum test that the algebraical content above indicated will not vanish because

$$\Sigma\,(a-b)^2\,(a-d)^2\,(a-c)\,(b-d)$$

is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of

$$\Sigma\,(a-b)\,(b-c)\,(c-d)\,(d-a)\,(a-c)\,(b-d)$$

is zero."]

CLIF.

*XXIX.

NOTES ON QUANTICS OF ALTERNATE NUMBERS, USED AS A MEANS FOR DETERMINING THE INVARIANTS AND COVARIANTS OF QUANTICS IN GENERAL*.

THE term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations:

$$\lambda_1^2 = 0, \quad \lambda_2^2 = 0, \quad \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \quad \ldots\ldots\ldots\ldots(1);$$

to which it is usual to add

$$\lambda_1\lambda_2 = 1.$$

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \ldots$

By a *Quadratic Form* is here meant an expression—lineo-linear in two sets of such numbers regarded as variables, say λ_1, λ_2; μ_1, μ_2, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \quad \ldots\ldots\ldots\ldots(2).$$

This may be also denoted, for shortness, by the symbol $a \,|\, 12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ.

* [From the *Proceedings of the London Mathematical Society*, Vol. x. No. 148, pp. 124—129. "This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in a letter to Professor Sylvester (see XXVIII.): 'I consider forms which are linear in a certain number of sets of k variables each....The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables.'" SP.]

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers,

$$(a \mid 12)^2 = -2\,(a_{11}a_{22} - a_{12}a_{21}) = -2D, \text{ suppose}\ldots\ldots(3).$$

But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ's by x, an ordinary, not an alternate, number, we get a linear function of the λ's, whose square vanishes in virtue of the relations (1). Thus we have $(a \mid 1x)^2 = 0$, and consequently such a product as $(a \mid 1x)(a \mid 1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$ since it vanishes when x and y represent the same point; in fact, if we write the expressions in full, thus,

$$(a \mid 1x) = (a_{11}x_1 + a_{12}x_2)\,\lambda_1 + (a_{21}x_1 + a_{22}x_2)\,\lambda_2,$$
$$(a \mid 1y) = (a_{11}y_1 + a_{12}y_2)\,\lambda_1 + (a_{21}y_1 + a_{22}y_2)\,\lambda_2,$$

actual multiplication gives

$$(a \mid 1x)(a \mid 1y) = \begin{array}{ll} a_{11}x_1 + a_{12}x_2, & a_{21}x_1 + a_{22}x_2 \\ a_{11}y_1 + a_{12}y_2, & a_{21}y_1 + a_{22}y_2, \end{array} = \begin{array}{ll} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{array} \times \begin{array}{ll} x_1, & x_2 \\ y_1, & y_2. \end{array}$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a's for its coefficients, say any form $(a \mid 1\gamma)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2\,(a \mid 1x)\,(a \mid 1y) = (a \mid 1\gamma)^2\,(xy)\,\ldots\ldots\ldots\ldots.(4).$$

Note.—The transformation

$$y_1, y_2 = \left| \begin{array}{ll} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{array} \right| (x_1, x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a \mid yx),$$

where the y's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1, z_2 = \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} (y_1, y_2)$$

may be represented by $(b \mid zy) = 0$; and the result of the two by

$$(b \mid zy)(a \mid yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a \mid 12)$ by (12), we get

$$(12)(a \mid 12) = a_{21} - a_{12} = s \text{ (suppose)} \qquad \ldots\ldots\ldots\ldots (5).$$

Also, if we suppose the coefficients to remain unaltered, viz., if $(a \mid 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \ldots$, then

$$(a \mid 12) + (a \mid 21) = -s(12) \qquad \ldots\ldots\ldots\ldots (6),$$

or
$$2(a \mid 12) = (a \mid 12) - (a \mid 21) - s(12),$$

the quantity s vanishing when the form is symmetrical, i.e. when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that

$$(a \mid 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \ldots,$$

then it will be found that

$$\left.\begin{aligned} (12)(a \mid 23) &= -(a \mid 13) \\ (13)(a \mid 12) &= -(a \mid 32) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (7).$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (i.e. from μ to λ), or from 1 to 3 (i.e. from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a, b be any two different forms, and if 1 be a set of variables in a and not in b, and 2 a set of variables in b and not in a, then, making the same supposition as in b, viz., that the constants a, b remain the same on both sides of the equation, we shall find that

$$(12)(a \mid 1\ldots)(b \mid 2\ldots) = (a \mid 2\ldots)(b \mid 2\ldots) = (a \mid 1\ldots)(b \mid 1\ldots).$$

With reference to the universal covariant $(12) = \lambda_1\mu_2 - \lambda_2\mu_1$, it may be remarked that we may consider such covariants for

more than two sets of variables; and we shall then obtain the following formulæ*:

$$(12)(23) = -(13), \quad (12)(23)(34) = (14), \ldots \quad \ldots\ldots(8),$$

of which the following is a consequence,

$$(12)(23)(31) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

It is clear that quadratic forms can only be combined in a *chain*†, which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12), (23), (34), (41); then, since

$$(12)(23)(34)(41) = (12)(23) \times (34)(41);$$

and since, by (3) and (4),

$$2(12)(23) = (a \mid 2\delta)^2 (31) = -2D(31),$$

$$2(34)(41) = (a \mid 4\delta)^2 (13) = -2D(13),$$

$$(31)(13) = 2,$$

it follows that

$$(12)(23)(34)(41) = 2D^2 \ldots\ldots\ldots\ldots\ldots (10).$$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2, according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2(-D)^n$.

* [This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," *Proceedings of the London Mathematical Society*, Vol. VII., p. 100. SP.]

† [I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the *American Journal of Pure and Applied Mathematics*, Vol. I. p. 127 [cf. XXVIII.]. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published. SP.]

A chain of $2n+1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows:

$$(a \mid x1)(a \mid 1y) = (a \mid xy)\, s - (xy)\, D;$$

or, with alternate numbers in the place of x, y,

$$(a \mid 12)(a \mid 23) = (a \mid 13)\, s - (13)\, D.$$

Multiplying now into $(a \mid 31)$, we get

$$(a \mid 12)(a \mid 23)(a \mid 31) = (a \mid 13)(a \mid 31)\, s + sD$$

in virtue of equation (5).

But $\quad (a \mid 13)(a \mid 31) = (a_{11}\lambda_1\mu_1 + \ldots)(a_{11}\mu_1\lambda_1 + \ldots)$

$$= (2a_{11}a_{22} - a^2_{12} - a^2_{21}) = 2D - s^2.$$

Hence $\quad (a \mid 12)(a \mid 23)(a \mid 31) = (3D - s^2)\, s \ldots\ldots\ldots (11).$

Again,

$$\begin{aligned}(a \mid 12)(a \mid 23)(a \mid 34) &= (a \mid 13)(a \mid 14)\, s - (13)(a \mid 34)\, D \\ &= (a \mid 14)\, s^2 - (14)\, sD + (a \mid 14)\, D \\ &= (a \mid 14)(D + s^2) - (14)\, sD;\end{aligned}$$

whence also

$$(a \mid 12)(a \mid 23)(a \mid 34)(a \mid 41) = (2D - s^2)(D + s^2) + s^2 D \ldots (12).$$

Suppose, in general, that

$$(a \mid 12)(a \mid 23) \ldots (a \mid lm) = A_m (a \mid 1m) - B_m (1m);$$

then

$$\begin{aligned}(a \mid 12)(a \mid 23) \ldots (a \mid mn) &= A_m (a \mid 1m)(amn) + B_m (a \mid 1n) \\ &= A_m \{(a \mid 1n)\, s - (1n)\, D\} + B_m (a \mid 1n) \\ &= (sA_m + B_m)(a \mid 1n) - A_m D\, (1n).\end{aligned}$$

Hence $\quad A_{m+1} = sA_m + B_m,\ B_{m+1} = A_m D;$

and consequently

$$A_{m+1} = sA_m + DA_{m-1} \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The form $\lambda_1\mu_1 + 0\lambda_1\mu_2 + s\lambda_2\mu_1 + D\lambda_2\mu_2$ is a form having s and D for invariants; and it may therefore be taken as the

canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following,

$$\sqrt{(s^2-4D)}\,(\lambda_1\mu_2+\lambda_2\mu_1)+s\,(\lambda\mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a; but we may also combine two having different coefficients, say, two forms a and b. Two such forms give rise to an invariant, namely, their product. In fact, we have

$$-(a\,|\,12)\,(b\,|\,12)=a_{11}b_{22}-a_{12}b_{21}-a_{21}b_{12}+a_{22}b_{11}=D_{ab},\text{ suppose};$$

and from (5), $\quad (a\,|\,12)\,(b\,|\,12)+(a\,|\,21)\,(b\,|\,12)=-s_a s_b,$

or $\quad (a\,|\,21)\,(b\,|\,12)=D_{ab}-s_a s_b \quad\ldots\ldots\ldots\ldots\ldots\ldots(14).$

The product of $(a\,|\,12)=a_{11}\lambda_1\mu_1+\ldots$ and $(b\,|\,13)=b_{11}\lambda_1\nu_1+\ldots$ gives a covariant; namely,

$$\begin{aligned}(a\,|\,12)\,(b\,|\,13) &= \begin{matrix} a_{11}\mu_1+a_{12}\mu_2, & a_{21}\mu_1+a_{22}\mu_2 \\ b_{11}\nu_1+b_{12}\nu_2, & b_{21}\nu_1+b_{22}\nu_2 \end{matrix}\\ &= \begin{vmatrix} a_{11}b_{21}-a_{21}b_{11}, & a_{11}b_{22}-a_{21}b_{12} \\ a_{12}b_{21}-a_{22}b_{11}, & a_{12}b_{22}-a_{22}b_{12} \end{vmatrix}(\mu_1,\,\mu_2)\,(\nu_1,\,\nu_2)\\ &= \vartheta_{ab}\,|\,23,\text{ suppose} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).\end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a\,|\,12)\,(a\,|\,13)=D_a(23),\quad (a\,|\,12)\,(a\,|\,23)=s_a(a\,|\,13)-D_a(13);$$

and, multiplying this by, or into, b, we get

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,43)=(b\,|\,43)\,(a\,|\,12)\,(a\,|\,13)=-D_a\,(b\,|\,42),$$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,34)=(b\,|\,43)\,(a\,|\,12)\,(a\,|\,13)=-D_a\,(b\,|\,24),$$

and $\quad (a\,|\,12)\,(a\,|\,13)\,(b\,|\,23)=D_a s_b,$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,32)=D_a s_b \ldots\ldots\ldots\ldots\ldots (16).$$

The remaining form to be investigated is

$$(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43).$$

The value of this is

$$\begin{vmatrix} (a_{11}b_{21}-a_{21}b_{11})\mu_1+(a_{12}b_{21}-a_{22}b_{11})\mu_2, & (a_{11}b_{22}-a_{21}b_{12})\mu_1+(a_{12}b_{22}-a_{22}b_{12})\mu_2 \\ a_{11}\rho_1+ \qquad a_{21}\rho_2, & a_{12}\rho_1+ \qquad a_{22}\rho_2 \end{vmatrix}$$

$$\begin{aligned}
&= \quad (-a_{21}a_{12}b_{11}+a_{11}a_{21}b_{12}+a_{11}a_{12}b_{21}-a_{11}a_{11}b_{22})\,\mu_1\rho_1 \\
&\quad +(-a_{21}a_{22}b_{11}+a_{21}a_{21}b_{12}+a_{11}a_{22}b_{21}-a_{11}a_{21}b_{22})\,\mu_1\rho_2 \\
&\quad +(\ \ a_{12}a_{12}b_{21}-a_{22}a_{12}b_{21}-a_{11}a_{12}b_{22}+a_{11}a_{22}b_{12})\,\mu_2\rho_1 \\
&\quad +(\ \ a_{22}a_{12}b_{21}-a_{22}a_{22}b_{11}-a_{21}a_{12}b_{22}+a_{21}a_{22}b_{12})\,\mu_2\rho_2 \\
&= \quad (-a_{11}D_{ab}+b_{11}D_{aa})\,\mu_1\rho_1 \\
&\quad +(-a_{21}D_{ab}+b_{21}D_{aa})\,\mu_1\rho_2 \\
&\quad +(-a_{12}D_{ab}+b_{12}D_{aa})\,\mu_2\rho_1 \\
&\quad +(-a_{22}D_{ab}+b_{22}D_{aa})\,\mu_2\rho_2 \\
&= D_{ab}\,(a\,|\,42)-D_{aa}\,(b\,|\,42) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).
\end{aligned}$$

Multiplying this into $(b\,|\,45)$, we obtain

$$(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43)\,(b\,|\,45) = D_{ab}\,(a\,|\,42)\,(b\,|\,45)-D_{aa}\,(b\,|\,42)\,(b\,|\,45)\,;$$

that is, referring to equation (15), and dropping the suffix ab,

$$(\vartheta\,|\,23)\,(\vartheta\,|\,35)=D_{ab}\,(\vartheta\,|\,25)-D_{aa}\,.\,D_{bb}\,(25).$$

But $$(\vartheta\,|\,23)\,(\vartheta\,|\,35)=s\,(\vartheta_{\vartheta}\,|\,25)-D_{\vartheta}\,(25).$$

Hence $$s_{\vartheta}=D_{ab},\quad D_{\vartheta}=D_{aa}\,.\,D_{bb}\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

as may be easily verified.

If, however, we multiply the equation

$$(a\,|\,12)+(a\,|\,21)=-s\,(12) \text{ by } (a\,|\,12),$$

we obtain

$$(a\,|\,12)\,(a\,|\,21)=2D_a-s_a^2\,;$$

hence $$(\vartheta\,|\,12)\,(\vartheta\,|\,21)=2D_{\vartheta}-s_{\vartheta}^2=2D_aD_b-D_{ab}^2\ \ldots\ldots\ (19),$$

and consequently

$$\left.\begin{aligned}\{(\vartheta\,|\,12)-(\vartheta\,|\,21)\}^2&=-4D_aD_b-2\,(2D_aD_b-D_{ab}^2) \\ &=-2\,(4D_aD_b-D_{ab}^2)\end{aligned}\right\}\ldots.(20).$$

The formula $s_{\vartheta}=D_{ab}$ is important as showing that ϑ is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of sym-

metrical forms, we are obliged to use the symmetrised form of ϑ, namely $(\vartheta \mid 12) - (\vartheta \mid 21)$. Thus, to adapt equation (17) to symmetrical forms, we have

$$(\vartheta \mid 23)\,(a \mid 43) = D_{ab}\,(a \mid 42) - D_{aa}\,(b \mid 42),$$

$$\begin{aligned}(\vartheta \mid 23)\,(a \mid 43) &= (a \mid 13)\,(b \mid 12)\,(a \mid 43)\\ &= D_a\,(14)\,(b \mid 12) = -D_a\,(b \mid 42).\end{aligned}$$

In general, we write $\bar{a}$ for the mean value of a, i.e., if

$$2\,(\bar{a} \mid 12) = (a \mid 12) - (a \mid 21),$$

we shall have $\quad (\bar{a} \mid 12) = (a \mid 12) + \tfrac{1}{2}s\,(12).$

Hence, since $(12)^2 = -2$,

$$(\bar{a} \mid 12)^2 = (a \mid 12)^2 + s^2 - \tfrac{1}{2}s^2 \text{ ....} (21),$$

and $\quad \bar{D} = D - \tfrac{1}{4}s^2,$

where $\bar{D}$ is the discriminant of the symmetrical function.

Similarly

$$\begin{aligned}\vartheta_{\bar{a}\bar{b}} = (\bar{a} \mid 12)\,(\bar{b} \mid 13) &= \{(a \mid 12) + \tfrac{1}{2}s_a\,(12)\}\,\{(b \mid 13) + \tfrac{1}{2}s_b\,(13)\}\\ &= \vartheta_{ab} - \tfrac{1}{2}\,\{s_a\,(b \mid 23) + s_b\,(a \mid 23)\} + \tfrac{1}{4}s_a s_b\,(23),\end{aligned}$$

and therefore

$$\left.\begin{aligned}2\vartheta_{\bar{a}\bar{b}} &= (\vartheta_{ab} \mid 23) - (\vartheta_{ab} \mid 32) - s_a s_b\,(23)\\ &= 2\,(\vartheta_{ab} \mid 23) - s_{\vartheta}\,(23) - s_a s_b\,(23)\end{aligned}\right\} \text{ ....} (22).$$

If we multiply this into $(a \mid 43)$, we have

$$\begin{aligned}2\,(\vartheta_{\bar{a}\bar{b}} \mid 23)\,(a \mid 43) &= 2\,(\vartheta \mid 23)\,(a \mid 43) - D_{ab}\,(a \mid 42) - s_a s_b\,(a \mid 42)\\ &= D_{ab}\,(a \mid 42) - 2D_a D_b\,(b \mid 42) - s_a s_b\,(a \mid 42).\end{aligned}$$

Putting $s_a = 0$, $s_b = 0$, we get the formula for symmetric functions.

XXX.

APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA*.

I PROPOSE to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the *Ausdehnungslehre,* and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers.* I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavours to determine the place of Quaternions and of what I have elsewhere† called Biquaternions in the more extended system, thereby *explaining* the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar* multi-

* [*American Journal of Mathematics Pure and Applied,* Vol. I. pp. 350—358.]
† *Proceedings of the London Mathematical Society* [XX. *supra*].

plication*; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *Scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

Consider now a system of n units $\iota_1, \iota_2, \ldots \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r, = \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the Ausdehnungslehre of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ; it is a thing totally different in kind from the vector ρ. The only way in

* [*American Journal of Mathematics*, Vol. I. p. 127 and p. 257, *supra*.]

which the factors q and ρ can be taken to be of the same kind, is to regard ρ as itself a special case of a quaternion, viz. a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations ρ and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors*; that is to say, they are operations which will turn a figure through a right angle in the three co-ordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points $\iota_0, \iota_1, \iota_2, \iota_3$, let us suppose that i turns the line $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that j turns $\iota_0 \iota_3$ into $\iota_0 \iota_1$; and that k turns $\iota_0 \iota_1$ into $\iota_0 \iota_2$. The turning of $\iota_0 \iota_2$ into $\iota_0 \iota_3$ is equivalent to a translation along the line at infinity $\iota_2 \iota_3$. We may, therefore, write $i = \iota_2 \iota_3$, and so $j = \iota_3 \iota_1$, $k = \iota_1 \iota_2$. Now i turns $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that is

$$i \,.\, \iota_0 \iota_2 = \iota_0 \iota_3,$$

or

$$\iota_0 \iota_3 = \iota_2 \iota_3 \,.\, \iota_0 \iota_2 = -\iota_2^2 \,.\, \iota_0 \iota_3.$$

We are therefore obliged to write $\iota_2^2 = -1$, and in a similar way we may find $\iota_1^2 = \iota_3^2 = -1$.

This at once enables us to find the rules of multiplication of the i, j, k. Namely, we have

$$jk = \iota_3 \iota_1 \,.\, \iota_1 \iota_2 = \iota_2 \iota_3 = i,$$

$$ki = \iota_1 \iota_2 \,.\, \iota_2 \iota_3 = \iota_3 \iota_1 = j,$$

$$ij = \iota_2 \iota_3 \,.\, \iota_3 \iota_1 = \iota_1 \iota_2 = k,$$

and finally

$$ijk = \iota_2 \iota_3 \,.\, \iota_3 \iota_1 \,.\, \iota_1 \iota_2 = -1.$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the square of each of our units equal to -1, as pointed out by Grassmann (*Math. Annalen*). But I venture to differ from his authority in thinking that the quaternion symbols do not in the first place answer to the "Elementargrösse" of the Ausdehnungslehre, but to

binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from factors of the same kind, and so of the product of two vectors, as a thing which might be thought of without regarding them as rectangular versors, grew upon Hamilton's mind, and led to the gradual replacement of the units i, j, k by the more general selective symbols S and V. To explain the laws of multiplication of i, j, k on this view, we must have recourse to the theory of "Ergänzung," or which comes to the same thing, *represent* an area ij by a vector k perpendicular to it. But the explanation in this case is by no means so easy; and it is instructive to observe that the distinction between a quantity and its "Ergänzung," i.e. between an area and its representative vector, which, for some purposes, it is so convenient to ignore, has to be reintroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors, which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

We have regarded the symbols i, j, k as rectangular versors operating on the quantities $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *rotors* (short for *rotators*) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation-velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A rotor is of the nature of the rotation-velocity of a rigid body, or of a force; it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b-a$); this is shewn by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols ι_1, ι_2, ι_3 may be taken to mean unit vectors along the axes. In fact, if we write $\iota_0+\iota_r=\alpha_r$, the points α will be situate on the axes at unit distance from the origin, and thus $\iota_r=\alpha_r-\iota_0$ will represent the unit vector from the origin to α_r.

The versors i, j, k will operate on these vectors in the same way as on the rotors $\iota_0\iota_1, \iota_0\iota_2, \iota_0\iota_3$. We find that

$$i\iota_2 = \iota_2\iota_3 . \iota_2 = \iota_3, j\iota_3 = \iota_1, k\iota_1 = \iota_2.$$

These rules of multiplication coincide with those for i, j, k if we write the latter in place of $\iota_1, \iota_2, \iota_3$. Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors $\iota_1, \iota_2, \iota_3$ are identical with the areas $\iota_2\iota_3, \iota_3\iota_1, \iota_1\iota_2$; it is only sometimes convenient to forget the difference between ι_1 and $\iota_2\iota_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points $\iota_0, \iota_1, \iota_2, \iota_3$ must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a *quadrant*. The product of four points $\alpha\beta\gamma\delta$ will then consist of three kinds of terms; (1) terms of the fourth order, being $\iota_0\iota_1\iota_2\iota_3$ multiplied by the determinant of the co-ordinates of the four points, which is proportional to $\sin(\alpha, \beta) \sin(\gamma, \delta) \cos(\alpha\beta, \gamma\delta)$; (2) terms of the second order, resulting from products of the form $\iota_0^2\iota_1\iota_2 = -\iota_1\iota_2$; (3) terms of order zero, resulting from products of the form $\iota_0^4\iota_0^2\iota_1^2$. Altogether we may arrange $\alpha\beta\gamma\delta$ in eight terms as follows:

$$\alpha\beta\gamma\delta = a + \Sigma b_{rs}\iota_r\iota_s + c\iota_0\iota_1\iota_2\iota_3. \quad (r, s \text{ different.})$$

And it is now easy to see that the product of any *even* number of linear factors will be of the same form. This form is what I have called a *biquaternion*, and may be easily exhibited as such. Namely, let us write ω for $\iota_0\iota_1\iota_2\iota_3$; then we have

$$i = \iota_2\iota_3, \ j = \iota_3\iota_1, \ k = \iota_1\iota_2,$$
$$\omega i = i\omega = \iota_1\iota_0, \ \omega j = j\omega = \iota_2\iota_0, \ \omega k = k\omega = \iota_3\iota_0,$$
$$\omega^2 = 1.$$

Therefore, the product of any even number of factors greater than two is a linear function of $1, i, j, k, \omega, \omega i, \omega j, \omega k$; that is to say, it is of the form $q + \omega r$, where q, r are quaternions.

* Dr Klein's names for the Geometry of a space of uniform positive or negative curvature. [Cf. p. 191 *supra*.]

While the multiplication of ω with i, j, k is scalar, its multiplication with ι_0, ι_1, ι_2, ι_3 is polar. The effect of multiplying by ω is to change any system into its polar system in regard to the absolute.

The chief classification of geometric algebras is into those of *odd* and *even* dimensions. The geometry of an elliptic space of n dimensions is the same as the geometry of the points at an infinite distance in a flat or parabolic space of $n+1$ dimensions; the theory of *points* and *rotors* in the former is the same as that of vectors and their products in the latter. Each requires a geometric algebra of $n+1$ units. Thus the algebra of four units, leading as above to biquaternions, is either that of points and rotors in an elliptic space of three dimensions, or of vectors and their products in a flat space of four dimensions. All geometric algebras having an even number of units are closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points ι_0, ι_1, ι_2, ι_3 by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0$, $\omega\iota_1$, $\omega\iota_2$, $\omega\iota_3$ by the symbols 1, ωi, ωj, ωk, but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n *units.*

In general, if we consider an algebra of n units, ι_1, ι_2, ... ι_n, such that $\iota_r^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain terms which are all of even order if m is even, and all of odd order if m is odd; for the substitution of -1 for any square factor of a term reduces the order of the term by 2.

A product of m units, all different, multiplied by any scalar is called a *term* of the order m. The sum of several terms of order m each multiplied by a scalar, is a *form* of order m. The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the m$^{\text{th}}$ *order is* $+1$ *or* -1 *according as the integer part of* $\frac{1}{2}(m+1)$ *is even or odd.* For the product $\iota_1\iota_2 \ldots \iota_m\iota_1\iota_2 \ldots \iota_m$ is transformed into $\iota_1^2\iota_2^2 \ldots \iota_m^2$ by $\frac{1}{2}m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2}m(m+1)$ is even or odd, which is equivalent to the rule stated.

The multiplication of a term P *of order* m *by a term* Q *of order* n, *having* k *factors common, is scalar or polar according as* mn $-$ k^2 *is even or odd.* Let $P = CP'$ and $Q = CQ'$, where C, P', Q' have no common factor; then the steps from $CP'CQ'$ to $CP'Q'C$, $CQ'P'C$, $CQ'CP'$ require respectively $k(n-k)$, $(m-k)(n-k)$, $k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn - k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k = 0$.)

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n = 2m + 1$, there are n terms of the order $n - 1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n - 2$ factors common.

We obtain in this way all the terms of the second order; and from them we can build up the terms of the fourth, sixth orders, &c. Let the product of all the units $\iota_1\iota_2 \ldots \iota_n$ be called ω, then these terms of the order $n-1$ shall be defined by the equations $k_r = \omega\iota_r$. It will follow that $k_1k_2 \ldots k_n = \mp 1$ according as m is even or odd, or, what is the same thing, according as the squares of the k are $+1$ or -1. By means of this formula, terms of order higher than m in the k, may be replaced by terms of order not higher than m. The multiplication of the k is always polar.

The terms of even order, regarded as compound units, constitute an algebra which is *linear* in the sense of Professor Peirce, viz. it is such that the product of any two of these terms is again a term of the system. The number of them is $2^{n-1} = 2^{2m}$; for the whole number of terms, odd and even, is

$$1 + n + \tfrac{1}{2}n(n-1) + \ldots + n + 1 = (1+1)^n = 2^n,$$

and the number of even terms is clearly equal to the number of odd terms.

I shall call the algebras whose units are the even terms formed with n elementary units $\iota_1\iota_2 \ldots \iota_n$, the *n-way geometric algebra.* Thus quaternions are the *three-way* algebra. We may regard the units of quaternions as expressed in either of two ways. First, in terms of the elementary units $\iota_1\iota_2\iota_3$; they are then $(1,\ \iota_2\iota_3,\ \iota_3\iota_1,\ \iota_1\iota_2)$. Secondly, we may write k_1, k_2 for the terms $\iota_2\iota_3$, $\iota_3\iota_1$, and the system may then be written $(1, k_1, k_2, k_1k_2)$. In this second form it is identical with the entire algebra of two elementary units, including both odd and even terms.

The five-way algebra depends upon the five terms k_1, k_2, k_3, k_4, k_5 and their products; the number of terms is sixteen. Now *we may obtain the whole of these sixteen terms by multiplying the quaternion set*

$$(1,\ k_1,\ k_2,\ k_1k_2)$$

by this other quaternion set

$$(1,\ k_4k_5,\ k_5k_3,\ k_3k_4).$$

For each of the sixteen products so obtained is a term of the even five-way algebra, and the products are all distinct. More-

over, *the two quaternion sets are commutative with one another.* For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients ωxyz are themselves quaternions of another set of units $(1, i_1, j_1, k_1)$, the i_1, j_1, k_1 being commutative with i, j, k.

I shall now extend this proposition, and shew that *the* $(2m+1)$ *way algebra is a compound of* m *quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_0 = k_1k_2$, and then

$$\begin{array}{ll} p_1 = k_1k_2k_6k_7 = p_0k_6k_7, & q_1 = k_3k_4k_5, \\ p_2 = p_1k_{10}k_{11} \qquad\qquad\quad , & q_2 = q_1k_8k_9, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ p_r = p_{r-1}\,k_{4r+2}\,k_{4r+3} \quad , & q_r = q_{r-1}\,k_{4r}\,k_{4r+1}. \end{array}$$

Consider now the quaternion sets

$$\begin{array}{l} 1,\ k_1,\ k_2,\ k_1k_2 \\ 1,\ k_4k_5,\ k_5k_3,\ k_3k_4 \\ 1,\ p_0k_6,\ p_0k_7,\ k_6k_7 \\ 1,\ q_1k_8,\ q_1k_9,\ k_8k_9 \\ 1,\ p_1k_{10},\ p_1k_{11},\ k_{10}k_{11} \\ \cdots\cdots\cdots\cdots\cdots \\ 1,\ q_{r-1}k_{4r},\ q_{r-1}k_{4r+1},\ k_{4r}k_{4r+1} \\ 1,\ p_{r-1}k_{4r+2},\ p_{r-1}k_{4r+3},\ k_{4r+2}k_{4r+3}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

viz.: a p-set and a q-set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical

with the terms of the $(2m+1)$ way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p-set and a q-set, because the q-set is of even order in the k, and no factors are common to the two sets. It remains only to examine the case of two p-sets and of two q-sets. Compare the two p-sets

$$1,\ p_{r-1}k_{4r+2},\ p_{r-1}k_{4r+3},\ k_{4r+2}k_{4r+3},$$
$$1,\ p_{s-1}k_{4s+2},\ p_{s-1}k_{4s+3},\ k_{4s+2}k_{4s+3},$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1}k_{4s+2}$, $p_{s-1}k_{4s+3}$, which are of odd order in the k; consequently the multiplication is scalar. The term $k_{4s+2}k_{4s+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q-sets

$$1,\ q_{r-1}k_{4r},\ q_{r-1}k_{4r+1},\ k_{4r}k_{4r+1},$$
$$1,\ q_{s-1}k_{4s},\ q_{s-1}k_{4s+1},\ k_{4s}k_{4s+1}.$$

Here again all the terms of the first sets are factors of $q_{s-1}k_{4s}$ and of $q_{s-1}k_{4s+1}$, and they have no factors in common with $k_{4s}k_{4s+1}$; since then all the terms are of even order in the k, the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: suppose this true for the first $m-1$ sets; that is to say, that no two of the products formed from them are identical or such that their product is $\pm k_1k_2 \dots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1k_2 \dots k_{2m-1}k_{2m}k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1}, so that it cannot be equal to ab. The product $abcd$ cannot be $\pm k_1 \dots k_{2m+1}$ unless cd is $k_{2m}k_{2m+1}$ and ab is $k_1k_2 \dots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first

$m-1$ sets are all distinct for the purposes of the $(2m-1)$ way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$ way algebra. But it is easy to see that the products of the first two sets are distinct.

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$ way algebra are also even terms of the $2m$-way algebra, and so also are their products by ω; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$-way algebra.

The multiplication of ω with the k of the $(2m-1)$ way algebra is scalar, because the k are factors in the ω, and they are both even terms.

Hence the $2m$-way algebra is a product of the $(2m-1)$ way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.

*XXXI.

BINARY FORMS OF ALTERNATE VARIABLES*.

INTRODUCTION.

1. ALTERNATE numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; i.e., if $\bar{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \ldots$, $\bar{\beta} = b_1\beta_1 + b_2\beta_2 + \ldots$, where the a, b are scalars, and the α, β alternate numbers, then we shall have $\bar{\alpha}\bar{\beta} = -\bar{\beta}\bar{\alpha}$, and $\bar{\alpha}^2 = 0 = \bar{\beta}^2$. If M, N are homogeneous functions of alternate numbers of degrees m, n respectively, the number of interchanges of consecutive letters necessary to pass from MN to NM is mn; thus we have

$$MN = (-)^{mn} NM.$$

Or the product of two functions changes sign when the order of the functions is changed *only when their degrees are both odd;* that is to say, forms of odd degree among themselves behave like alternate numbers, forms of even degree in all cases like scalars. It follows that the square of any form of odd degree is zero.

Determinants of alternate numbers.

2. In expanding a determinant of alternate numbers, the order of the *rows* must be followed in multiplication; that is to say, in every term of the expanded determinant the

* [Communicated to the London Mathematical Society (June 12, 1879) by Dr Spottiswoode, P.R.S., and subsequently printed in the *Proceedings*, Vol. x. pp. 214—221. See note at end of this paper.]

constituent from the first row must be written first, that from the second row second, and so on. Thus, in expanding

$$\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3 \\ \nu_1, & \nu_2, & \nu_3 \end{vmatrix} = (\lambda\mu\nu),$$

the terms are of the form $\pm\lambda_r\mu_s\nu_t$, where rst is a permutation of 123 and the signs follow the ordinary rule. An interchange of two *columns* will then alter the sign of the determinant but an interchange of two rows will leave it unaltered. For the change of sign caused by the interchange is in the latter case counteracted by the change in the order of multiplication. Thus the determinant $(\lambda\mu\nu)$ written above is a symmetrical function of the $\lambda\mu\nu$.

3. Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space), and in that case it is convenient to regard the product of all the numbers of any set as equal to unity. Hence the products of all but one of the numbers make a new set, the *reciprocal* numbers. The n^{th} power of a determinant of *even* order n is $-\underline{|n}$; the $(n-1)^{\text{th}}$ power is the determinant of the reciprocal numbers. Considering especially determinants of the second order, we have an important theorem of their multiplication, viz., $(\lambda\mu)(\mu\nu) = -(\lambda\nu)$. For

$$\begin{aligned}(\lambda_1\mu_2 - \lambda_2\mu_1)(\mu_1\nu_2 - \mu_2\nu_1) &= \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &= (\lambda_2\nu_1 - \lambda_1\nu_2)\,\mu_1\mu_2 \\ &= -(\lambda_1\nu_2 - \lambda_2\nu_1).\end{aligned}$$

Hence $\quad (\lambda\mu)(\mu\nu)(\nu\rho)\dots(\sigma\tau)\ \{n \text{ factors}\} = (-)^{n-1}(\lambda\tau).$

An analogous theorem holds for determinants of the n^{th} order; viz., if we denote a determinant with r rows of λ and s rows of μ by $(\lambda^r\mu^s)$, where $r+s=n$; then

$$(\lambda^r\mu^s)(\mu^r\nu^s)\dots(\sigma^r\tau^s)\ \{k \text{ factors}\} = (-)^{s(n+r)k}(\lambda^r\tau^s) = (-)^{sk}(\lambda^r\tau^s),$$

since $n + r = 2r + s \equiv$ (mod. 2), and so $s(n+r) \equiv s^2 \equiv s$ (mod. 2).

Multipartite Forms.

4. A homogeneous function linear in each of n sets of k alternate numbers is called a k-ary multipartite form of the n^{th} order, or shortly, a k-ary form of the n^{th} order. We may consider also forms of any order lower than k in any of the sets; but for the present we restrict ourselves to the case in which the forms are linear. Consider now k forms, each linear in regard to the k alternate numbers $\lambda_1, \lambda_2, \dots \lambda_k$; viz.,

$$F_a = a_1\lambda_1 + a_2\lambda_2 + \dots + a_k\lambda_k$$
$$\vdots$$
$$F_k = k_1\lambda_1 + k_2\lambda_2 + \dots + k_k\lambda_k,$$

where the coefficients $a, b, \dots k$ are themselves k-ary multipartite forms of alternate numbers.

The product $F_aF_b \dots F_k = \Pi F$ *is an invariant;* that is to say, if for the λ we substitute k linear functions of them, say the μ, then the functions F will be transformed into functions of the μ; and if we form the same function of the new coefficients that ΠF is of the old coefficients, one will be equal to the other multiplied by the determinant of transformation.

For the product is $I \,.\, \lambda_1\lambda_2 \dots \lambda_k$, whether we regard the λ as linear functions of the μ or not; but in the latter case

$$\lambda_1\lambda_2 \dots \lambda_k = D \,.\, \mu_1\mu_2 \dots \mu_k,$$

where D is the determinant of transformation.

But it is also to a constant factor *près* the only function possessing this property. For let I be such a function, and calculate it for the linear forms $\lambda_1, \lambda_2, \dots \lambda_k$. As all the coefficients are here either unity or zero, I must be represented by a constant, say I_0. Now, expressing the λ in terms of the μ, we have, by hypothesis, $I_\mu = DI_0 = \lambda_1 \dots \lambda_k \,.\, I_0$.

It is to be remarked that the coefficients of transformation may themselves be forms involving alternate numbers to any *even* order. Otherwise the μ would not be alternate numbers, which is implied.

Moreover, we may regard the λ as no longer linear forms, but forms of any odd order; the new invariant will then be equal to the old one multiplied by a *commutant* of transformation. This leads to a useful theorem in the comparison of invariants; e.g.,

$$c \mid 4568 \,.\, a \mid 82 \,.\, b \mid 8'3 \,.\, c \mid 4568' = (c \mid 4568 \mid)^2 \,.\, a \mid 8'2 \,.\, b \mid 8'3.$$

The proposition may be further extended by considering forms which involve the λ to an order higher than the first, but less than k; i.e., linear functions of their products r together. Let F_a, F_b, ...F_k be forms such that the sum of their orders in the λ is equal to k; then their product is an invariant (and the only one) in regard to linear transformations of the λ.

If the sum of the orders is less than k, $= h$ suppose, the product is a covariant; viz., it is a linear function of the products of the λ, h together, which, whether derived from the original forms before or after the λ are replaced by linear functions of the μ, has the same value. In this case the r-products of the λ are replaced by linear functions of the r-products of the μ, the coefficients being determinants of the r^{th} order formed with the coefficients of the μ.

Now suppose any number of forms $F_a \ldots F_n$ to involve any number of sets of alternate numbers $\lambda\mu \ldots \tau$, yet so that the sum of the orders in any one set is not greater than the number of alternates in that set; then the product of the forms is an invariant or covariant in regard to each of the sets taken separately, in the sense explained above; and it is the only function which possesses this property.

Expression of Unsymmetrical forms in terms of Symmetrical forms and Determinants of the Variables.

5. The binary form in two sets of variables,

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = a12,$$

will be called *symmetrical* when $a_{12} = a_{21}$. In that case the

interchange of the variables λ, μ only alters the sign of the form; we have $a12 = -a21$. We have, in general,

$$a12 + a21 = (a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1) = s(\lambda\mu).$$

The factor $a_{12} - a_{21}$ is an invariant when both sets of variables are transformed by the same substitutions; in fact, we have

$$(\lambda_1\mu_2 - \lambda_2\mu_1)\, a12 = a_{21} - a_{12} = -s,$$

which exhibits it as a product of the form by the universal covariant $(\lambda\mu)$ or (12). We may make a symmetrical form from $a12$ by adding $-a21$ to it; half this sum shall be called the mean value of $a12$ and denoted by $\overline{a12}$. Thus, we have

$$\overline{a12} = \tfrac{1}{2}(a12 - a21) = a_{11}\lambda_1\mu_1 + \tfrac{1}{2}(a_{12} + a_{21})(\lambda_1\mu_2 + \lambda_2\mu_1) + a_{22}\lambda_2\mu_2;$$

but also

$$\tfrac{1}{2}(a12 + a21) = \tfrac{1}{2}(a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1),$$

therefore $$a12 = \overline{a12} + \tfrac{1}{2}s\,.\,(12),$$

where $$-s = (12)\,a12.$$

It is easy to apply this to forms involving more sets of variables, if we remember that in these results the coefficients may themselves be such forms. We have, for example,

$$a123 = \overline{a123} - \tfrac{1}{3}\{(23)\,a\,.\,(23) + (13)\,a\,.\,(13) + (12)\,a\,.\,(12)\},$$

and so, generally,

$$a12\ldots k = \overline{a12\ldots k} - \frac{\Sigma\,(12)\,a\,.\,(12)}{n} + \frac{\Sigma\,(12)\,(34)\,a\,.\,(12)\,(34)}{\tfrac{1}{2}n\,(n-1)} - \ldots,$$

the coefficients being the reciprocals of the binomial coefficients*.

Theory of Quadratic Forms.

6. In connection with the quadratic form $a12$, we have already considered the invariant

$$a12 + a21 = s_a\,(12), \text{ where } (12)\,a12 = -s_a.$$

We have thus the formula

$$a21 = -a12 + s_a\,.\,(12) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

* [In the last two equations of Par. 5, the symbols a are to be considered as abbreviations for $a23$, $a13$, $a12$; $a1234$, &c. Sp.]

To this if we add

$$(13)\, a12 = -a32, \quad (23)\, a12 = -a13 \ldots\ldots\ldots\ldots (2),$$

we shall have exhausted all the invariants and covariants of the first order in the coefficients.

The *discriminant* which is the only invariant of the second order, is given by the square of the quadratic form. We may write

$$a12 \, . \, a12 = -2D_{aa} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $D_{aa} = a_{11}a_{22} - a_{12}a_{21}$.

We have

$$\begin{aligned} a21 \, . \, a21 &= \{-a12 + s_a \, . \, (12)\} \{-a12 + s_a \, . \, (12)\} \\ &= a12 \, . \, a12 - 2s_a (12)\, a12 + s_a^2 \, . \, (12)^2 \\ &= a12 \, . \, a12 + 2s_a^2 - 2s_a^2 = a12 \, . \, a12, \end{aligned}$$

as it obviously should; and this may be regarded as a proof of the formula (2).

Moreover,

$$\begin{aligned} a12 \, . \, a21 = a12 \{-a12 + s_a (12)\} &= -a12 \, . \, a12 - s_a^2 \\ &= 2D_{aa} - s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4); \end{aligned}$$

and again since

$$\overline{a12} = a12 - \tfrac{1}{2}s \, . \, (12),$$

$$\begin{aligned} \overline{a12} \, . \, \overline{a12} &= a12 \, . \, a12 + s_a^2 + \tfrac{1}{4}s_a^2 (12) \\ &= a12 \, . \, a12 + \tfrac{1}{2}s_a^2; \end{aligned}$$

therefore $\overline{D_{aa}} = D_{aa} - \tfrac{1}{4}s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$

where $\overline{D_{aa}}$ is the corresponding invariant of the symmetrical function $\bar{a}$.

Since the product of two *even* forms is independent of their order, we have

$$a12 \, . \, a13 = a13 \, . \, a12.$$

But

$$\begin{aligned} a12 \, . \, a13 + a13 \, . \, a12 &= -(23)\, a12 \, . \, a13 \, . \, (23) \\ &= a13 \, . \, 13\, (23) = -2D_{aa} (23), \end{aligned}$$

therefore $a12 \, . \, a13 = -D_{aa} (23) \ldots\ldots\ldots\ldots\ldots\ldots (6).$

Hence

$$a12\,.\,a31 = a12\,\{-a13 + s_a\,(13)\} = D_{aa}\,.\,(23) - s_a\,.\,a32 \,\ldots\ldots (7),$$

$$\begin{aligned} a21\,.\,a31 &= \{-a12 + s_a\,(12)\}\,\{-a13 + s_a\,(13)\} \\ &= -D_{aa}\,(23) + s_a\,(a32 + a23) - s_a^2\,(23) \\ &= -D_{aa}\,(23) = a12\,.\,a13 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8), \end{aligned}$$

$$a21\,.\,a13 = a21\,\{-a31 + s_a\,(13)\} = D_{aa}\,(23) - s_a\,.\,a23 \ldots (9).$$

In the case of a symmetrical function $s = 0$, and the formulæ reduce themselves to the following:

$$a12\,a12 = -2D_{aa},\quad a21 = -a12,\quad a12\,.\,a13 = -D_{aa}\,.\,(23).$$

The following formulæ are also worth noticing:

$$a12\,.\,a34 + a13\,.\,a24 = \{D_{aa}\,.\,(14) - s_a\,.\,a14\}\,(23) \,\ldots\ldots (10),$$

$$\begin{aligned} a12\,.\,a34 - a21\,.\,a43 &= s_a\,\{a23\,.\,(14) - a14\,.\,(23)\} \\ &= s_a\,\{a41\,.\,(23) - a32\,.\,(14)\} \ldots\ldots (11); \end{aligned}$$

these enable us to make a single or bifid substitution in a product of the form by itself with two different sets of variables.

Passing now to two different forms a, b, we have in the first place the covariant

$$\vartheta_{ab}23 = a12.b13 = \begin{vmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12} \end{vmatrix} (\mu_1\mu_2)(\nu_1\nu_2) \ldots (12),$$

and next the invariant

$$D_{ab} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = -a12\,.\,b12 \,\ldots\ldots (13).$$

We may now make the following analysis of the one-place and two-place products of a and b:

$$\left.\begin{aligned} a12\,.\,b13 &= \vartheta 23, \\ a12\,.\,b31 &= a12\,\{-b13 + s_b\,(13)\} = -\vartheta 23 - s_b\,.\,a32, \\ a21\,.\,b13 &= \{-a12 + s_a\,(12)\}\,b13 = -\vartheta 23 - s_a\,.\,b23, \\ a21\,.\,b31 &= \{-a12 + s_a\,(12)\}\,\{-b13 + s_b\,(13)\} \\ &= \vartheta 23 + s_b\,.\,a32 + s_a\,.\,b23 - s_a s_b\,.\,23 \end{aligned}\right\} (14),$$

$$\left.\begin{aligned} a12\,.\,b12 &= -D_{ab}, \\ a12\,.\,b21 &= a12\,\{-b12 + s_b\,.\,(12)\} = D_{ab} - s_a s_b \end{aligned}\right\} \ldots\ldots (15).$$

Let us now apply to ϑ all those formulæ which we have proved for a single form. Thus from (1), we have

$$\vartheta 23+\vartheta 32=-(23)\vartheta 23(23)=-(23)a12.b13.(23)=a13.b13.(23)$$
$$=D_{ab}(23) \text{ or } s_{\vartheta}=D_{ab} \quad\ldots\ldots(16).$$

Next, from (3),

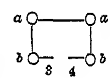
$$-2D_{\vartheta\vartheta}=\vartheta 23\,.\,\vartheta 23=a12\,.\,b13\,.\,a42\,.\,b43=D_{aa}D_{bb}\,(14)^2$$
$$=-2D_{aa}D_{bb} \text{ or } D_{\vartheta\vartheta}=D_{aa}D_{bb} \quad\ldots\ldots(17).$$

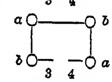
Hence
$$\left.\begin{aligned}\vartheta 23\,.\,\vartheta 32&=2D_{aa}D_{bb}-D_{ab}^2,\\ \overline{\vartheta 23}&=\vartheta 23-\tfrac{1}{2}D_{ab}(23),\\ D_{\overline{\vartheta\vartheta}}&=D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2,\end{aligned}\right\}\quad\ldots\ldots(18),$$
and

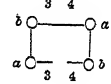
which last agrees with the known formula.

We have next the one-place products of ϑ by itself; namely,

$$\left.\begin{aligned}\vartheta 23.\vartheta 24&=-D_{aa}D_{bb}(34)=\vartheta 32.\vartheta 42=a12.b13.a52.b54\\ &=a13.b12.a54.b52\\ \vartheta 23.\vartheta 42&=+D_{aa}D_{bb}(34)-D_{ab}.\,\vartheta 43=a12.b13.a54.b52\\ \vartheta 32.\vartheta 24&=+D_{aa}D_{ab}(34)-D_{ab}.\,\vartheta 34=a13.b12.a52.b54\end{aligned}\right\}(19)$$







but it is important also to calculate these for the mean value $\overline{\vartheta}$; namely, we have

$$\overline{\vartheta 23}\,.\,\overline{\vartheta 24}=\{\vartheta 23-\tfrac{1}{2}D_{ab}\,(23)\}\,\{\vartheta 24-\tfrac{1}{2}D_{ab}\,(24)\}$$
$$=-D_{aa}D_{bb}\,(34)+\tfrac{1}{2}D_{ab}\,(\vartheta 43+\vartheta 34)-\tfrac{1}{4}D_{ab}^2\,(34)$$
$$=-(D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2)\,(34)=-D_{\overline{\vartheta\vartheta}}\,.\,(34),$$

as it ought to be, confirming equation (18).

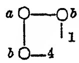
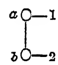
Lastly, from (10) and (11), we get

$$\vartheta 12\,.\,\vartheta 34+\vartheta 13\,.\,\vartheta 24=\{D_{aa}\,.\,D_{bb}\,.\,(14)-D_{ab}\,.\,\vartheta 14\}\,(23)$$
$$\vartheta 12\,.\,\vartheta 34-\vartheta 21\,.\,\vartheta 43=D_{ab}\,\{23\,.\,(14)-\vartheta 14\,(23)\}$$
$$=D_{ab}\,\{\vartheta 41\,.\,(23)-\vartheta 32\,.\,(14)\}\ldots\ldots(20).$$

There are four one-place products of a, a, b; namely,

$$\left.\begin{aligned}
&a12.a32.b34 = D_{aa}.b14, \quad a12.a32.b43 = D_{aa}.b41,\\
&a12.a23.b34 = -D_{aa}.b14 + s_a.\vartheta 14 + s_a^2.b14 = (s_a^2 - D_{aa})b14 + s_a.\vartheta 14,\\
&a12.a23.b43 = -D_{aa}.b41 - s_a.\vartheta 14 - s_a s_b.a41 - s_a^2 b14 + s_a s_b (14)\\
&\qquad = (s_a^2 - D_{aa}).b41 - s_a.\vartheta 14 - s_a s_b.a41,
\end{aligned}\right\} \quad (21$$

When the forms are symmetrical, all these reduce to

 $= \pm 2D_{ab}$  $= \pm D_{aa}.b14$  $= \pm \vartheta 12.$

Three forms a, b, c give rise to the new invariant

$$\begin{aligned}
R_{abc} = a12.b23.c31 = &+ a_{11}(b_{21}c_{22} - b_{22}c_{12})\\
&- a_{12}(b_{11}c_{22} - b_{12}c_{12})\\
&- a_{21}(b_{21}c_{21} - b_{22}c_{11})\\
&+ a_{22}(b_{11}c_{21} - b_{12}c_{11}) \quad \ldots\ldots\ldots\ldots (22),
\end{aligned}$$

and the remaining closed products are of the type

$$a21.b23.c31 = \{a12 + s_a(12)\}b23.c31 = -R_{abc} + s_a D_{bc} - s_a s_b s_c.$$

R calculated for the symmetrical functions $\bar{a}$, $\bar{b}$, $\bar{c}$ is

$$\begin{aligned}
\overline{R_{abc}} &= \{a12 - \tfrac{1}{2}s_a(12)\}\{b23 - \tfrac{1}{2}s_b(23)\}\{c31 - \tfrac{1}{2}s_c(31)\}\\
&= R_{abc} + \tfrac{1}{2}(s_a D_{bc} + s_b D_{ca} + s_c D_{ab}) - \tfrac{1}{2}s_a s_b s_c \quad \ldots\ldots\ldots (23).
\end{aligned}$$

It may be mentioned in this place that the determinant made with the coefficients of four forms a, b, c, d has the value

$$\text{Det. } (a, b, c, d) = s_a R_{bcd} - s_b R_{cda} + s_c R_{dab} - s_d R_{abc} = s_a \overline{R_{bcd}} - \&c.,$$

and that no additional invariants are introduced by a greater number of forms.

We now consider the open one-place products of 3 forms. There are evidently four such products having b in the middle; and they are in the first place connected by the formulæ

$$a12.b23.c34 + a42.b23.c31 = -R_{abc}(14),$$

$$a12.b23.c34 + a12.b32.c34 = -s_b a13c34 = s_b \vartheta_{ac} 14 + s_a s_b.c14,$$

$$a12\ b23.c34 - a34.b23.c12 = -c32\,a34.b21 + c23.a21.b43.$$

[Here the paper abruptly ends. It has been found very difficult to devise a atisfactory way of dealing with the other notes on *graphs* on account of the ragmentary condition in which they were left. Prof. H. J. S. Smith says 'the)apers tied up with the fragment on *Quadric forms* are little more than mere ottings. They serve to show that Clifford had applied the method to the Quintic and Sextic. They should be preserved.. .If the letters by which Clifford lenotes certain concomitants are those used in Clebsch's *Binäre Formen*, or in ıny other accessible work or memoir, it might be possible to rescue the pages ·elating to the Quintic and Sextic.' By the advice, and at the expense, of Dr Spottiswoode, these fragments have been lithographed, and fifty copies have been printed off for circulation among the principal libraries.]

XXXII.

ON MR SPOTTISWOODE'S CONTACT PROBLEMS*.

THE present communication consists of two parts.

The first part treats of the contact of conics with a given surface at a given point; this class of questions was first treated by Mr Spottiswoode in his paper "On the Contact of Conics with Surfaces," and general formulæ applicable to all such questions were given.

The results of that paper are here reproduced with some additions; with the exception of a few collateral theorems, these are all contained in the following Table:—

†Number of five-point conics through fixed point .... = 6
†Order of surface formed by five-point conics through fixed axis .... = 8
Number of six-point conics through fixed axis .... = 9 ‡
†Number of seven-point conics .... = 70

[* From the *Philosophical Transactions* of the Royal Society of London, Vol. CLXIV. Part 2.]

† These results constitute the additions.

‡ [In the Memoir quoted by Professor Clifford, it was stated that the number of conics passing through a given axis and having six-pointic contact with a surface at a given point is ten. In making this statement I overlooked the fact that, in order to put in evidence that a certain quantity was a factor of the equation which determines the positions of the planes of the conics, the equation was multiplied by a quantity D which is a linear function of the position. In reckoning the degree of the equation this factor must of course be discarded. The degree is consequently less by unity than that stated in the Memoir; viz. it is 9, as proved by Professor Clifford.—SP. July 3, 1873.]

The second part treats of the contact of a quadric surface with a surface of the order n; and in particular it determines the number of points at which a quadric (other than the tangent plane reckoned twice) can have four-branch contact with the surface. In his paper "On the Contact of Surfaces," Mr Spottiswoode proves that at an arbitrary point on a surface there is no other solution than the doubled tangent plane, and gives the conditions that must be satisfied by those points at which another solution is possible.

The method here adopted is an extension of that applied by Joachimstal to the contact of lines with curves and surfaces. The co-ordinates of a point on a conic are expressed in terms of a single parameter, those of a point on a quadric by two parameters. To determine the intersection with a given surface we have an equation in the parameter or parameters, and the conditions of contact are expressed in terms of the coefficients of that equation. The special case of the intersection of a quadric with a cubic surface is treated by the method of representation on a plane.

PART I.—THE CONTACT OF CONICS WITH SURFACES OF ORDER n.

I.

The current plane-coordinates being denoted by

$$X_1,\ X_2,\ X_3,\ X_4,$$

let the equations of the three points A, B, C be respectively

$$0 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 \equiv \Sigma aX,$$
$$0 = \Sigma bX,$$
$$0 = \Sigma cX.$$

The quantities a_i, b_i, c_i ($i = 1, 2, 3, 4$) are the coordinates of the points A, B, C. The symbol A itself I shall use indifferently, as denoting either the form ΣaX or the differential operator

$$a_1\partial_{x_1} + a_2\partial_{x_2} + a_3\partial_{x_3} + a_4\partial_{x_4} \equiv \Sigma a\partial_x,$$

where x_1, x_2, x_3, x_4 are the current point-coordinates. It will be seen in the sequel that this double meaning is useful, while it does not introduce any confusion. Similar interpretations are of course to be given to the symbols B, C, and the like.

Consider now the point

$$P \equiv A + \theta B + \theta^2 C,$$

whose coordinates are $a_i + \theta b_i + \theta^2 c_i$ ($i = 1, 2, 3, 4$). If we suppose θ to take all possible values, the point P will describe a conic section whose tangential equation is

$$0 = 4AC - B^2 \equiv K_2.$$

To the value $\theta = 0$ corresponds the point A, to $\theta = \infty$ the point C; while the equation shews that B is the intersection of the tangents at A and C. (Fig. 51.)

To find the point at which this conic intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation in θ of the order $2n$, the solution of which will give the values of θ belonging to the $2n$ points of intersection.

If in this equation the term independent of θ vanishes, then $\theta = 0$ is a root of the equation; consequently the point A is one of the intersections, or the surface u_n passes through the point A. If also the coefficient of θ vanishes, another root of the equation coincides with zero, and *two* points of intersection are at A. And generally if the coefficient of θ^m is the first that does not vanish, m roots of the equation coincide with zero, and m points of intersection are united at A.

The result of substituting the coordinates of any point P in u_n may be conveniently represented by means of the differential operator P. It is known, in fact, that

$$(p_1\partial_{x_1} + p_2\partial_{x_2} + p_3\partial_{x_3} + p_4\partial_{x_4})^n \cdot (x_1, x_2, x_3, x_4)^n \equiv \underline{|n} \cdot (p_1, p_2, p_3, p_4)^n;$$

or, which is the same thing, $P^n u_n$ is $\underline{|n}$ times the result of substituting the coordinates of P in u_n. Our result may therefore be stated in the following form :—

The necessary and sufficient conditions that the conic $K_2 \equiv 4AC - B^2$ *may have* m-*point contact with the surface* u_n

at the point A, *are that in the expansion of* $(\mathrm{A} + \theta\mathrm{B} + \theta^2\mathrm{C})^{\mathrm{n}} . \mathrm{u}_{\mathrm{n}}$ *in powers of* θ, *the* m*th power of* θ *is the lowest whose coefficient does not vanish.*

II.

Equating to zero the coefficients of 1, θ, θ^2 in this expansion, we obtain

$$0 = A^n . u_n,$$
$$0 = nA^{n-1}B . u_n,$$
$$0 = \tfrac{1}{2}n(n-1)A^{n-2}B^2 . u_n + nA^{n-1}C . u_n.$$

Before proceeding further with these equations, it is convenient to make the following remarks upon their nature, which will serve to simplify the expression of them.

In the first place, then, we have here a series of relations among the coordinates of the points A, B, C and the coefficients of the surface u_n; and the determination of the coordinates and coefficients so as to satisfy a certain number of the relations presents us with the solution of various geometrical problems. These problems fall naturally into three classes.

1. The surface u_n and the point A are given. In this case the unknowns are the ratios of the eight quantities b, c, a singly infinite number of solutions corresponding to each conic; and we are accordingly able to satisfy seven of the equations*. The problem here is to find the number of conics which have seven-point contact at a given point of a given surface.

We may, however, impose beforehand certain restrictions upon the values of the unknowns, and so consider problems which involve a less number of the equations. While the number of the septactic conics is definite, the sextactic conics form a singly infinite series; and we may ask what is the number of them:

(*a*) whose planes pass through a given point,
(*b*) which meet a given line, or
(*c*) which touch a given plane.

* Viz. six besides the first, which is satisfied identically.

The quintactic conics, again, form a doubly infinite system, and we may inquire about the number of them which satisfy two conditions; *e.g.* which pass through a given point.

2. The surface u_n is given, but not the point A. In this case, as the point A is only restricted to be a point on the surface, we have two more unknowns, making nine in all. The problems here are, to find the order of the curve on the surface at every point of which there is a conic having eight-point contact, and to find the number of points at which there is a conic having nine-point contact.

As before, however, there are certain derived problems coming under this head which involve a less number of equations. We may seek the order of the curve traced out by points of contact of septactic conics satisfying one condition, sextactic satisfying two, &c.; or we may seek the number of septactic conics satisfying two conditions, sextactic satisfying three, &c.

3. The surface is not wholly given. We may here assign a number of relations sufficient to eliminate the quantities a, b, c, leaving one or more relations among the coefficients of u_n. The problems here are such as:—to find the number of surfaces in a pencil $u_n + \lambda v_n$ which admit of ten-point contact with some conic, or one of whose nine-point conics meets a given line.

In the present communication only problems of class 1 will be considered; the formulæ in this case may be very considerably simplified. The quantities a and the coefficients of u_n, then, are data of the problem; so that the first of our equations, $A^n u_n = 0$, is satisfied by hypothesis. The next equation, $A^{n-1} B u_n = 0$, signifies that the point B lies in the tangent plane at A, as it obviously must if the conic touch the surface at A. We shall suppose this also to be satisfied from the commencement; that is, we shall regard B as a point moving in the tangent plane, and to be determined by construction in that plane. This may be effected analytically if we substitute for B, $\lambda A + \mu B + \nu B'$, where now B, B' are regarded

as fixed points in the tangent plane, and the three unknowns λ, μ, ν take the place of the four quantities b. There is, however, as will be seen, no occasion to make the substitution explicitly.

This being so, any relation involving B only beside the data must be regarded as the equation of a curve in the tangent plane. For example, $A^{n-2}B^2u_n=0$, expressing that B lies on the quadric polar of A, is the equation of the two chief tangents at that point. Generally, $B^nu_n=0$ is the equation of the intersection with u_n of the tangent plane; and the curves $AB^{n-1}u_n=0$, $A^2B^{n-2}u_n=0$, &c. are the successive polars of A in regard to that intersection.

The terms entering into our equations are of the general form

$$\theta^{p+2q}\,.\,\frac{\underline{|n}}{\underline{|n-p-q}\;\underline{|p}\;\underline{|q}}\,A^{n-p-q}B^pC^q\,.\,u_n.$$

Any term, therefore, is completely determined by the two numbers p and q, and might, for any thing that has yet appeared, be denoted by (p, q). In view, however, of subsequent substitutions, we shall keep in evidence the manner in which B and C are involved, and denote the term in question by the symbol $\theta^{p+2q}(B^pC^q)$.

As we do not consider any higher than seven-point contact, we have only the five equations:

$$0=(B^2)+(C) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$0=(B^3)+(BC) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

$$0=(B^4)+(B^2C)+(C^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$0=(B^5)+(B^3C)+(BC^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$0=(B^6)+(B^4C)+(B^2C^2)+(C^3) \quad\ldots\ldots\ldots\ldots \quad (7).$$

III. *Conics through a fixed point.*

Combining equation (3) successively with (4) and (5), so as to obtain results homogeneous in B, we find

$$0=(B^3)(C)-(BC)(B^2),$$

$$0=(B^4)(C)^2-(B^2C)(B^2)(C)+(C^2)(B^2)^2.$$

If we regard C as a fixed point, these are equations of a cubic and a quartic curve in the tangent plane, on each of which B must lie if the conic K_2 has five-point contact. But these curves have a common node at A, and common tangents at it; for every term in each has at least one factor of the form (B^m), which we know to represent a polar of A in regard to the intersection of u_n by the tangent plane—that is, a curve touched by the chief tangents at A. Of their 12 intersections, then, 6 coincide with the point A; and there remain *six conics having five-point contact at* A *which pass through an arbitrary point* C.

IV. *Conics meeting a fixed axis through* A.

If the point C, instead of being altogether given, is movable on a fixed straight line through A, we may represent it by $A + \lambda C$; where now C is really a fixed point, and λ a quantity to be determined. When we make this substitution in our equations, they become*

$$0 = (B^2) + \lambda (C),$$
$$0 = (B^3) + \lambda (BC),$$
$$0 = (B^4) + (B^2) + \lambda (B^2 C) + 2\lambda (C) + \lambda^2 (C^2),$$
$$0 = (B^5) + (B^3) + \lambda (B^3 C) + 2\lambda (BC) + \lambda^2 (BC^2),$$
$$0 = (B^6) + (B^4) + \lambda (B^4 C) + (B^2) + 2\lambda (B^2 C) + \lambda^2 (B^2 C^2) + 3\lambda (C) + 3\lambda^2 (C^2) + \lambda^3 (C^3).$$

The last three admit of obvious simplifications by aid of the previous ones; and the system may finally be written

$$0 = (B^2) + \lambda (C) \quad\ldots\ldots\ldots\ldots (3'),$$
$$0 = (B^3) + \lambda (BC) \quad\ldots\ldots\ldots\ldots (4'),$$
$$(B^2) = (B^4) + \lambda (B^2 C) + \lambda^2 (C^2) \quad\ldots\ldots\ldots\ldots (5'),$$
$$(B^3) = (B^5) + \lambda (B^3 C) + \lambda^2 (BC^2) \quad\ldots\ldots\ldots\ldots (6'),$$
$$(B^4) - \lambda^2 (C^2) = (B^6) + \lambda (B^4 C) + \lambda^2 (B^2 C^2) + \lambda^3 (C^3) \quad\ldots\ldots (7').$$

* It must be remembered that (B) and terms involving A only vanish by hypothesis. The formulæ have also been simplified by the omission of certain coefficients depending on n which do not affect the final results.

Locus of poles of axis in regard to four-point conics.

If we select any plane through the line AC, there will be a singly infinite number of conics in the plane having four-point contact with the surface at A. The line AC will, as is well known, have the same pole in regard to all these conics—that is to say, the point B will be the same for the whole system. If we now allow the plane to turn round the axis, the point B will trace out a curve in the tangent plane. The equation to this curve is got by eliminating λ between equations (3′) and (4′); namely, it is

$$0 = (B^2)(BC) - (B^3)(C) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4'').$$

We see, therefore, that *the locus of the poles of an axis in regard to all the four-point conics whose planes pass through it is a cubic curve in the tangent plane touching the chief tangents at* A, *which point is therefore a node on the curve.*

We might have inferred this from the fact that on any line through A there is only one point B, while this point coincides with A in the case of the two chief tangents; since at a point of inflexion all the four-point conics contain the inflexional tangent.

Number of six-point conics through an axis.

We have now to determine B so that the equations (3′), (4′), (5′), (6′) may be simultaneously satisfied. We know already that B must lie on the cubic (4″); it is necessary therefore to find some other locus on which it has to lie. First of all, then, we must eliminate λ between (3′) and (5′) and between (3′) and (6′); the results are,

$$(B^2)(C)^2 = (B^4)(C)^2 - (B^2C)(B^2)(C) + (B^2)^2(C^2) \quad \equiv p_4, \text{ say};$$
$$(B^3)(C)^2 = (B^5)(C)^2 - (B^3C)(B^2)(C) + (B^2)^2(BC^2) \equiv q_5, \text{ say}.$$

Here $p_4 = 0$ and $q_5 = 0$ are curves touching the chief tangents at A, and of the degrees four and five respectively. But the equations are not homogeneous; in fact only the ratios and not the absolute values of the quantities a were determined by the

fixing of the point A, and they may be regarded as involving an arbitrary factor whose square affects the left-hand side of the equations. It is, however, at once eliminated, and we obtain the homogeneous result,

$$(B^2) \cdot q_5 = (B^3) \cdot p_4.$$

This is a curve of order 7 having two branches in each of the chief directions at A. Of its 21 intersections with the cubic (4″), then, 12 coincide with A, and there remain *nine positions of* B *which give sextactic conics through the fixed axis;* or we may say, *of the sextactic conics at the point* A, *there are nine whose planes pass through an arbitrary point* C.

System of five-point conics through an axis.

Since there is one five-point conic in every plane, if we consider all the planes through a fixed axis we shall obtain a singly infinite number of five-point conics. Of this system there is only one conic whose plane passes through an arbitrary point, viz. the conic determined by the plane through that point and the axis.

There are eight conics of the system which meet an arbitrary line.

The number of conics which meet an arbitrary line is clearly the same as the order of the surface which they trace out. Now, since through every point on the axis can be drawn six conics of the system (as proved in the last section), the axis is a six-fold line on the surface. The section of the surface, then, by a plane through the axis is made up of the axis taken six times over and the conic in that plane; or it is of the order *eight*. Q. E. D.

V. *Conics not subject to any condition.*

In order to get rid of the restriction of meeting a fixed axis, we must again modify our fundamental equations. We have to put them into a form in which they will represent *any* conic touching the surface u_n at the point A. For this

purpose it is necessary and sufficient that C should be movable over a plane passing through A; since every conic through A must cut the plane in one other point, but this may be any point of the plane. We attain this analytically by substituting for λC in the second set of equations, $\lambda C + \mu D$, where C and D are now fixed points not in the tangent plane and not in any straight line through A. This is equivalent to still considering the conics which meet a given axis, but allowing that axis to move over a fixed plane.

The transformed equations are :—

$$0 = (B^2) + \lambda(C) + \mu(D) \quad \ldots\ldots (3''') ,$$

$$0 = (B^3) + \lambda(BC) + \mu(BD) \quad \ldots\ldots (4''') ,$$

$$(B^2) = (B^4) + \lambda(B^2C) + \mu(B^2D) + \lambda^2(C^2) + 2\lambda\mu(CD) + \mu^2(D^2) \quad (5''') ,$$

$$(B^3) = (B^5) + \lambda(B^3C) + \mu(B^3D) + \lambda^2(BC^2) + 2\lambda\mu(BCD) + \mu^2(BD^2) \quad \ldots\ldots (6''') ,$$

$$\left.\begin{aligned} &(B^4) - \lambda^2(C^2) - 2\lambda\mu(CD) - \mu^2(D^2) = (B^6) + \lambda(B^4C) \\ &+ \mu(B^4D) + \lambda^2(B^2C^2) + 2\lambda\mu(B^2CD) + \mu^2(B^2D^2) \\ &+ \lambda^3(C^3) + 3\lambda^2\mu(C^2D) + 3\lambda\mu^2(CD^2) + \mu^3(D^3) \end{aligned}\right\} \ldots\ldots (7''') .$$

Locus of poles of axis in given plane in regard to sextactics.

From the first two of these equations we obtain

$$1 : \lambda : \mu = (BD)(C) - (BC)(D) : (B^3)(D) - (B^2)(BD) : (B^3)(C) - (B^2)(BC) = n_1 : l_3 : m_3, \text{ say};$$

here $n_1 = 0$ is the equation to a straight line passing through A, while $l_3 = 0$, $m_3 = 0$ are cubics touching the chief tangents at A.

Substituting these values in $(5''')$ and $(6''')$, we obtain

$$n_1^2(B^2) = n_1^2(B^4) + n_1 l_3(B^2C) + n_1 m_3(B^2D) + l_3^2(C^2) + 2l_3m_3(CD) + m_3^2(D^2) = u_6, \text{ say};$$

$$n_1^2(B^3) = n_1^2(B^5) + n_1 l_3(B^3C) + n_1 m_3(B^3D) + l_3^2(BC^2) + 2l_3m_3(BCD) + m_3^2(BD^2) = v_7, \text{ say}.$$

Here the curves $u_6 = 0$, $v_7 = 0$ are of the sixth and seventh orders respectively, each of them having *one* branch at A in

each of the chief directions, and one other branch different for the two curves.

The equations

$$n_1^2 (B^2) = u_6,$$
$$n_1^2 (B^3) = v_7$$

must hold for six-point contact, but they are not homogeneous. Eliminating n_1^2, however, there results

$$(B^2) \,.\, v_7 = (B^3) \,.\, u_6,$$

a curve of the ninth order, locus of the poles in regard to the sextactic conics of an axis moving in a fixed plane. This curve has *two* branches at A in each of the chief directions, and *one* other branch; and consequently is met by the plane ACD in *five* points coinciding with A, and in *four* other points. Now the plane ACD does not in general contain a sextactic conic; the pole of the axis can therefore only be in this plane when the axis itself is in the tangent plane. In this case there is a certain number of proper sextactic conics in planes through the axis, and it is clear that the pole of the axis in regard to any such conic is the point A. These conics, therefore, correspond to the *five* intersections of the plane ACD with the locus of poles which coincide with A; or *through an axis in the tangent plane can be drawn five proper sextactic conics.* In the tangent plane itself there are four improper conics having six-point contact; viz. the pair of chief tangents, which (as a sharp conic or line-pair reckoned among conics given tangentially) counts for two, and each chief tangent doubled.

Number of septactic conics.

There is a finite number of septactic conics at the point A; each of these meets the plane ACD in a determinate point $A + \lambda C + \mu D$, and fixes thereby a position of the point B. These positions of the point B must necessarily lie in the 9thic locus just investigated; it remains only to find a homogeneous relation which shall determine another locus for B, and to count the number of their intersections.

To this end we must first substitute for $1 : \lambda : \mu$ their values $n_1 : l_3 : m_3$ in equation (7‴). The result is

$$n_1^3 (B^4) - n_1 l_3^2 (C^2) - 2n_1 l_3 m_3 (CD) - n_1 m_3^2 (D^2)$$
$$= n_1^3 (B^6) + n_1^2 l_3 (B^4 C) + n_1 m_3 (B^4 D)$$
$$+ n_1 l_3^2 (B^2 C^2) + 2n_1 l_3 m_3 (B^2 CD) + n_1 m_3^2 (B^2 D)$$
$$+ l_2^3 (C^3) + 3l_3^2 m_3 (C^2 D) + 3l_3 m_3^2 (CD^2) + m_3^3 (D^3),$$

or $$n_1 t_5 = w_9, \text{ say.}$$

The curve w_9 has *one* branch at A in each of the chief directions and one other branch. The curve t_6 has one branch in each of the chief directions and *two* other branches.

The equations (5), (6), (7) have now become

$$n_1^2 (B^2) = u_6,$$
$$n_1^2 (B^3) = v_7,$$
$$n_1 t_6 = w_9.$$

The first two of these give us the curve already considered,

$$(B^2) . v_7 = (B^3) . u_7,$$

which has at A two branches in the chief directions and one other. The first and third give

$$n_1 . (B^2) . w_9 t_6 u_6,$$

a curve of order 12, having two branches in each of the chief directions and two other branches. Of the 108 intersections of these curves, then, $24 + 8 + 4 + 2 = 38$ coincide with A, leaving 70 for the number of septactic conics.

PART II. THE CONTACT OF QUADRIC SURFACES WITH SURFACES OF ORDER n.

I. *Conditions of contact.*

Let A, B, C, D be four points forming a tetrahedron, whose tangential equations are

$$0 = \Sigma aX, \ \Sigma bX, \ \Sigma cX, \ \Sigma dX$$

respectively, their coordinates being a_i, b_i, c_i, d_i $(i=1, 2, 3, 4)$. Then the point

$$P \equiv A + \theta B + \phi C + \theta\phi D,$$

whose coordinates are

$$p_i \equiv a_i + \theta b_i + \phi c_i + \theta\phi d_i \ (i=1, 2, 3, 4),$$

will, if we suppose θ and ϕ to take all possible values, trace out a quadric surface whose tangential equation is

$$0 = AD - BC \equiv Q_2.$$

To the pair of values

$$\begin{array}{llll} \theta=0\ , & \phi=0\ , & \text{corresponds the point} & A, \\ \theta=\infty, & \phi=0\ , & \quad\text{,,} \qquad\qquad\text{,,} & B, \\ \theta=0\ , & \phi=\infty, & \quad\text{,,} \qquad\qquad\text{,,} & C, \\ \theta=\infty, & \phi=\infty, & \quad\text{,,} \qquad\qquad\text{,,} & D. \end{array}$$

The equation shews that AB, AC, BD, CD are generating lines.

If, now, we wish to find the nature of the curve in which this quadric intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation which is of the order n in θ and in ϕ separately. If we regard θ and ϕ as coordinates of a point on the quadric surface, the equation just found is that of the curve of intersection.

Suppose that in this equation the term independent of θ and ϕ vanishes, then the equation is satisfied by the pair of values $\theta=0$, $\phi=0$, or the curve of intersection passes through the point A. Now the various directions in which we may start from the point A are determined by the initial value of the ratio $\theta : \phi$ when we move along them. The direction in which the intersection-curve starts from A is therefore that obtained by neglecting in the equation terms of higher order than the first; and we see that there is only one such direction.

If, however, not only the constant term but the coefficients of θ and ϕ in the equation vanish, the initial directions are

obtained by equating to zero the terms of the second order, *i. e.* by neglecting in the equation all terms of a higher order than the second. In this case, then, there are two such directions, the intersection-curve has a double point at A, and the quadric has with the surface u_n an ordinary or *two-branch* contact.

Again, if the coefficients of the terms of the third order are the first that do not vanish, the initial directions are obtained by neglecting all the terms of higher order, and there are consequently three of them. Thus the intersection-curve has a *triple* point at A, and the two surfaces have a *three-branch* contact.

And so generally, if the coefficients of the terms of the m^{th} order are the first that do not vanish, the intersection-curve has at A a multiple point of order m, and the two surfaces have at that point an m-branch contact.

The result of these considerations may be stated as follows:

The necessary and sufficient conditions that the quadric $\mathrm{Q}_2 \equiv \mathrm{AD} - \mathrm{BC}$ *may have* m*-branch contact with the surface* u_n *at the point* A, *are that in the expansion of*

$$(\mathrm{A} + \theta\,\mathrm{B} + \phi\mathrm{C} + \theta\phi\mathrm{D})^{\mathrm{n}} \,.\, \mathrm{u}_\mathrm{n}$$

in powers and products of θ and ϕ, the terms of order m *in θ and ϕ are the lowest whose coefficients do not vanish.*

II. *Quadrics of four-branch contact.*

The equations we shall have to employ are so simple in form that it is unnecessary to employ the abridged notation of the former Part. We shall merely omit the operand u_n, and any common factor of the binomial coefficients.

The conditions for ordinary contact are then

$$0 = A^n, \quad 0 = A^{n-1} B, \quad 0 = A^{n-1} C.$$

The first of these expresses that A is a point on the surface u_n, the second and third that B and C are on the tangent plane at A.

The further conditions for three-branch contact are

$$0 = A^{n-2} B^2, \quad 0 = A^{n-2} C^2, \quad 0 = A^{n-1} D + (n-1) A^{n-2} BC.$$

The first two of these shew that B and C are points on the chief tangents at A. If we regard the absolute values of the coordinates of A as given, then it appears from the third equation that B and C may be chosen arbitrarily on the chief tangents and D anywhere in space, the equation giving the relation between the absolute values of their coordinates which determines the particular surface $AD - BC = 0$.

For four-branch contact we have the additional equations,

$$0 = A^{n-3} B^3, \qquad 0 = A^{n-3} C^3,$$
$$0 = 2A^{n-2} BD + (n-2) A^{n-3} B^2 C,$$
$$0 = 2A^{n-2} CD + (n-2) A^{n-3} BC^2.$$

The first two of these indicate that B and C lie on the polar cubic of A in regard to the section u_n by the tangent plane. Now this polar cubic has a node at A, whose tangents are the chief tangents. Each of these lines therefore meets the cubic in three points at A, and cannot have any other point on the curve unless it be itself a part of the cubic. But the points B and C have to lie one on each of the chief tangents. In order, therefore, that all the equations may be satisfied, the polar cubic in question must break up into the two chief tangents and some other line.

This condition may be put into another form. For if we seek the points in which the line AB meets the surface, by substituting the coordinates of $A + \lambda B$ in $u_n = 0$, the conditions $A^n u_n = 0$, $A^{n-1} B u_n = 0$, $A^{n-2} B^2 u_n = 0$, $A^{n-3} B^3 u_n = 0$ indicate that *four* roots of the equation are equal to zero, or that the line meets the surface in four consecutive points at A. We find, therefore, that

Those points of a surface at which a quadric may have four-branched contact are the points at which each chief tangent meets the surface in four consecutive points, or, which is the same thing, *the points whose polar cubic contains the chief tangent.*

The number of these points has been counted by Clebsch, *Crelle*, LXIII. 14*, and turns out to be

$$n(41n^2-162n+162).$$

A point of this nature being given, one quadric surface having four-branch contact at it may be drawn through another arbitrary point.

The coordinates of the point A being given as to their absolute values, let us substitute for B and C, $A+\lambda B$, $A+\mu C$; where now B and C are fixed points on the chief tangents, whose coordinates are given absolutely. This being so, the following equations are satisfied *ex hypothesi*:—

$$A^n u_n=0,\qquad A^{n-1}Bu_n=0,\quad A^{n-1}Cu_n=0,$$
$$A^{n-2}B^2u_n=0,\quad A^{n-2}C^2u_n=0,\quad A^{n-3}B^3u_n=0,\quad A^{n-3}C^3u_n=0;$$

from which it follows at once, for example, that

$$A^{n-3}(A+\lambda B)^3u_n=0.$$

For D also let us substitute νD, where the coordinates of D are now given absolutely. Our three remaining equations are (omitting for shortness the mention of A)

$$0=\nu D+(n-1)\lambda\mu\,.\,BC \qquad\qquad (1),$$
$$0=\nu D+(n-2)\lambda\mu\,.\,BC+\nu\lambda\,.\,BD+\tfrac{1}{2}(n-2)\lambda^2\mu\,.\,B^2C \qquad (2),$$
$$0=\nu D+(n-2)\lambda\mu\,.\,BC+\nu\mu\,.\,CD+\tfrac{1}{2}(n-2)\lambda\mu^2\,.\,BC^2 \qquad (3);$$

and it remains only to shew that these equations determine uniquely λ, μ, ν.

If we subtract (1) from (2) and (3) successively, we obtain

$$0=-\mu\,.\,BC+\nu\,.\,BD+\tfrac{1}{2}(n-2)\lambda\mu\,.\,B^2C \qquad (4),$$
$$0=-\lambda\,.\,BC+\nu\,.\,CD+\tfrac{1}{2}(n-2)\lambda\mu\,.\,BC^2 \qquad (5),$$

the values $\lambda=0$, $\mu=0$ not being admissible. But if we substitute in (4) and (5) the value of $\lambda\mu$ derived from (1), we obtain two simple equations which determine the ratios $\lambda:\mu:\nu$; after which the absolute values are uniquely determined from (1).

* The investigation is given by Salmon, *Geom. Three Dim.* p. 444.

It is otherwise obvious that if $0 = q_2$ be the point-equation to a four-branch quadric, and $0 = t_1$ to the tangent plane, $0 = q_2 + \rho t_1^2$ will be the equation to a pencil of quadrics having four-branch contact, one of which may be made to pass through an arbitrary point.

Special investigation for n = 3.

In the case in which u_n is a cubic surface, the points in question are clearly the 135 points of contact of the 45 triple tangent planes, namely, the intersections of the 27 lines. This case may be conveniently studied by means of the representation of that surface on a plane. The plane sections of the surface here correspond to a system of cubics having six common points; the quadric sections therefore to sextics having nodes at these points. The problem is then to draw a sextic curve having six given nodes and a quadruple point elsewhere.

The twenty-seven lines of the cubic are represented by

the 6 fixed points,
the 15 lines joining them, and
the 6 conics each through five of them.

It shall now be proved that the sextic must include two of these; and that for each pair that meet there is a singly infinite number of sextics.

The sextic cannot be a proper curve; for six nodes and a quadruple point are equivalent to twelve nodes, and a proper sextic can have only ten. Nevertheless we may apply a quadric transformation to it, whose principal points are the quadruple point and two of the nodes. The sextic is thus reduced to a quartic, passing 2, 0, 0 times through the principal points respectively and having four other nodes. But a quartic having five nodes must be made up of a conic and two straight lines. Now, if the node at a principal point is the intersection of the two lines, the original sextic was made up of two lines, passing each through two of the six points, and a quartic having nodes

at their intersection and the remaining two points and passing through those four. Let a, b, c, d, e, f be the six points, p the intersection of ab, cd; then the sextic is in this case made up of

lines ab, cd,

quartic $p^2e^2f^2abcd$.

If, however, in the transformed figure the node at a principal point was the intersection of a line with the conic, the original sextic was made up of a line, a conic, and a nodal cubic, viz. if p be the intersection of the line ab and the conic $bcdef$, the nodal cubic is p^2acdef.

Now we know that we can draw a singly infinite number of quartics with three fixed nodes and four fixed points, or of cubics with a fixed node and five fixed points. Hence in both these cases the sextic includes two representatives of straight lines in the cubic, together with another curve which may be arbitrarily chosen from a pencil.

XXXIII.

ON THE CLASSIFICATION OF LOCI*.

Part I. Curves.

By a *curve* we mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and then, if these be connected by $k-1$ equations of any order, the curve is either the whole aggregate of common solutions of these equations, or, when this breaks up into algebraically distinct parts, the curve is one of these parts. It is thus convenient to employ still further the language of geometry, and to speak of such a curve as the complete or partial intersection of $k-1$ loci in flat space of k dimensions, or, as we shall sometimes say, in a k-flat. If a certain number, say h, of the equations are linear, it is evidently possible by a linear transformation to make these equations equate h of the coordinates to zero; it is then convenient to leave these coordinates out of consideration altogether, and only to regard the remaining $k-h-1$ equations between $k-h$ coordinates. In this case the curve will, therefore, be regarded as a curve in flat space of $k-h$ dimensions. And, in general, when we speak of a curve as in flat space of k dimensions, we mean that it cannot exist in flat space of $k-1$ dimensions.

The whole aggregate of linear complexes may be regarded

[* From the *Philosophical Transactions of the Royal Society*.—Part II. 1878, pp. 663—681.]

as constituting a space of five dimensions, in which the *special* complexes, or straight lines, constitute a quadric locus. A ruled surface, or scroll, will be thus regarded as a curve lying in a quadric locus in a flat space of five dimensions. If, however, the generators of the scroll all belong to the same linear complex, the scroll must be regarded as a curve lying in a quadric locus in a flat space of *four* dimensions. And if, further, the scroll has two linear directrices, so that the generators belong to a linear congruence, then the scroll may be regarded as a curve lying on an ordinary quadric surface in three dimensions. Thus, for example, quartic scrolls having two linear directrices correspond either to quadri-quadric curves of deficiency 1 (that is, they are *elliptic* curves whose coordinates may be expressed as elliptic functions of one variable), or to the curves of deficiency 0 which are the partial intersections of a quadric and a cubic surface (that is, they are unicursal curves).

This view of ruled surfaces is made excellent use of by Voss, "Zur Theorie der windschiefen Flächen," *Math. Annalen*, Vol. VIII. p. 54.

Similar considerations apply to surfaces. By a *surface* we shall mean, in general, a continuous two-dimensional aggregate (which may also be called a *two-spread* or *two-way locus*) of any elements whatever, curves, surfaces, complexes, &c., defined by the whole or a portion of the system of solutions of $k-2$ equations among k coordinates. We shall assume that none of these equations are linear, and then shall speak of the surface as in a flat space of k dimensions. We shall in certain cases go further, and speak of an h-spread or h-way locus, viz., a locus determined by the whole or an algebraically separate portion of the system of solutions of $k-h$ equations among k coordinates; if none of these equations are linear, the h-way locus will be said to be in k dimensions. The general point of view is that of Professor Cayley, "On the Curves which satisfy given Conditions," *Phil. Trans.*, Vol. 158 (1868), pp. 75—144; the methods of enumeration are those of Dr Salmon, "Solid Geometry," p. 261.

THEOREM A. *Every proper curve of the* n^{th} *order is in a flat space of* n *dimensions or less.* For through $n+1$ points of it we can draw a flat space of n dimensions, which must therefore contain the curve, since it meets it in a number of points greater than its order.

Thus, for example, there is no curve of the second order, in space of any number of dimensions, except a plane conic. If, therefore a system of curves, in a plane or on any surface, is such that two curves of the system can be drawn through an arbitrary point, then the coordinates of a varying curve of the system may be represented by $x_i + 2\theta y_i + \theta^2 z_i$ $(i = 1, 2, 3 \ldots k)$, and the envelope of the system is, in the case of plane curves, a curve having the equation $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, where U, V, W are three curves of the system; in the case of curves on a surface, it is the intersection of the surface with another having an equation of that form*.

* Professor Henrici has kindly written for me the following notes in elucidation of this argument:—

"In the first sentence of the paper it is stated that by a *curve* is meant any one-dimensional aggregate of any sort of elements. The definitions given are algebraical, but the reasoning later on becomes more and more geometrical.

"In this note the connexion between the algebraical definition and the geometrical reasoning will be shewn in the case where the elements are plane curves of order n.

"If we suppose a curve given by its equation in point coordinates we may take the coefficients as homogeneous coordinates of the curve.

"As there are $\frac{n(n+3)}{2}$ ratios of these coefficients, it follows that all curves of order n in a plane constitute a $\frac{n(n+3)}{2}$ spread, and this will be a *flat* spread as no relation has been supposed between the coordinates.

"To determine in this spread a k-flat, $k < \frac{n(n+3)}{2}$, we have to assume a sufficient number of equations between the coordinates, or denoting by n_1, n_2, ... curves of order n we may write down the equation of one element in the k-flat in the form $a_1 u_1 + a_2 u_2 + \ldots + a_{k+1} u_{k+1} = 0$, and take the k ratios of the a as the coordinates of a variable curve.

"For $k=2$ we get a *net* as the flat space of two dimensions or as the *plane* in this space, and for $k=1$ a pencil corresponding to the *line*.

"If, on the other hand, we assume in the k-flat $k-1$ equations between the coordinates a, there remains a singly infinite number of curves, that is according

To particularise still further, a system of conics having the characteristic $\mu = 2$ must always have quadruple contact with a quartic curve; and the different species may be enumerated by studying the successive degeneration of the curve, ending with the fundamental system $\nu = 1$, when it breaks up into four straight lines.

So again, there is no quadric scroll, in any number of dimensions, except the ordinary quadric surface which is in flat space of three dimensions.

A curve of the third order must be either the known skew cubic in three dimensions, or a plane cubic. Hence, if a system

to Professor Clifford a *curve* (with curves as elements), according to the usual nomenclature a series of curves.

"To determine the order of this curve we have to find the number of elements on it which satisfy a linear relation between the coordinates. In our case the condition that a curve shall pass through a given point gives such a relation, and the number of curves through a point is the *order* in question.

"Hence, if we wish to extend a theorem relating to a curve (in the ordinary sense with points as elements, but in any number of dimensions) to a proposition relating to a series of curves, or if we wish to illustrate in a plane a theorem relating to a curve in more dimensions than three, we have instead of a point on the curve to take a curve in the series, and to replace the order of the curve by the index of the series.

"The theorem that every curve of order two is a *plane* curve becomes thus—the curves in a series of index 2 belong to a *net*.

"Further, the coordinates of a point on a conic may be represented as expressions of the second degree in a variable parameter, say by $x_i + 2\theta y_i + \theta^2 z_i$; where $i = 1, 2, 3$, if the coordinates are taken in the plane of the conic, but if they are taken in space we have to take $i = 1, 2, 3, 4$, and so on for more dimensions. The locus of these points, that is, the conic, is then given by an equation of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

where U, V, W are three of the points.

"If we apply this to our series we obtain the results stated in the text, viz., the coordinates of any curve of a series such that two curves pass through a given point are of the form quoted, and the equation of the envelope is of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

U, V, W being three of the curves.

"Similarly, if the series is such that three pass through any point, then the series may be considered as a 'curve' of order three, and the statements made in the text follow at once from the known properties about cubic curves, which are either unicursal (twisted, or plane nodal, cubics) or they are plane curves of deficiency one."—January, 1879.

of curves be such that three of them can be drawn through an arbitrary point, the equation of any curve of the system is of one of the two forms—

$$U + 3Vt \quad + 3Wt^2 + Xt^3 = 0,$$
$$U + V\,\mathrm{sn}^2 u + 2W\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u = 0,$$

where t, u are parameters. Hence it is easy to write down the equations to the envelopes in the two cases, and to enumerate the distinct species.

A cubic scroll must be of the nature of the skew cubic, because it is a curve (with complexes for elements) which is obliged to lie on a quadric locus (that of the special complexes, or straight lines).

THEOREM B. *A curve of order* n *in flat space of* k *dimensions (and no less) may be represented, point for point, on a curve of order* n − k + 2 *in a plane.*

The proposition is obvious when $k = 3$. The cone standing on a curve of order n (in ordinary space of three dimensions), and having its vertex at a point of the curve, is of order $n-1$; if then we cut this cone by a plane, we have the tortuous curve represented, point for point, on a plane curve of order $n-1$.

Now this process is applicable in general. Starting with an arbitrary point, P, of a curve in any number of dimensions, let us join this point to all the other points of the curve; we shall thus get a *cone* of order $n-1$. For any flat locus of $k-1$ dimensions drawn through the point P must meet the curve in n points, of which P is one; and therefore it must meet the cone in $n-1$ lines. Hence, if we cut this cone by such a flat $(k-1)$ way locus *not* passing through P, we shall get a curve of order $n-1$ in flat space of $k-1$ dimensions, which is a point-for-point representation of the original curve. By continuing this process we may go on diminishing the order of the curve and the number of dimensions by equal quantities, until we have subtracted $k-2$ from each; when we are left with a curve of order $n-k+2$ in a plane.

The reduction may, however, be effected in one step. A

flat $(k-2)$ way locus may be drawn through $k-1$ arbitrary points. Suppose it to contain $k-2$ consecutive points of the curve at P, and another variable point, Q, of the curve. Such a locus will meet an arbitrary plane in one point, R. As Q then moves about on the curve, R will trace out on the plane a curve which corresponds to it, point for point. But this curve is of order $n-k+2$, for a flat $(k-1)$ way locus, passing through $k-2$ consecutive points of the original curve at P, will meet that curve in $n-k+2$ other points, and therefore will meet also the locus of R in $n-k+2$ points. This locus is, therefore, of order $n-k+2$, as was to be proved.

The fixed points through which the variable $(k-2)$ way locus passes need not all be united at P, but they may be any $k-2$ arbitrary points on the curve.

We will now consider some examples of this remark.

1. *Unicursal curve of order* n *in* n-*dimensional space.*

A curve of order n *in flat space of* n *dimensions (and no less) is always unicursal.*—We may prove this independently by considering a variable $(n-1)$ flat which passes through $n-1$ fixed points on the curve. Its equation will be of the form $A+tA'=0$, where t is a variable parameter, and it will meet the curve in one other point, which is thus associated with a value of t.

The equations to such a curve may always be written in the form—

$$0=\left\| \begin{matrix} A, & B, & C \dots K \\ B, & C, & D \dots L \end{matrix} \right\| \dots\dots\dots\dots\dots\dots(1),$$

where the $A, B, C \dots K, L$ are linear functions of the coordinates, and the number of columns is $=n$. For the $n+1$ homogeneous coordinates are proportional to rational integral functions of t of the n^{th} order. Solving these $n+1$ equations for $1, t, t^2 \dots t^n$ we find

$$1, t, t^2 \dots t^n = A, B, C \dots L,$$

which is equivalent to the system written down above.

The more general system of equations—

$$0 = \left\| \begin{matrix} A, & B \dots K \\ A', & B' \dots K' \end{matrix} \right\| \dots\dots\dots\dots\dots\dots\dots (2),$$

where the $A \dots K$, $A' \dots K'$ are linear functions as before, may always and easily be reduced to the former, for they are got by eliminating t from the n equations.

$$\begin{aligned} A + tA' &= 0, \quad \dots\dots\dots\dots\dots\dots\dots\dots (3). \\ B + tB' &= 0, \\ &\vdots \\ K + tK' &= 0. \end{aligned}$$

We may, however, solve these equations for the ratios of the coordinates, which will thus be expressed as rational functions of t of the n^{th} order. Solving these for $1, t, t^2 \dots t^n$ we come back to the previous system.

The equations (3) exhibit the curve as the locus of the intersection of corresponding elements in n projective pencils.

The equation to the $(n-1)$ flat which passes through the n points whose parameters are $t_1, t_2 \dots t_n$, is easily seen to be—

$$0 = \begin{vmatrix} A, & B, & C, & \dots L \\ 1, & t_1, & t_1^2, & \dots t_1^n \\ 1, & t_2, & t_2^2, & \dots t_2^n \\ \vdots & & & \\ 1, & t_n, & t_n^2, & \dots t_n^n \end{vmatrix}.$$

But this equation is manifestly divisible by the coefficient of L, which is the product of the differences of all the t. If we write—

$$\begin{aligned} \Sigma_1 &= t_1 + t_2 + t_3 + \dots + t_n, \\ \Sigma_2 &= t_1 t_2 + t_1 t_3 + t_2 t_3 + \dots + t_{n-1} t_n, \\ \text{etc.} &= \text{etc.} \\ \Sigma_n &= t_1 t_2 \dots t_n, \end{aligned}$$

then the equation is

$$0 = L - K\Sigma_1 + \dots \pm B\Sigma_{n-1} \mp A\Sigma_n \dots\dots\dots\dots (4).$$

If we omit the suffixes of the t in this formula we obtain

the equation to the osculant $(n-1)$ flat at the point t. Namely (beginning at the other end), it is—

$$0 = At^n - nBt^{n-1} + \tfrac{1}{2}n(n-1)\,Ct^{n-2} - \ldots \pm nKt \mp L \ldots\ldots (5),$$

and we see at once that *the class of such a curve is always equal to its order.*

We thus obtain a very useful representation (*Abbildung*) of the points of the n-dimensional space by means of groups of n points on such a unicursal curve, namely, each point in the space is represented by the points of contact of the n osculant $(n-1)$ flats which pass through it. The use of such a representation of ordinary three-dimensional space by means of a skew cubic was pointed out by Dr Hirst, and the corresponding representation of a plane by means of a conic has been used by M. Darboux ('Sur une classe remarquable de courbes et de surfaces algébriques,' Paris, 1873, Note II., p. 183), and by me ("On the Transformation of Elliptic Functions," *Proc. Lond. Math. Soc.*, Vol. VII. (1875) [xxii., xxiii., pp. 205—228, supra]). It may be worth while to mention that an extension to all space of the theory of the in-and-circumscribed polygon may be obtained by this means.

A curve of this kind determines also a dualistic correspondence in the space of n dimensions. Through every point may be drawn n osculant $(n-1)$ flats, and through their points of contact another $(n-1)$ flat, which shall be called the *polar* of the point. If the point moves along a straight line its polar will pass through a fixed $(n-2)$ flat, the *polar* of the line. And generally if the point lies in any k flat the polar will pass through a fixed $(n-k-1)$ flat.

When $n=2$ we have the ordinary system of polar reciprocals in regard to a plane conic. When $n=3$ we have that system in regard to a skew cubic which is described by Schröter, *Crelle*, Vol. LXV. p. 39. These two systems are typical respectively of the cases in which n is even and odd. When n is even, the relation between the coordinates of two points, which expresses that each lies in the polar of the other, is a symmetrical one; consequently those points which lie in their own polars are points on a certain quadric locus, and the system is merely

that of the poles and polars in regard to this quadric locus upon which the curve lies. The equation to this locus is at once obtained by equating to zero the quadrinvariant of the form $(1, t)^n$ which occurs in the equation (5) of the osculant $(n-1)$ flat, namely, it is

$$0 = AL - nBK + \tfrac{1}{2}n(n-1)\,CH - \text{etc.} \qquad (6).$$

To prove this, observe that if in the equation (5) we substitute the coordinates of any point p, the values of t which satisfy the equation are the parameters of the points of contact of the osculant $(n-1)$ flats which pass through the point. If $t_1, t_2, \ldots t_n$ be these values, the equation (4) represents the $(n-1)$ flat which passes through the points of contact, that is to say, the polar of the point. Now if we denote by A', B', ... the results of substituting the coordinates of the point p in A, B, ... then we shall have—

$$\begin{aligned} A'\Sigma_1 &= nB' \qquad (7), \\ A'\Sigma_2 &= \tfrac{1}{2}n(n-1)\,C', \\ &\vdots \\ A'\Sigma_n &= L' \end{aligned}$$

so that, when n is even the equation of the polar is—

$$0 = AL' + A'L - n(BK' + B'K) + \tfrac{1}{2}n(n-1)(CH' + C'H) - \text{etc.} \ldots (8),$$

that is, it is simply the polar of the point in regard to the quadric (6).

It is to be observed that the quadric is completely determined when the curve is given. I reserve the question of the conditions to which the curve is subject when the quadric locus is given, or, say, the discussion of the problem to represent the relation of poles and polars in regard to a quadric locus (in space of an even number of dimensions) by means of a unicursal curve.

But when n is odd, the last term of equation (4) is negative, and the equation of the polar is—

$$0 = AL' - A'L - n(BK' - B'K) + \tfrac{1}{2}n(n-1)(CH' - C'H) - \text{etc.} \ldots (9),$$

that is, it is skew symmetrical, and *every point lies upon its polar*. It is convenient to use the term *co-flat* for $n + 1$ points,

which are in the same $(n-1)$ flat; with this nomenclature we may say that *when* n *is odd every point is co-flat with the* n *points of contact of the osculant* (n − 1) *flats, which can be drawn through it.* This will be recognised as an extension of the property of a skew cubic, that every point in space is co-planar with the points of contact of the three osculating planes which can be drawn through it.

A case of this skew symmetrical relation is given by any arbitrary state of motion of the whole space as a rigid body, the relation between two points being that the line joining them moves perpendicularly to itself. The polar of any point is an $(n-1)$ flat drawn through it perpendicular to the direction of its motion. When n is even there is always one point which remains at rest, and all the polars pass through this point. Thus the general motion of a solid in an even number of dimensions always depends in this simple way on the motion in one dimension less. In an odd number of dimensions, however, every point moves in the general case; but if any point is at rest, then all the points in a certain straight line are at rest.

Besides its order and class, a curve has, in general, characteristic numbers intermediate to these, which may be called its first rank, second rank, etc. The first rank is the order of the locus traced out by straight lines through two consecutive points of the curve; the second rank, of that traced out by planes through three consecutive points; and generally the k^{th} rank is the order of the $(k+1)$ wide locus traced out by k-flats through $k+1$ consecutive points. For the curve just considered these numbers are $2(n-1)$, $3(n-2)$, ... $(k+1)(n-k)$; it is convenient to derive them from the corresponding numbers for its projection, the unicursal curve of order n in $n-1$ dimensions, to which we now proceed.

2. *Unicursal curve of order* n *in* n − 1 *dimensions.*

Every curve of order n *in flat space of* n − 1 *dimensions is either unicursal or elliptic.* For it may be represented point-for-point on a plane cubic.

We shall treat these two cases in succession. They are exemplified by the two species of quartics in ordinary tri-dimensional space.

The coordinates of a point on the unicursal curve are proportional to rational integral functions of a parameter t. This representation may be simplified in a manner due to Rosanes, *Crelle*, Vol. LXXV. p. 166. We have n binary quantics of order n; now these may be linearly combined in n different ways so as to produce a perfect n^{th} power. Hence the original quantics may be expressed each as a linear function of the n^{th} powers of the same n linear quantics. Thus, for example, three binary cubics may be simultaneously reduced to the forms

$$\begin{gathered} au^3 + b\,v^3 + c\,w^3, \\ a'u^3 + b'\,v^3 + c'\,w^3, \\ a''u^3 + b''v^3 + c''w^3, \end{gathered}$$

where $u + v + w = 0$ identically. If the x, y, z of a point in a plane are respectively proportional to these cubics, we may, by solving the equations for u^3, v^3, w^3, obtain three linear functions X, Y, Z of the coordinates, which are respectively proportional to u^3, v^3, w^3. Transforming them to the new triangle whose sides are $X = 0$, $Y = 0$, $Z = 0$ we must have the equation of a unicursal cubic expressed in the form

$$X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0.$$

It is clear that the lines $X = 0$, $Y = 0$, $Z = 0$ are tangents at the three points of inflexion.

In general, let the n quantics be

$$\begin{gathered} a_0 + na_1t + \dots + a_nt^n, \\ b_0 + nb_1t + \dots + b_nt^n, \\ \vdots \\ h_0 + nh_1t + \dots + h_nt^n, \end{gathered}$$

then the linear quantics u, v, w ... are the factors of

$$\begin{vmatrix} t^n, & -nt^{n-1}, & \frac{1}{2}n(n-1)\,t^{n-2}, & \dots\, t \\ a_0, & a_1, & a_2, & \dots\, a_n \\ b_0, & b_1, & b_2, & \dots\, b_n \\ \vdots & \vdots & \vdots & \vdots \\ h_0, & h_1, & h_2, & \dots\, h_n \end{vmatrix}.$$

Since there are $n-2$ identical relations between n linear quantics, the $n-2$ equations of the unicursal curve may be written in the form

$$\left\|\begin{matrix} X_1^{\frac{1}{n}}, & X_2^{\frac{1}{n}}, & \dots X_n^{\frac{1}{n}} \\ \alpha_1, & \alpha_2, & \dots \alpha_n \\ \beta_1, & \beta_2, & \dots \beta_n \end{matrix}\right\| = 0;$$

it is evident that the equations $X_1, X_2, \dots X_n = 0$ represent stationary osculant $(n-2)$ flats, that is to say, $(n-2)$ flats which pass through n consecutive points of the curve.

The properties of this curve may be very conveniently studied by regarding it as a projection of the curve considered in the last section. If all the points of that curve be joined to a point O, not situated upon it, the joining lines will form a cone of order n; and on cutting this cone by an $(n-1)$ flat we shall obtain the curve now under discussion.

The n points of superosculation, whose existence has just been proved, are then clearly the projections of the points of contact of osculant $(n-1)$ flats to the full-skew curve drawn through the point O. It follows that *when* n *is odd, these* n *points of superosculation are on the same* (n − 2) *flat;* but when n is even this is not the case, unless the point O lies on the quadric locus associated with the full-skew curve, in which case we have a special variety of the projection. Thus the three points of inflexion of a nodal cubic are in one straight line; but a unicursal skew quartic in ordinary space has not in general the property that the points of contact of its four stationary osculating planes are in one plane. The property established above for the full-skew curve shews that this will be the case if the four points form an equianharmonic system, or if the quadrinvariant of the quartic which determines them is equal to zero. And generally when n is even, the n points of superosculation will be co-flat if, and only if, the quantic in t which determines them has its quadrinvariant zero.

By using the values of the coordinates of a variable point of the curve expressed in terms of a parameter t, we may obtain an expression of this quadrinvariant and also of its product by

the discriminant in terms of the roots of the quantic. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the values of t which belong to the points of superosculation, and $x_1, x_2, \ldots x_n$ the coordinates of a variable point on the curve. Then we may write

$$x_i = (t-\alpha_i)^n, \qquad i = 1, 2, \ldots n,$$

and the coordinates of the point α_i are $(\alpha_i - \alpha_1)^n$, $(\alpha_i - \alpha_2)^n$, $\ldots (\alpha_i - \alpha_n)^n$. If for shortness we write (hk) instead of $\alpha_h - \alpha_k$, then the condition that the n points shall be co-flat is

$$0 = \begin{vmatrix} 0\ , & (12)^n, & (13)^n, & \ldots (1n)^n \\ (21)^n, & 0\ , & (23)^n, & \ldots (2n)^n \\ (31)^n, & (32)^n, & 0\ , & \ldots (3n)^n \\ \vdots & \vdots & \vdots & \vdots \\ (n1)^n, & (n2)^n, & (n3)^n, & \ldots\ 0 \end{vmatrix}.$$

This is obviously always satisfied if n is odd, for then the determinant is skew symmetrical, and being of odd order it necessarily vanishes. If, however, n is even, the determinant is a symmetrical function of the roots which vanishes when any two of them are equal; and consequently it must contain as a factor the product of the squares of their differences. Now the determinant is of the order $2n$ in each root, and the discriminant is of order $2(n-1)$; therefore the remaining factor is of order 2 in each root, and being a symmetrical invariant must be a function of the squares of their differences. It can therefore be no other than $\Sigma (\alpha_1 - \alpha_2)^2 (\alpha_3 - \alpha_4)^2 \ldots (\alpha_{n-1} - \alpha_n)^2$; this is, to a factor *près*, equal to the quadrinvariant of the form

$$(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_n).$$

The equation to the $(n-2)$ flat passing through two consecutive points of the curve at t, and through $n-3$ other points $p\, q \ldots u$, is clearly

$$0 = \begin{vmatrix} x\, dx\, p\, q \ldots y \\ 1\ 2\ 3\ 4 \ldots n \end{vmatrix},$$

where the y are current coordinates, and the determinant is expressed in umbral notation. Writing in this for x_i, $(t-\alpha_i)^n$, and for dx_i, $n(t-\alpha_i)^{n-1}dt$, we may observe that the determinant

$$\begin{vmatrix} (t-\alpha_1)^n\ , & (t-\alpha_2)^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1} \end{vmatrix} = (\alpha_2 - \alpha_1)(t-\alpha_1)^{n-1}(t-\alpha_2)^{n-1},$$

so that the equation is of order $2(n-1)$ in t. It thence follows that $2(n-1)$ different $(n-2)$ flats may be drawn through $n-2$ arbitrary points to touch the curve; or that the developable traced out by the tangent lines is of the order $2(n-1)$.

Similarly, from the value of the determinant

$$\begin{vmatrix} (t-\alpha_1)^n, & (t-\alpha_2)^n, & \dots (t-\alpha_{k+1})^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1}, & \dots (t-\alpha_{k+1})^{n-1} \\ \vdots & \vdots & \vdots \\ (t-\alpha_1)^{n-k}, & (t-\alpha_2)^{n-k}, & \dots (t-\alpha_{k+1})^{n-k} \end{vmatrix},$$

which is equal to the product of the differences of $\alpha_1, \alpha_2, \dots \alpha_{k+1}$ multiplied by

$$\{(t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_{k+1})\}^{n-k},$$

we may conclude that the number of $(n-2)$ flats which can be drawn through k consecutive points of the curve and through $n-k$ other arbitrary points is $(k+1)(n-k)$; or that the k-wide locus which is traced out by $(k-1)$ flats passing through k consecutive points is of the order $(k+1)(n-k)$. For the equation of an $(n-2)$ flat passing through k consecutive points is clearly

$$0 = \begin{vmatrix} x\, dx\, d^2x \dots d^{k-1}x\, p\, q \dots y \\ 1\ 2\ \ 3\ \dots k, \qquad \dots n \end{vmatrix},$$

where we must substitute for the x_i, dx_i, d^2x_i, etc., the descending powers of $t-\alpha_i$ beginning at the n^{th}. Making k equal to $n-1$ we obtain the equation of the osculant $(n-2)$ flat at any point of the curve; it is

$$0 = \frac{P_1 y_1}{(t-\alpha_1)^2} + \frac{P_2 y_2}{(t-\alpha_2)^2} + \dots + \frac{P_n y_n}{(t-\alpha_n)^2},$$

where P_i = product of the differences of all the α except α_i. Thus the class of the curve is $2(n-1)$.

3. *Unicursal curve of order* n *in* n − k *dimensions.*

The characteristic numbers belonging to this curve may at once be obtained by regarding it as a projection of the full-skew curve. The number of ranks is $n-k-2$, and the numerical values of them are respectively $2(n-1)$, $3(n-2)$, $\dots k(n-k+1)$; the class is $(k+1)(n-k)$; and the number of points of superosculation is $(k+2)(n-k-1)$. For example, the unicursal

quintic in three dimensions is of rank $2 \cdot 4, = 8$, and of class $3 \cdot 3, = 9$, and it has $4 \cdot 2, = 8$ superosculant planes.

Convenient forms of the equations may be got by eliminating some of the variables from the equations of the full-skew curve; but care must be taken to select these variables so that the resulting system is sufficiently general.

4. *Elliptic (or bicursal) curve of order* n *in* n − 1 *dimensions.*

We have proved already that a curve of order n in $n-1$ dimensions can be represented, point for point, on a plane cubic. If, therefore, it is not unicursal, its coordinates can be expressed in terms of elliptic functions of a single parameter. Now, it follows from the investigations of Clebsch, *Crelle*, Vol. LXIV. (1864), pp. 210—270, that if n points of the curve are co-flat, the sum of their parameters will differ from a certain constant by a sum of integer multiples of the two periods of the elliptic function. Let the periods be ω and ω', then if $t_1, t_2, \ldots t_n$ are the parameters of the points,

$$t_1 + t_2 + \ldots + t_n = c + a\omega + b\omega',$$

where c is a constant, and a, b are integers. To find the points of superosculation, we must suppose the n points to become identical, or the t, still satisfying this equation, to become equal. We thus obtain

$$nt = c + a\omega + b\omega'$$

$$t = \frac{c}{n} + \frac{a}{n}\omega + \frac{b}{n}\omega',$$

and values of t, representing distinct points, will be got by giving to the numbers a, b the values $0, 1, \ldots n-1$ independently. Hence *there are* n^2 *points of superosculation.*

Thus a plane cubic has nine inflexional tangents, and a quadri-quadric curve has sixteen superosculant planes.

Propositions hold good in general in regard to the grouping of these points, which are analogous to those which relate to the inflexions of a cubic. Thus, *an* $(n-2)$ *flat drawn through* $n-1$ *of them will always pass either through another besides, or through the tangent line at one of the* $n-1$. This is obvious

from the values already given for the parameters of points of superosculation.

Through any point of the curve can be drawn $(n-1)^2$ osculant $(n-2)$ flats. This is proved in the same way as the preceding proposition, which is, in fact, the projection of it; for if through the given point we draw a cone containing the curve, and cut it by an $(n-2)$ flat, the section will be an elliptic curve of order $n-1$ in $n-2$ dimensions, and the projections of the points whose osculant $(n-2)$ flats pass through the given point will be points of superosculation on the projected curve. Hence, also, the lines joining the given point to the points of contact are grouped in respect of co-flatness in the same way as the points of superosculation in the curve of next lower order.

More generally, through k given points of the curve there can be drawn $(n-k)^2$ $(n-2)$ flats which have $(n-k)$ pointic contact with the curve. If $u_1, u_2, \dots u_k$ are the parameters of the k given points, those of the required points are given by

$$u = \frac{1}{n-k}(u_1 + u_2 + \dots u_k + a\omega + b\omega'),$$

where the integers a, b may take independently the values

$$0, 1, \dots n-k-1.$$

From these results we may now determine the various ranks and the class of the curve. Suppose that we know the number of $(n-2)$ flats which can be drawn through $n-2$ arbitrary points in space—or, which is the same thing, through an arbitrary $(n-3)$ flat P— to touch a certain curve. Then, if the arbitrary $(n-3)$ flat meets the curve in any point, *two* of these will coincide at that point. For taking an $(n-4)$ flat in the $(n-3)$ flat, and joining it to all the points of the curve by $(n-3)$ flats, we may cut this figure by a plane or 2 flat. Every $(n-3)$ flat will cut this plane in a single point. The problem is then reduced to drawing tangents from a point (viz., the intersection of P by the plane) to a plane curve; and we know that when this point lies on the curve, two of the tangents coincide at it.

In general, a certain number of $(n-2)$ flats can be drawn

through an arbitrary $(n-k-1)$ flat to have k-pointic contact with a given curve; this number is, in fact, the $(k-1)^{\text{th}}$ rank of the curve. If the arbitrary $(n-k-1)$ flat meets the curve at any point, then k of these $(n-2)$ flats coincide at that point. For we may project the whole figure from an $(n-k-2)$ flat lying in the $(n-k-1)$ flat on to a k flat. The problem is then reduced to drawing $(k-1)$ flats through a given point to have k-pointic contact with a curve in k dimensions. Now we know, from the example of the full-skew curve, that, when the point lies on the curve, k of these coincide at the point.

If the arbitrary $(n-k-1)$ flat meet the curve in more points than one, k of the osculants will coincide at each of them; and this result is not affected by the union of the points into one. In particular, if it meet the curve in $n-k$ coincident points, the number of osculants which there coincide is $k(n-k)$.

Applying now these general considerations to the elliptic curve, we find at once that the $(k-1)^{\text{th}}$ rank of it is nk. For we must add to the number $k(n-k)$, just obtained, the number, k^2, given by the theory of elliptic functions for the $(n-2)$ flats drawn through $n-k$ consecutive points of the curve to have k-pointic contact elsewhere. In particular, the class of the curve is $n(n-1)$; we have observed already that the number of superosculants is n^2.

Thus, a plane cubic is of order 3, class 6, and has 9 inflexions; a quadri-quadric is of order 4, rank 8, class 12, and has 16 superosculant planes. We learn, moreover, that a quintic curve in four dimensions, when not unicursal, is of first rank 10, second rank 15, class 20, with 25 points of superosculation. Hence a quintic in three dimensions, with five apparent dps., is of rank 10, class 15, and has 20 superosculant planes; this follows by projection from the former case.

A curve of this kind, viz., an elliptic curve of order n in an $(n-1)$ flat has its coordinates $x_1, x_2 \dots x_n$ determined by the equations

$$x_1, x_2, \dots x_n = 1, t, t', t^2, tt' \dots$$

(the last term on the right being $t^{(\frac{1}{2}n-1)}t'$ or else $t^{\frac{1}{2}n}$, according as n is odd or even), where $t = \operatorname{sn}^2(u+iK')$ and

$$t' = \frac{dt}{du} = 2\operatorname{sn}(u+iK')\operatorname{cn}(u+iK')\operatorname{dn}(u+iK')$$
$$= \sqrt{2t(1-t)(1-k^2t)}.$$

{If n is even, we may write $t = \operatorname{sn}^2 u$ instead of $\operatorname{sn}^2(u+iK')$.} The condition for n points $u_1, u_2 \dots u_n$ to be co-flat is then

$$u_1 + u_2 + \dots + u_n = 0.$$

See Lindemann; Clebsch's *Lectures on Geometry*, vol. II.

{*Theory of Derived Points on an Elliptic* (*or Bicursal*) *Curve.*

Sylvester's theory of derived points on a plane cubic is as follows:—Starting from any given point on the curve, we may construct its *tangential*, or point where the tangent at the original point meets the curve again; similarly we may construct the tangential of the tangential, or second tangential, and so on. By joining any two non-consecutive points on this series, we can find their *residual*, the point where the joining line meets the curve again. In this way we obtain an infinite group of points derived from (and including) the original point, such that the line joining any two of them is either tangent at one of these or passes through a third point of the group. It is to be observed that all points on the curve uniquely derived from the given point by any geometrical process (*e g.*, the point where the conic of five-pointic contact meets the curve again, the point where cubics of eight-pointic contact meet the curve again, &c.) are included in the group.

The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, and the order of the functions to which they are proportional is always a square number. Thus the three coordinates of the tangential are proportional to quartic functions of the coordinates of the original. If the square root of the order of these functions be called the order of the derived point, then we have the theorem that when three derived points are in a straight

line, the order of one of them is equal to the sum of the orders of the other two. It is observed that there is no derived point whose order is divisible by 3. By help of this observation it is easy to make out a scheme of the orders; for when we join two points, the order of their residual must be the sum or the difference of the orders of the points, and one or the other of these is always divisible by 3.

This theory is really a geometric representation of the multiplication of elliptic functions. The coordinates x_1, x_2, x_3 of any point on the cubic curve may by proper choice of axes be made proportional to elliptic functions of a parameter u, so that $x_1 : x_2 : x_3 = 1 : \mathrm{sn}^2(u+iK') : \mathrm{sn}(u+iK')\,\mathrm{cn}(u+iK')\,\mathrm{dn}(u+iK')$. This being so, if u, v, w are parameters of three points in a straight line, we shall have $u+v+w=0$. If v be the tangential of u, the three points u, u, v are in a straight line, and $2u+v=0$, or $v=-2u$. Hence the series of tangentials has for parameters

$$u,\ -2u,\ +4u,\ -8u,\ \&c.:$$

and in general the parameter of any derived point is of the form nu, where n is a positive or negative integer. The number n, taken positively, coincides with what was called the order of the derived point. For the elliptic functions of nu are of the order n^2 in the elliptic functions of u.

In this way all points of the theory are explained, excepting the fact that no derived point has its order divisible by 3.

Moreover, we see at once that the theory can be extended to other curves of deficiency 1; as, for example, the quadri-quadric curve. Starting with any point on this curve, we may find the point where the osculating plane at that point meets the curve again; then repeat the process with the point so found, and so on. The plane joining any three of these points will meet the curve in another derived point, or else touch it at one of the three points. The plane drawn through one derived point to touch the curve at another derived point will meet it again in a derived point, *or* touch at the first point, *or* osculate at the second. The coordinates of any derived point are of the order n^2 in those of the original point, where $\pm n$ may be called the

order of the derived point. In this case the order of no derived point is divisible by 2.

I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* (in the sense explained in the beginning of this paper) of deficiency greater than 1. For this purpose I considered scrolls. Taking an arbitrary generator on a quartic scroll having two linear directrices, we may draw a one-sheeted hyperboloid through three consecutive generators at that place; this will meet the quartic scroll in one other generator, which is thus uniquely derived from the given one. Similarly on a quintic scroll contained in a linear complex, the two tractors of four consecutive generators meet the scroll in two other points lying on a generator. And on a sextic scroll not contained in a linear complex, the linear complex having five-line contact at a given generator (containing five consecutive generators) will contain one other generator of the scroll. In these three cases, then, from any three, four, or five generators we may uniquely derive a fourth, fifth, or sixth generator respectively; and the whole theory of derived elements may be applied to the generators of these scrolls.

Unfortunately, however, each of the scrolls considered is at most of deficiency 1, so that we merely get more illustrations of the multiplication of elliptic functions. And it may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1.

Suppose that it has no singular points, and that $k-1$ points on it being given, there is uniquely determined one other point. If this is effected (as in the above examples) by drawing a flat space through the $k-1$ points, which meets the curve in one other point, then it must be of the order k. Moreover, it must be in a flat space of so many dimensions that the flat of one dimension less is determined by $k-1$ points. Now a $(k-2)$ flat is determined by $k-1$ points; therefore, the curve is in a $(k-1)$ flat.

Thus the impossibility of extending the theory of derivation

to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k-1$ dimensions is at most of deficiency 1. This failure was the starting point of the present paper.

It remains to explain why, in the group of numbers expressing the orders of the derived points, only certain forms present themselves. Let that number which, with $k-1$ other numbers, makes up zero, be called the *residual* of those numbers; it is, in fact, their sum taken negatively. Then the process of forming the group is to start from unity, and add the residual of every $k-1$ numbers of the group, repetitions being allowed. I say that by this process we shall only get numbers of the form $mk+1$. For let m_1k+1, m_2k+1, &c. be $k-1$ such numbers, then their residual is $-(m_1+m_2+..)k-k-1$, which is a number of the same form. Now as unity, with which we start, is of this form, it follows that all the numbers of the group must be of the form $mk+1$.—January, 1879.}

CURVES OF DEFICIENCY p.

Theorems relating to Abelian Functions.

It will be convenient to put together shortly those propositions relating to the application of Abelian functions to curves which will be wanted in the sequel.

The aggregate of the real and imaginary points on a curve constitutes a two-way spread, or surface, which may be transformed, by stretching without tearing, into the surface of a body with p holes in it. On this surface there are $2p$ distinct closed curves which cannot without breaking be shrunk into a point, namely, one round each hole, and one through each hole. Any other irreducible circuit must be made up of combinations of these.

If any rational function of the coordinates be integrated from one point to another along the curve-spread, the value of the integral will depend upon the path of the integration. If the integral becomes infinite at any points, it may be altered in

value by making the path go round one or more of these; but in any case it may be altered by incorporating into the path any of the $2p$ closed circuits just mentioned. It is found that there are p distinct rational functions of the coordinates whose integrals do not become infinite at any point of the curve-spread. Any other integral which is everywhere finite must be a linear combination of these. Of such linear combinations it is convenient to take a certain set as the *normal* set; they are so chosen that each of them, when integrated along a closed path which goes *round* a hole, gives zero for all the holes but one, and πi for that one; thus, the p integrals, which we may call $u_1, u_2, u_3, \ldots u_p$, are associated one by one with the p holes $1, 2, \ldots p$. If they are integrated along a closed curve passing *through* the hole h, let the values be called a_{h1}, a_{h2}, a_{hp}; then it is found that $a_{hk} = a_{kh}$, or the integral of u_h through the hole k is equal to the integral of u_k through the hole h.

If we now take all the integrals from a point x to a point y along the same path, and if $u_1, u_2, \ldots u_p$ are the set of values for one such path, and $U_1, U_2, \ldots U_p$ for another path, then we must have

$$\begin{aligned}
U_1 &= u_1 + m_1\pi i + q_1 a_{11} + q_2 a_{12} + \ldots + q_p a_{1p}, \\
U_2 &= u_2 + m_2\pi i + q_2 a_2 + q_2 a_{22} + \ldots + q_p a_{2p}, \\
&\vdots \\
U_p &= u_p + m_p\pi i + q_1 a_{p1} + q_2 a_{p2} + \ldots + q_p a_{pp},
\end{aligned}$$

where the numbers m, q are integers; namely, m_h is the additional number of times the new path has gone round the hole h, and q_h is the additional number of times it has gone through that hole. We shall write these equations thus

$$U_1, U_2, \ldots U_p \equiv u_1, u_2 \ldots u_p \pmod{\pi i, a},$$

and shall say that the quantities U are *congruent* to the quantities u in respect of the periods πi, a.

Suppose now that the curve has no actual nodes, and that a locus of any order intersects it in the points $x_1, x_2, \ldots x_m$. Then, if another locus of the same order intersects it in the points $y_1, y_2, \ldots y_m$, and we take any one of the integrals, say u,

from x_1 to y_1, from x_2 to y_2,... from x_m to y_m, the sum of these results will be congruent to zero. That is to say

$$\Sigma \int_x^y du_h \equiv 0.$$

Here the Σ refers to the suffixes of the x and y, not to h. There are p such equations. This is Abel's Theorem.

When the curve is in a k-flat and of the order n, we shall use this theorem chiefly for its n intersections with a $(k-1)$ flat. If we regard the lower limits of the integrals (the point x) as fixed, the integrals for any point y may be regarded as parameters belonging to that point, and then Abel's Theorem gives us p equations between the parameters of n points which lie on a $(k-1)$ flat. The truth of these equations is *necessary* to the points lying on a $(k-1)$ flat, but it may not be sufficient. Thus in a bicircular quartic curve, $p=1$, we have one equation to express that four points are in a straight line, and if the points are collinear the equation is true. But it does not follow from the equation that the points are collinear; in fact, the equation holds equally good if the points are in a circle.

If the sums of the parameters of p points are given, that is, if we have the p equations—

$$\left(\int^{x_1} + \int^{x_2} + \ldots + \int^{x_p}\right) du_h = v_h \quad \{h = 1, 2, \ldots p\},$$

the v_h being arbitrary constants, and the lower limits of the integrals being supposed constant; then the upper limits $x_1, x_2, \ldots x_p$ may be expressed in terms of the quantities v_h —namely, they are the roots of an equation of degree p whose coefficients are products of ϑ-functions of the v. If

$$\phi(m_1, m_2, \ldots m_p) = \Sigma m_h m_k a_{hk} + 2\Sigma m_h v_h,$$

then
$$\vartheta(v_1, v_2, \ldots v_p) = \Sigma^p e^{\phi(m)},$$

the Σ^p indicating that each of the p integers $m_1, m_2, \ldots m_p$ is to take all integral values positive and negative. When the lower limits are so chosen that the sum of the parameters is zero for the complete intersection by any locus, this ϑ-function has remarkable properties. If we sum each of the parameters

for any $p-1$ points on the curve, the ϑ-function whose arguments are these sums will vanish. That is if

$$v_h = \left(\int^{x_1} + \int^{x_2} + \dots \int^{x_{p-1}}\right) du_h,$$

then $\vartheta(v_1 v_2 \dots v_p) = 0$. If these sums are taken for any $p-2$ points, not only will the ϑ vanish, but also its differential coefficient in regard to any one of the points. And generally, if we take for the v the sums of the parameters for $p-r$ points, the ϑ and its first $r-1$ differential coefficients in regard to any of the points will vanish.

Conversely, if the p quantities v are such that $\vartheta(v)$ and its first $r-1$ differential coefficients vanish, then it is possible to find $p-r$ points x such that

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{p-r}}\right) du_h = v_h.$$

Although here the number of equations is greater by r than the number of unknown quantities, yet it is possible to satisfy them all in virtue of the relations existing between them.

Relation between the Order and Deficiency of a Curve.

We shall now apply these theorems to the study of curves existing in k dimensions, of the order n and deficiency p. A $(k-1)$ flat cuts such a curve in n points, such that the sum of each of the p parameters, for the n points, is zero. But a $(k-1)$ flat is determined by k points; so that, k arbitrary points being chosen on the curve, it is always possible to find $n-k$ other points, so that the sum of each parameter for the whole n points shall be zero. Let then $-v_1, -v_2, \dots -v_p$ be the sums of the parameters for the given k points; then to find the remaining $n-k$ points we have the p equations

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{n-k}}\right) du_h \equiv v_h.$$

If p is not greater than $n-k$, we know that these equations can be solved, although the solution may be indeterminate. But if $p > n-k$, the equations cannot be solved unless certain

conditions are satisfied by the v. Let $p-n+k=r$, then r conditions must be satisfied; namely the v must be sums of the parameters of not more than $p-r$ (or $n-k$) points. But they are sums of the parameters of k points; therefore k is not greater than $n-k$, or $2k$ is not greater than n. We have proved then that

If p > n − k, *then* 2k *is not greater than* n.

Conversely, if k > ½n, p *is at most equal to* n − k.

We may also state the proposition in this way. *A curve of order* n *and deficiency* p, *not greater than* ½n, *can at most exist in* n − p *dimensions.*

{It appears, therefore, that the theorems at the beginning of the paper may be extended, and that in n dimensions we have the curve of order n which is unicursal, the curve of order $n+1$, and deficiency at most 1, of order $n+2$, and deficiency at most 2, and so on till we come to the order $2n$, which is the first case of exception, and may have deficiency $n+1$. This curve is the natural geometric representation of the general Abelian functions, its multiple tangent flats playing the same part as the double tangents of the quartic curve in Riemann's beautiful paper on the case $p=3$. H. Weber has noticed that in four dimensions this curve is the complete intersection of three quadric loci.—January, 1879.}

(ABSTRACT OF XXXIII)*.

"A CURVE," is to be understood to mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and if these be connected by $k-1$ equations

* [From the *Proceedings of the Royal Society*, No. 187, 1878. This abstract, I believe, contains the communication on the same subject made to the London Mathematical Society, Feb. 8th, 1877.]

of any order, the curve is either the aggregate of common solutions, or, when this breaks up into algebraically distinct parts, the curve is one of these parts.

In the paper, of which this is an abstract, theorems are established relating to the nature of the space in which such curves can exist, to the mode of representing them in flat space of lower dimensions, and to some of their properties. The following are the leading theorems:—

I. Every proper curve of the nth order is in a flat space of n dimensions or less.

II. A curve of order n in flat space of k dimensions (or less) may be represented, point for point, on a curve of order $n-k+2$ in a plane.

III. A curve of order n, in flat space of n dimensions (and no less), is always unicursal.

From this the author obtains a representation of the points of an n dimensional space by means of groups of n points on such a unicursal curve, corresponding to the methods of Hirst and Darboux for three-dimensional space.

When n is even, the system corresponds to that of poles and polars in regard to a quadric locus upon which the curve lies.

When n is odd, every point is co-flat (i.e. $n+1$ points lie in the same $n-1$ flat), with the n points of the osculant $(n-1)$ flats which can be drawn through it.

IV. Every curve of order n in flat space of $n-1$ dimensions is either unicursal or elliptic.

V. When the curve is unicursal, and n is odd, the n points of superosculation, or points of stationary osculant $(n-2)$ flats, are on the same $(n-2)$ flat. But when n is even, this will be the case only under a certain condition.

VI. When the curve is elliptic (or bi-cursal) the class of the curve is $n(n-1)$, and the number of superosculants n^2.

If we consider a curve of the order n and deficiency p, existing in k dimensions, a $(k-1)$ flat cuts such a curve in n points, such that the sum of each of the p parameters (Abel's theorem gives p equations between the parameters of n points which lie on a $(k-1)$ flat), for these n points is zero. And we obtain the theorem.

VII. A curve of order n and deficiency p, not greater than $\frac{1}{2}n$, can at most exist in $n-p$ dimensions.

*XXXIV.

ON THE POWERS OF SPHERES.*

DEF. The *power* of two spheres (or of one sphere in regard to the other) is the squared distance of their centres less the sum of the squares of their radii.

Or if d be the distance of the centres, r_1, r_2 the radii, it is

$$d^2 - r_1^2 - r_2^2,$$

which is the same thing as

$$-2r_1r_2 \cos\theta,$$

where θ is the angle of intersection.

If the power vanishes, the spheres cut at right angles.

Let the equations of two spheres be

$$P \equiv a(x^2+y^2+z^2) + 2bx + 2cy + 2dz + e = 0,$$
$$Q \equiv a'(x^2+y^2+z^2) + 2b'x + 2c'y + 2d'z + e' = 0,$$

then their power is equal to

$$\frac{ae' + a'e - 2bb' - 2cc' - 2dd'}{aa'} \quad \text{.................} (1).$$

Let the numerator of this fraction be denoted by (PQ), and let O stand for the sphere $a=b=c=d=\theta$, $e=1$, (the sphere at infinity); then the fraction may be written

$$\frac{(PQ)}{(PO)(QO)} \quad \text{.........................} (2).$$

* [I think this paper must be the one more than once promised me by Prof. Clifford, and that it contains what he communicated to the London Mathematical Society, Feb. 27, 1868, *Proc.* Vol. II. 61.]

Under the name *sphere* must be included as particular cases points and planes. The power of a point in regard to a sphere is the squared tangent to it; the power of two points is the squared distance between them. The powers of spheres in regard to planes are infinite quantities which are proportional to the distances of their centres from the planes. The powers of planes in regard to one another are infinities of the second order which are proportional to the cosines of the angles of intersection.

The symbol (PQ) is linear in regard to the coefficients of the two spheres involved, and therefore distributive. That is to say, we have

$$(P, \lambda Q + \mu R) = \lambda (PQ) + \mu (PR).$$

It follows directly from this that if we form a determinant whose constituents are such symbols, of the form

$$\begin{pmatrix} ABCDE \\ PQRST \end{pmatrix} \equiv \begin{vmatrix} (AP), & (AQ), & (AR), & (AS), & (AT) \\ (BP), & \&c. & \ldots & \ldots & \ldots \\ (CP), & & & & \\ (DP), & & & & \\ (EP), & & & & (ET) \end{vmatrix}$$

(the letters denoting any ten spheres) it must be equal to the product of the determinants $(ABCDE)$, $(PQRST)$, multiplied by a linear factor. The ordinary theorem for multiplication of determinants shews that we have in fact

$$\begin{pmatrix} ABCDE \\ PQRST \end{pmatrix} \equiv 8\,(ABCDE)\,(PQRST) \quad \ldots\ldots\ldots\ldots (3).$$

If we divide the rows of the determinant by (AO), (BO), etc. and the columns by (PO), (QO), etc., its constituents become actually the powers of the spheres involved. Now, for the right-hand side of the equation (3), we have

$$\frac{(ABCDE)}{(AO)\,(BO)\,(CO)\,(DO)\,(EO)}$$

$$= \frac{(ABCD\,.\,E)}{(ABCDO)\,(EO)} \cdot \frac{(ABCDO)}{(AO)\,(BO)\,(CO)\,(DO)} .$$

The quantity

$$\frac{(ABCDO)}{(AO)\,(BO)\,(CO)\,(DO)},$$

is easily seen to be 48 times the volume of the tetrahedron whose vertices are at the centres of the four spheres. The other factor

$$\frac{(ABCDE)}{(ABCDO)\,(EO)}$$

is clearly the power of the sphere E in regard to a sphere cutting A, B, C, D orthogonally. We see then that *given any five spheres, the product of the tetrahedron whose vertices are at the centres of any four into the power in regard to the fifth of a sphere cutting them orthogonally is a symmetrical function of the five spheres.* I shall call this the *aospheric function* of the five spheres; it vanishes when they are all orthotomic of the same sphere.

This being so, the theorem (3) informs us that *the determinant formed with the powers of two sets of five spheres is equal to* 6144 ($= 8 \times 48 \times 48$) *times the product of their aospheric functions.*

The corresponding determinant formed with two sets of six spheres vanishes identically; or we have always

$$\begin{pmatrix} ABCDEF \\ PQRSTU \end{pmatrix} \equiv 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Power-Coordinates.

DEF. Coordinates of a sphere are quantities proportional to certain multiples of its powers in respect of five fixed spheres.

It is convenient for many purposes to take as these multiples the reciprocals of the radii of the fixed spheres. Thus if the coordinates of a sphere X are x_1, x_2, x_3, x_4, x_5, its powers in respect of the fundamental spheres are x_1r_1, x_2r_2, etc., where

r_1, r_2, etc., are their radii. Using the symbols 1, 2, 3, 4, 5 for the fundamental spheres, we have by (4)

$$\left(\frac{1\ 2\ 3\ 4\ 5\ X}{1\ 2\ 3\ 4\ 5\ X}\right) \equiv 0,$$

or, expanding and dividing rows and columns by the $r_1, r_2, \ldots$

$$\phi_r = \begin{vmatrix} 1, & \cos 12, & \cos 13, & \cos 14, & \cos 15, & x_1 \\ \cos 21, & 1, & \cos 23, & \&c. & & x_2 \\ \cos 31, & \cos 32, & 1, & & & x_3 \\ \cos 41, & \&c. & & 1, & & x_4 \\ \cos 51, & & & & 1, & x_5 \\ x_1, & x_2, & x_3, & x_4, & x_5, & -2r^2 \end{vmatrix} = 0 \ldots (5),$$

where r is the radius of the sphere X. This equation may be regarded as giving the radius of a sphere in terms of its co-ordinates. Let ϕ be the value of the determinant ϕ_r when $r = 0$, or when the sphere reduces to a point. Then whenever the coordinates x represent a point, they must satisfy the homogeneous equation of the second order

$$\phi\,(xx) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If we choose to regard the x as abbreviations for expressions of the form

$$\frac{(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2}{r},$$

then the equation (6) is identically satisfied; and from this point of view it is a form of the identical relation connecting the equations of any five spheres. But from our point of view the x are primarily coordinates of a sphere, and (6) is only satisfied when the sphere reduces to a point.

An expression for the power of two spheres X and Y is obtained in the same manner. Namely, the equation

$$\left(\begin{matrix}1\ 2\ 3\ 4\ 5\ X\\ 1\ 2\ 3\ 4\ 5\ Y\end{matrix}\right) = 0,$$

gives at once

$$\text{power of } X \text{ in regard to } Y = \frac{\phi\,(xy)}{(1\ 2\ 3\ 4\ 5)^2}.$$

If this vanishes, X is orthotomic of Y. Hence any linear equation

$$\Sigma u_i x_i = 0 \quad (i = 1, 2, 3, 4, 5)$$

expresses that X is orthotomic of a fixed circle whose coordinates y are to be found by solving the equations

$$u_i = \frac{\partial \phi\,(yy)}{\partial y_i} = \frac{\partial \phi\,(xy)}{\partial x_i}.$$

If therefore we write

$$\Phi\,(u, u) = \Sigma u_i^2 + 2\Sigma u_i u_j \cos(ij),$$

we shall have

$$(1\ 2\ 3\ 4\ 5)\, y_i = \frac{\partial \Phi\,(uu)}{\partial u_i}.$$

We may draw five new spheres, each orthotomic to four of the fundamental spheres; and it is clear that the quantities u form a system of coordinates relating to these reciprocal spheres. The system is not quite of the same kind as the original one; for the quantities u are proportional to the powers of the sphere U divided not by the radii of the reciprocal spheres, but by the volumes of the tetrahedra whose vertices are at the centres of the fundamental spheres. In one very important case, however, these ratios coincide, and the reciprocal spheres are the same as the fundamental spheres; namely, if these form an orthogonal system. In that case

$$\phi\,(xx) = \Sigma x_i^2 = \Phi\,(xx),$$
$$\phi\,(xy) = \Sigma x_i y_i = \Phi\,(xy),$$

and the coefficients in the equation of a sphere are also its coordinates.

*XXXV.

A FRAGMENT ON MATRICES.

[In explanation of the following paper observe that the matrix $\phi, = \begin{pmatrix} a\ , & b\ , & c \\ a'\ , & b'\ , & c' \\ a'', & b'', & c'' \end{pmatrix}$ substitutes for the point having the coordinates (α, β, γ), the point having the coordinates

$$(\phi \between \alpha, \beta, \gamma) = a\alpha + b\beta + c\gamma,\ \ a'\alpha + b'\beta + c'\gamma,\ \ a''\alpha + b''\beta + c''\gamma\,;$$

in particular for the points i, j, k coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, it substitutes the points whose coordinates are (a, a', a''), (b, b', b''), (c, c', c'') respectively. An indeterminate matrix (or more definitely, a matrix indeterminate in the first degree) is a matrix the determinant of which vanishes, but for which the first minors do not all of them vanish: such a matrix substitutes for a given point a point in a determinate line called in the paper, the axis: but for one position of the given point, called the null-point, the position of the substituted point is altogether arbitrary. A matrix indeterminate in the second degree is a matrix for which all the first minors vanish, or what is the same thing, one for which the second and third rows are mere multiples of the first row. C.]

An indeterminate matrix ϕ substitutes for the fundamental points i, j, k three collinear points α_1, α_2, α_3. As these must satisfy an identical equation $l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3 = 0$, there is a point $l_1 i + l_2 j + l_3 k$ which is reduced by ϕ to zero. But forasmuch as upon the line α there is a (1, 1) correspondence (viz. to

every point $x_1 i + x_2 j + x_3 k$ there corresponds one point $\Sigma x\alpha$, to every point $\Sigma x\alpha$ on the line corresponds the straight line

$$(x_1 + el_1)\, i + (x_2 + el_2)\, j + (x_3 + el_3)\, k$$

which has *one* point on the line α), there must be two united points, say μ, ν. Thus if λ represents the null-point and λ, μ, ν are now taken as fundamental points, the matrix ϕ is reduced to the form

$$\begin{vmatrix} 0, & 0, & 0 \\ 0, & g_2, & 0 \\ 0, & 0, & g_3 \end{vmatrix}.$$

This clearly corresponds to that mode of projection in which the centre of projection is taken on one of the two planes but not on the other.

Now consider two indeterminate matrices ϕ, ψ, and let λ, ρ be their null-points, μ, ν and σ, τ the respective united points. The product $\phi\psi$ will have ρ for a null-point, and $\mu\nu$ for axis; while $\psi\phi$ will have λ for null-point and $\sigma\tau$ for axis. In general therefore the product will still be indeterminate in the first degree only. But if λ is on the line $\sigma\tau$, i.e. if the null-point of ϕ is on the axis of ψ, then there is a certain line through ρ any point on which ψ will convert into λ; any point on which, therefore, $\phi\psi$ will destroy. Hence in this case $\phi\psi$ is indeterminate in the second degree.

Let
$$\phi = \begin{vmatrix} \phi_{11}, & \phi_{12}, & \phi_{13} \\ \phi_{21}, & \phi_{22}, & \phi_{23} \\ \phi_{31}, & \phi_{32}, & \phi_{33} \end{vmatrix},$$

i.e.
$$\phi\, (x_1 i_1 + x_2 i_2 + x_3 i_3) = (\phi_{11} x_1 + \phi_{12} x_2 + \phi_{13} x_3)\, i_1 + \ldots$$

say
$$\phi\, (ix) = \Sigma\Sigma \phi_{hk} i_h x_k.$$

Then if ϕ is indeterminate, $|\phi| = 0$, and the points $\phi_1 i$, $\phi_2 i$, $\phi_3 i$ are all on the axis. The axis is therefore

$$\begin{Vmatrix} \phi_{11}, & \phi_{21}, & \phi_{31} \\ \phi_{12}, & \phi_{22}, & \phi_{32} \end{Vmatrix},$$

and the null-point is

$$\begin{vmatrix} i_1, & i_2, & i_3 \\ \phi_{11}, & \phi_{12}, & \phi_{13} \\ \phi_{21}, & \phi_{22}, & \phi_{23} \end{vmatrix}.$$

If then the null-point of ϕ is on the axis of ψ, we must have

$$0 = \begin{vmatrix} \phi_{11}\psi_{11} + \phi_{12}\psi_{21} + \phi_{13}\psi_{31}, & \phi_{11}\psi_{12} + \phi_{12}\psi_{22} + \phi_{13}\psi_{32} \\ \phi_{21}\psi_{11} + \phi_{22}\psi_{21} + \phi_{23}\psi_{31}, & \phi_{21}\psi_{12} + \phi_{22}\psi_{22} + \phi_{23}\psi_{32} \end{vmatrix},$$

i.e. the first minors of $|\phi| \times |\psi'|$ must vanish, the rows of $|\phi|$ being multiplied into the columns of $|\psi|$. If the null-point of ψ is on the axis of ϕ, the first minors of $|\phi'| \times |\psi|$ must vanish, the columns of $|\phi|$ being multiplied into the rows of $|\psi|$*. Hence it is possible for $\phi\psi$ to be indeterminate in the second degree, while $\psi\phi$ is so only in the first. But $\phi\psi$ and $\psi\phi$ may both be doubly indeterminate without being equivalent. If however the null-point of each matrix is a united point on the axis of the other, and the remaining united points coincide at the intersection of the axes, then ϕ, ψ are commutative and their product is indeterminate in the second order.

In general two matrices are commutative when and only when they have the same united points. For let λ be a united point of ψ and not of ϕ, and suppose ϕ alters it to λ'. Then $\phi\psi$ will change λ to λ', but $\psi\phi$ will not do this unless λ' be another united point of ψ. Now there can be no cycle of changes, because the powers of ϕ have the same united points as ϕ. Hence, &c. Q.E.D.†

From this Cayley's theorem follows, because any set of n quantities $g_1 \ldots g_n$ can be expressed linearly in terms of n other sets, $1, \phi, \phi^2 \ldots \phi^{n-1}$. Consequently any matrix commutative with ϕ is of the form

$$a + b\phi + \ldots + k\phi^{n-1}.$$

To form a matrix which shall have three given points for united points. Let the points be l, m, n; viz. $l = l_1 i_1 + l_2 i_2 + l_3 i_3$,

* [Observe that the rows of the unaccented symbol are multiplied into the columns of the accented symbol, the accents being used only for the purpose of marking this distinction. C.]

† [The argument seems to be, if ϕ and ψ are commutative, then $\phi\lambda = \lambda'$ a united point of ψ, and in like manner $\phi^2\lambda$, $\phi^3\lambda$, &c. are all of them united points of ψ, and being thus finite in number, there is some power $\phi^m\lambda$ which is $=\lambda$: viz. λ is a united point of ϕ^m, and therefore a united point of ϕ. C.]

$m = \&c.$, $n = \&c.$ Then $(lmn)\, i_1 = (m_2 n_3)\, l + (n_2 l_3)\, m + (l_2 m_3)\, n$, etc. Thus $(lmn)\, \phi i_1 = \lambda\, (m_2 n_3)\, l + \mu\, (n_2 l_3)\, m + \nu\, (l_2 m_3)\, n$ and

$$(lmn)\, \phi\, (xi) = (xmn)\, \lambda l + (xnl)\, \mu m + (xlm)\, \nu n$$
$$= \{(xmn)\, \lambda l_1 + (xnl)\, \mu m_1 + (xlm)\, \nu n_1\}\, i_1 + \dots$$

$$\therefore\ (lmn)\, \phi_{hk} = \lambda l_h \frac{\partial\,(lmn)}{\partial l_k} + \mu m_h \frac{\partial\,(lmn)}{\partial m_k} + \nu n_h \frac{\partial\,(lmn)}{\partial n_k},$$

$$\phi = \begin{pmatrix} \Lambda_1, & \mathrm{M}_1, & \mathrm{N}_1 \\ \Lambda_2, & \mathrm{M}_2, & \mathrm{N}_2 \\ \Lambda_3, & \mathrm{M}_3, & \mathrm{N}_3 \end{pmatrix} \between \begin{pmatrix} \lambda l_1, & \lambda l_2, & \lambda l_3 \\ \mu m_1, & \mu m_2, & \mu m_3 \\ \nu n_1, & \nu n_2, & \nu n_3 \end{pmatrix}.$$

[Some lines, inserted apparently by way of verification, are omitted. The value obtained for the matrix ϕ is incorrect: we ought to have

$$(\phi \between l_1,\ l_2,\ l_3) = \lambda\, (l_1,\ l_2,\ l_3),$$
$$(\phi \between m_1,\ m_2,\ m_3) = \mu\, (m_1,\ m_2,\ m_3),$$
$$(\phi \between n_1,\ n_2,\ n_3) = \nu\, (n_1,\ n_2,\ n_3),$$

where ϕ is the required matrix

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix},$$

and (λ, μ, ν) are arbitrary constants.

There are thus nine linear equations for the determination of the nine coefficients of ϕ, and the solution is contained in the formula

$$\phi = \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} . \begin{pmatrix} \lambda, & 0, & 0 \\ 0, & \mu, & 0 \\ 0, & 0, & \nu \end{pmatrix} . \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix}^{-1}.$$

In fact starting from this formula, and substituting for ϕ its signification we have

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} . \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} = \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} . \begin{pmatrix} \lambda, & 0, & 0 \\ 0, & \mu, & 0 \\ 0, & 0, & \nu \end{pmatrix},$$

or forming the products of the two pairs of matrices

$$\begin{array}{c|ccc} & (l_1,\ l_2,\ l_3), & (m_1,\ m_2,\ m_3), & (n_1,\ n_2,\ n_3) \\ \hline (a\ ,\ b\ ,\ c\ ) & ,, & ,, & ,, \\ (a',\ b',\ c'\ ) & ,, & ,, & ,, \\ (a'',\ b'',\ c'') & ,, & ,, & ,, \end{array}$$

$$= \begin{array}{c|ccc} & (\lambda,\ 0,\ 0), & (0,\ \mu,\ 0), & (0,\ 0,\ \nu), \\ \hline (l_1,\ m_1,\ n_1) & ,, & ,, & ,, \\ (l_2,\ m_2,\ n_2) & ,, & ,, & ,, \\ (l_3,\ m_3,\ n_3) & ,, & ,, & ,, \end{array}$$

that is

$$\begin{aligned} &(a\ ,\ b\ ,\ c\ )(l_1,\ l_2,\ l_3) = (l_1,\ m_1,\ n_1)(\lambda,\ 0,\ 0), = l_1\lambda, \\ &(a',\ b',\ c'\ )(\quad ,, \quad) = (l_2,\ m_2,\ n_2)(\quad ,, \quad), = l_2\lambda, \\ &(a'',\ b'',\ c'')(\quad ,, \quad) = (l_3,\ m_3,\ n_3)(\quad ,, \quad), = l_3\lambda, \end{aligned}$$

and these are precisely the equations $(\phi)(l_1,\ l_2,\ l_3) = \lambda\,(l_1,\ l_2,\ l_3)$, &c. which were to be satisfied. C.]

*XXXVI.

ON TRICIRCULAR SEXTICS.

THE focal properties of n-circular curves of order $2n$ are most easily studied by inverting their plane in regard to an external point; the plane then becomes a sphere, and the inverse curve is the general intersection of this sphere by a surface of the n^{th} order, so that the problem is reduced to the study of the intersection of a quadric with a surface of order n in its relations with a certain plane section of the quadric. The method is equivalent to either of the two following :—

(1) The method of representation of a unicursal surface upon a plane used by Clebsch and Cremona. Given a quadric surface (sphere), its plane sections are unicursal curves involving three variables, any two of which intersect in two points; they may therefore be fitly represented by plane conics passing through two fixed points (circles). The two fixed points represent two generators on the surface, and the line joining them represents the point of intersection of the generators; these constitute the exceptional system in the surface and in the plane. A section of the quadric surface by a surface of order n is represented by a plane curve order $2n$ passing n times through each of the fixed points; because the curve on the surface is met n times by each of the special generators; and conversely a curve of order $2n$ in the plane is in general represented by a curve of order $4n$ on the surface, but when the plane curve passes n times through each of the fixed points, the two special generators are each thrown off n times from the curve on the surface; this makes a reduction $2n$ in the order, and there remains only a curve of order $2n$ meeting every generator in n points, that is

to say, the intersection of the quadric by an n^{thic} surface. A tricircular sextic is in this way regarded as the representation on a plane of a quadricubic curve.

(2) The method of circular co-ordinates. If we write

$$\begin{aligned} X &= x^2 + y^2 + 2a_1x + 2b_1y + c_1, \\ Y &= x^2 + y^2 + 2a_2x + 2b_2y + c_2, \\ Z &= x^2 + y^2 + 2a_3x + 2b_3y + c_3, \\ W &= x^2 + y^2 + 2a_4x + 2b_4y + c_4, \end{aligned}$$

then the equation of every circle can be put into the form $lX + mY + nZ + sW = 0$. The quantities $XYZW$ satisfy identically a homogeneous equation of the second order, which may be written as follows, (XY) meaning the cosine of the angle of intersection of the circles X and Y: viz.,

$$\begin{vmatrix} 1, & (XY), & (XZ), & (XW), & Xr_1^{-1} \\ (YX), & 1, & (YZ), & (YW), & Yr_2^{-1} \\ (ZX), & (ZY), & 1, & (ZW), & Zr_3^{-1} \\ (WX), & (WY), & (WZ), & 1, & Wr_4^{-1} \\ Xr_1^{-1}, & Yr_2^{-1}, & Zr_3^{-1}, & Wr_4^{-1}, & 0 \end{vmatrix} = 0$$

where r_1, r_2, r_3, r_4 are the radii of the four circles. If they cut orthogonally this reduces to

$$X^2r_1^{-2} + Y^2r_2^{-2} + Z^2r_3^{-2} + W^2r_4^{-2} = 0.$$

In any case we may write this relation $\Omega_2 = 0$. An n-circular curve of order $2n$ may always be represented by an equation of order n in $XYZW$, say $U_n = 0$, and this in an infinity of ways. For if $U_n = 0$ represent any curve, $U_n + K_{n-2}\Omega_2 = 0$ will represent the *same* curve, whatever quantic of order $n-2$ in X, Y, Z, W is represented by K_{n-2}. But regarding $U_n = 0$, $\Omega_2 = 0$, since they contain four variables, as equations to surfaces, this amounts to saying that we have only to pay attention to their curve of intersection. So that this method is merely a translation into analytical language of the former one. It will however exhibit the results in a convenient form; for if we succeed in drawing through the curve of intersection some surface of the n^{th} order whose equation is of a simple form, this amounts to finding an equation for the plane curve of order

$2n$, which is the same as that of the surfaces except that what there meant the distance of a point from a fixed plane will here be interpreted to mean the squared tangent to a fixed circle.

The chief problem is to find the series of doubly tangent circles to the curve, among which the foci are included as special cases. They represent sections of the quadric surface by planes doubly tangent to the curve of intersection, and the foci correspond to those points of the quadric which are touched by such planes. Our problem is therefore, to study the torse which is the envelope of planes doubly tangent to the curve of intersection of a quadric with an n^{thic} surface, and especially the common tangent planes to their torse and the quadric.

The reciprocal problem is that of the determination of the focal curve of a surface of the n^{th} class. In elliptic space this curve is the nodal curve of the torse enveloped by planes tangent to the surface and a certain proper quadric, called the absolute; in parabolic space the absolute is a plane conic, and as this is part of the nodal curve (it is in fact an n-tuple curve on the torse) the degree of the focal curve is reduced. For some purposes it is convenient to consider the problem in its original form, and for others in the reciprocal form.

In any case, however, it is quite sufficiently complicated for tricircular sextics and surfaces of the third class; and I have only attempted it in this case of $n = 3$.

II.

A quadricubic curve, the intersection of a quadric with a cubic surface, when looked at from an arbitrary point in space, appears to be a sextic curve with six nodes and no cusps. It could only appear to have a cusp if the arbitrary point of observation were on the torse generated by tangents to the curve. To prove that it has six apparent nodes, we have only to remember that it is a point-for-point representative of a plane sextic with two triple points, and is therefore of

deficiency 4. The same thing appears by considering the cone standing on the curve and converging to a point of it. This is of the fifth order and has two nodal lines, namely the generators of the quadric through the point. The tangent cone, then, drawn through an arbitrary point of space, being of the sixth order with six nodal lines and no cusps, is of class $18 = 30 - 12$; and this is also the class of the tricircular sextic. In the latter case the number of tangents which can be drawn to the curve from one of the triple points is $12 = 18 - 6$, 2 tangents being swallowed up by each of the 3 branches through the triple point. Consequently the number of foci is 144, of which only 12 are real.

Returning to the quadricubic curve, we see that the tangent cone from an arbitrary point must have $36, = 6 \times 12 - 6 \times 6$ inflexions, and $96, = \frac{1}{2}(18 \times 17 - 3 \times 36 - 6)$ double tangents. The former number is the class of the torse traced out by tangents to the curve, and the latter is the class of the torse enveloped by planes doubly tangent. In the reciprocal problem, this latter number is the order of the nodal curve of the torse generated by common tangent planes to a quadric and a surface of the third class; that is to say, *the focal curve of a surface of the third class is of order* 96.

* XXXVII.

ON BESSEL'S FUNCTIONS.

1. I CONSIDER the function

$$f(x) = 1 + x + \frac{x^2}{(\Pi 2)^2} + \dots + \frac{x^k}{(\Pi k)^2} + \dots \textit{ ad inf.};$$

the series is evidently one-valued and convergent for all values real or complex of the variable x.

2. The n^{th} derived function of $f(x)$ is

$$f_n(x) = \frac{1}{\Pi n} + \frac{x}{\Pi(n+1)} + \frac{x^2}{\Pi(n+2) \, . \, \Pi 2} + \dots + \frac{x^k}{\Pi(n+k) \, . \, \Pi k} + \dots \textit{ ad inf.};$$

and if we integrate n times from 0 to x we obtain

$$f_{-n}(x) = \frac{x^n}{\Pi n} + \frac{x^{n+1}}{\Pi(n+1)} + \frac{x^{n+2}}{\Pi(n+2) \, . \, \Pi 2} + \dots + \frac{x^{n+k}}{\Pi(n+k) \, . \, \Pi k} + \dots = x^n f_n(x).$$

3. Hence we derive the differential equations

$$\partial_x^{2n} \{x^n f_n(x)\} = f_n(x),$$
$$\partial_x \{x^{n+1} \partial_x f_n(x)\} = x^n f_n(x).$$

4. We may generalize the definitions of (2) so as to obtain a value of $f_n(x)$ for all values of n, if we take Πn to mean $\Gamma(n+1)$, according to Gauss's notation. The differential equations of (3) will hold good in the general case.

5. By multiplying together the series for ϵ^{xy} and $\epsilon^{\frac{x}{y}}$, as follows:

$$\begin{array}{l}
1 + xy + \dfrac{x^2y^2}{\Pi 2} + \dfrac{x^3y^3}{\Pi 3} + \dfrac{x^4y^4}{\Pi 4} + \\
+ \\
xy^{-1} + x^2 + \dfrac{x^3y}{\Pi 2} + \dfrac{x^4y^2}{\Pi 3} + \dfrac{x^5y^3}{\Pi 4} + \\
+ \\
\dfrac{x^2y^{-2}}{\Pi 2} + \dfrac{x^3y^{-1}}{\Pi 2} + \dfrac{x^4}{(\Pi 2)^2} + \dfrac{x^5y}{\Pi 3 . \Pi 2} + \dfrac{x^6y^2}{\Pi 4 . \Pi 2} + \\
+ \\
\dfrac{x^3y^{-3}}{\Pi 3} + \dfrac{x^4y^{-2}}{\Pi 3} + \dfrac{x^5y^{-1}}{\Pi 3 . \Pi 2} + \dfrac{x^6}{(\Pi 3)^2} + \dfrac{x^7y}{\Pi 4 . \Pi 3} + \\
\dfrac{x^4y^{-4}}{\Pi 4} + \dfrac{x^5y^{-3}}{\Pi 4} + \dfrac{x^6y^{-2}}{\Pi 4 . \Pi 2} + \dfrac{x^7y^{-1}}{\Pi 4 . \Pi 3} + \dfrac{x^8}{(\Pi 4)^2} +
\end{array}$$

we find

$$e^{x\left(y+\frac{1}{y}\right)} = f(x^2) + \left(y + \frac{1}{y}\right) . xf_1(x^2) + \left(y^2 + \frac{1}{y^2}\right) . x^2f_2(x^2) + \dots$$
$$+ \left(y^k + \frac{1}{y^k}\right) . x^kf_k(x^2) + \dots$$

6. In this formula write $-ix$ for x, and iy for y; then

$$e^{x\left(y-\frac{1}{y}\right)} = f(-x^2) + \left(y - \frac{1}{y}\right) . xf_1(-x^2) + \left(y^2 + \frac{1}{y^2}\right) . x^2f_2(-x^2) - \dots$$
$$+ \left\{y^k + \left(-\frac{1}{y}\right)^k\right\} x^k . f_k(-x^2) + \dots$$

a result which may also be obtained by direct multiplication.

7. In the equation of (5) and (6), put $y = e^{i\phi}$; thus we obtain

$$e^{2x\cos\phi} = f(x^2) + 2\cos\phi . xf_1(x^2) + 2\cos 2\phi . x^2f_2(x^2) + \dots$$
$$+ 2\cos k\phi . x^kf_k(x^2) + \dots$$

$$e^{2ix\sin\phi} = f(-x^2) + 2i\sin\phi . xf_1(-x^2) + 2\cos 2\phi x^2 f_2(-x^2) + \dots$$
$$+ 2i\sin(2k-1)\phi . x^{2k-1}f_{2k-1}(-x^2) + 2\cos 2k\phi . x^{2k}f_{2k}(-x^2) + \dots$$

and separating real and imaginary parts of the last equation,

$$\cos(2x\sin\phi) = f(-x^2) + 2\cos 2\phi\,.\,x^2 f_2(-x^2) + \dots$$
$$+ 2\cos 2k\phi\, x^{2k} f_{2k}(-x^2) + \dots$$
$$\sin(2x\sin\phi) = 2\sin\phi\,.\,x f_1(-x^2) + 2\sin 3\phi\,.\,x^3 f_3(-x^2) + \dots$$
$$+ 2\sin(2k-1)\phi\,.\,x^{2k-1} f_{2k-1}(-x^2) + \dots$$

8. Hence by Fourier's theorem we derive the definite integrals,

$$\int_0^\pi e^{2x\cos\phi}\cos n\phi d\phi = \pi x^n f_n(x^2),$$
$$\int_0^\pi \cos(2x\sin\phi)\cos 2n\phi d\phi = \pi x^{2n} f_{2n}(-x^2),$$
$$\int_0^\pi \cos(2x\sin\phi)\cos(2n+1)\phi d\phi = 0,$$
$$\int_0^\pi \sin(2x\sin\phi)\sin 2n\phi d\phi = 0,$$
$$\int_0^\pi \sin(2x\sin\phi)\sin(2n+1)\phi d\phi = \pi x^{2n+1} f_{2n+1}(-x^2),$$

and by addition of the last set of four

$$\int_0^\pi \cos(2x\sin\phi - n\phi)\,d\phi = \pi x^n f_n(-x^2).$$

[The above paper owed its origin to a communication made by Lord Rayleigh to the Mathematical Society (January 10th, 1878, *Proceedings*, Vol. IX. pp. 61—64). A few days after the meeting Prof. Clifford told me that he had a short note to lay before the Society, but I heard nothing further on the matter. Amongst the MSS. were two papers on the subject, the one printed above, the other entitled "Note on Bessel's and Laplace's functions." The former has been inserted in its entirety on Prof. Cayley's recommendation: I give an extract or two in this place from the latter. It commences—"Lord Rayleigh's interesting paper 'On the relation between the functions of Laplace and Bessel' has led me to examine whether a certain simple expression which I had found for Bessel's functions could not be extended to those of Laplace. It appears that the two functions may be derived by a very simple transformation from the exponential and binomial series respectively, and that the passage from one to the other is in fact equivalent to the well-known passage from $\left(1+\frac{x}{n}\right)^n$ to ϵ^x, when n is made indefinitely great."

He then points out that

$$J_n(2x) = x^n f_n(-x^2),$$

that the equation

$$\partial_x f_n(x) = f_{n+1}(x)$$

gives the formula for the fluxion of a Bessel's function, and derives the relation between three consecutive Bessel's functions from the identity

$$f_{m-1} - m f_m + x^2 f_{m+1} = 0.$$

From the equations for $\epsilon^{2x\cos\phi}$, $\epsilon^{2x\sin\phi}$, $\epsilon^{2\iota x\cos\phi}$ and $\epsilon^{2\iota x\sin\phi}$ he derives the first equation in 8 of the preceding paper. The remaining two pages are fragmentary, the only clear result being, that taking

$$\frac{1}{\sin\theta}\,\partial_\theta\,(\sin\theta\,\partial_\theta P) + n(n+1)\,P = 0$$

and assuming

$$z = \tan^2\tfrac{1}{2}\theta, \quad u = P \div \cos^{2n}\tfrac{1}{2}\theta,$$

he obtains

$$(1+z)^2 z\partial_z^2 u + \{(1+z)^2 - 2nz(1+z)\}\partial_z u + \{n(n+1) - n\}(1+z)\,u = 0,$$

or

$$(1+z)\,z\partial_z^2 u + (1+z-2nz)\,\partial_z u + n^2 u = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is satisfied, as Prof. Cayley points out, by

$$u = 1 - \left(\frac{n}{1}\right)^2 z + \left(\frac{n\,.\,n-1}{1\,.\,2}\right)^2 z^2 - \ldots,$$

viz. the resulting expression for P is

$$P = \cos^{2n}\tfrac{1}{2}\theta\left\{1 - \left(\frac{n}{1}\right)^2 \tan^2\tfrac{1}{2}\theta + \left(\frac{n\,.\,n-1}{1\,.\,2}\right)^2 \tan^4\tfrac{1}{2}\theta - \ldots\right\},$$

which is a known formula, cf. Todhunter, *Functions of Laplace, Lamé and Bessel*, p. 11.]

* XXXVIII.

ON GROUPS OF PERIODIC FUNCTIONS.

I. Introduction.

The simplest periodic functions are the circular and elliptic functions; these may be regarded as built up out of exponential and θ-functions, and the latter again as built up of circular functions. Let us write *

$$\kappa u = e^u + e^{-u},$$

then $$\vartheta u = \Sigma_0^\infty q^{nn}\, \kappa(nu) = \Sigma_{-\infty}^{+\infty} e \,|\, n^2 Q + nu,$$

where $$q = e^Q.$$

In the same way if we write

$$\kappa(u, v) = e^u \vartheta(v + a) + e^{-u}\vartheta(v - a),$$

we may form the function

$$\vartheta(u, v) = \Sigma_0^\infty p^{mm}\, \kappa(mu, v) = \Sigma\Sigma_{-\infty}^{+\infty} e \,|\, m^2 P + n^2 Q + 2mna + mu + nv,$$

where it is to be understood that $\kappa(mu, v)$ means $e^{mu}\vartheta(v + ma) +$. By proceeding in this manner, we may form an ascending series of functions κ, ϑ, of an increasing number of variables. Let $m_1, m_2, \ldots m_p$ be p whole numbers, and let

$$\phi m = \Sigma Q_i m_i^2 + 2\Sigma Q_{ij} m_i m_j;$$

then we may form a function ϑ of the p arguments $u_1, u_2, \ldots u_p$ by adding together all the exponentials of the form

$$e^{\phi(m) + 2\Sigma mu},$$

* Rosenhain's Method, *Mem. Div. Sav.* t. XI.

obtained by giving to the whole numbers m all possible values from $-\infty$ to $+\infty$. That is, we shall write

$$\vartheta(u_1, u_2, \ldots u_p) = \Sigma^p e \,|\, \phi(m) + 2\Sigma mu \,|,$$

from this we may form a function κ with another argument u_0 as follows,

$$\kappa(u_0; u_1, u_2, \ldots u_p) = e^{2u_0}\vartheta(u_1 + a_1, u_2 + a_2, \ldots u_p + a_p) \\ + e^{-2u_0}\vartheta(u_1 - a_1, \ldots u_p - a_p).$$

This being so, the functions ϑ and κ are unaltered if we increase any of the arguments u by any multiple of πi; so that the functions ϑ are p-fold periodic*, the functions κ are $(p+1)$ fold periodic.

In any one term of the function ϑ let us increase the number m_i by unity. The alteration in $\phi(m)$ is $\partial_{m_1}' \phi(m) +$, which is

$$2Q_i m_i + 2\Sigma Q_{ij} m_j + Q_i,$$

the increase in $2\Sigma mu$ is simply $2u_i$. Thus the same effect will be produced if we increase every argument u_κ by the quantity Q_{ik}, and then add $2u_i + Q_i$. Now the change from m_i to $m_i + 1$ merely passes from one term to another in the summation, and does not alter the function ϑ. Hence we have

$$\vartheta(u_1 \ldots u_p) = e^{2u_i + Q_i}\vartheta(u_1 + Q_{i1}, \ldots u_p + Q_{ip}).$$

Thus for simultaneous addition of one row of the quantities Q to the arguments, ϑ is quasi-periodic; it reproduces itself multiplied by an exponential factor.

The exponent in each term of ϑ, namely,

$$\phi(m) + 2\Sigma mu,$$

may be reduced to the form of a sum of squares of linear functions of the (m), less a quantity of the second order in the u. That is to say, we may have

$$\phi(m) + 2\Sigma mu = \Sigma_i (v_i + 2\Sigma m_j B_{ij})^2 - \Sigma v^2,$$

where the v are linear functions of the u. If then we multiply the ϑ-function by an exponential $e^{\Sigma v^2}$, we shall obtain the con-

* [Periodic as to these periods of the arguments u. C.]

venient form considered by Göpel, in which I now replace the letters v by the u;

$$G(u_1 \dots u_p) = \Sigma^p e \,|\, \Sigma_i (u_i + 2\Sigma m_j B_{ij})^2 \,|.$$

It is clear that this is unaltered if we simultaneously increase the arguments $u_1 \dots u_p$ by the quantities $2B_{1j} \dots 2B_{pj}$, where j may be any of the numbers 1, 2, p. There is also a set of quantities A in regard to which the function is quasi-periodic, or reproduces itself with an exponential factor. In regard to those quantities the quotient of G by $e \,|\, \Sigma u^2 \,|$ is actually periodic; viz., if we write

$$G(u_1 \dots u_p) = e \,|\, \Sigma u^2 \,|\, \Theta(u_1 \dots u_p),$$

then we shall have

$$\Theta(u_1 + 2A_{1j}, \dots u_p + 2A_{pj}) = \Theta(u_1 \dots u_p),$$

provided that

$$4\Sigma_j A_{ij} B_{ij} = \pi i, \quad 4\Sigma_j A_{ij} B_{kj} = 0,$$

and these equations suffice to determine the quantities A. We have then

$$\begin{aligned} G(u_1 + 2A_{1j} \dots u_p + 2A_{pj}) &= e \,|\, \Sigma_i (u_i + 2A_{ij})^2 \,|\, \Theta(u_1 \dots u_p) \\ &= e \,|\, 4\Sigma_i A_{ij}^{\,2} + 4\Sigma_i u_i A_{ij} \,|\, G(u_1 \dots u_p). \end{aligned}$$

We shall now leave out in all cases the first suffix, writing for instance $G(u + A_j + B_k)$ instead of

$$G(u_1 + A_{1j} + B_{1k}, \dots u_p + A_{pj} + B_{pk}),$$

and so in similar cases.

II.

A linear function of the quantities A, B appertaining to a particular argument u, the coefficients being either 0 or 1, is called a *quadrant*, e.g. for u_1 a certain quadrant is

$$A_{11} + A_{12} + B_{12} + B_{13} + B_{1p}.$$

Since for every argument there are p A's and p B's, the whole number of quadrants is 2^{2p}, if 0 be included among them. We have now to consider the 2^{2p} functions $G(u + X)$, where X

is any quadrant. The quantities A, B themselves may be distinguished as *prime* or *elementary* quadrants.

If the prime quadrants of u are disposed in two rows, thus

$$\begin{array}{l} A_1, A_2, A_3, \ldots A_p, \\ B_1,\ B_2,\ B_3, \ldots B_p; \end{array}$$

they may be considered as forming the p *pairs* A_1B_1, $A_2B_2, \ldots A_pB_p$. Thus every quadrant will be composed of a certain number of pairs, a certain number of single A's and a certain number of single B's.

If the number of pairs in a quadrant X is odd, then $G(u+X)$ is an odd function of the u, and $G(X)=0$.

Let*

$$X = a_1A_1 + \ldots + a_pA_p + b_1B_1 + \ldots + b_pB_p,$$

where the a and the b are each of them either 0 or 1. We have

$$\begin{aligned} G(u+X) &= \Sigma e \,|\, \Sigma\, (u + X + 2\Sigma mB)^2 \,| \\ &= \Sigma e \,|\, \Sigma\, \{u + \Sigma aA + \Sigma\, (2m+b)\, B\}^2 \,|, \\ G(-u+X) = G(u-X) &= \Sigma e \,|\, \Sigma\, \{u - \Sigma aA + \Sigma\, (2m+b)\, B\}^2 \,| \\ &\qquad (m+1 \text{ written for } m). \end{aligned}$$

Each exponent on the right is the same in these two expressions, except for the term $2\Sigma_k a_i b_i A_{ki} B_{ki}$ in the first, and $-2\Sigma_k a_i b_i A_{ki} B_{ki}$ in the second. The difference is $4a_ib_i\Sigma_k A_{ki}B_{ki} = a_ib_i\pi i$. Hence we have

$$e^{-4\Sigma Aau}\, G(u+X) = e\,|\,\pi i \Sigma a_i b_i\,|\, G(-u+X),$$

and therefore $e^{-2\Sigma Aau}\, G(u+X)$ is an odd or even function of u according as $\Sigma a_i b_i$ is odd or even.

The number of odd functions is $2^{p-1}(2^p-1)$. We have to make

$$a_1b_1 + a_2b_2 + \ldots a_pb_p = 2n+1,$$

where the a, b are either 0 or 1. Let k of the a be zero, the remaining $p-k$, unity; then the b belonging to the first k may be either 0 or 1, which gives 2^k combinations.

* Clebsch and Gordan, p. 260.

The sum of the remaining $p-k$ b's must be odd, so that the last is determined when all the others are chosen, which may be done in 2^{p-k-1} ways; thus we have 2^{p-1} systems of the b. But the k a's which are zero may be chosen in $\frac{*[p]^k}{\lfloor k}$ ways; and k may have all values from 0 to $p-1$. Hence the whole number of odd functions is

$$2^{p-1}\left(1+p+\frac{*[p]^2}{2}+\frac{1}{\lfloor 3}*[p]^3+\ldots+p\right)=2^{p-1}\,(2^p-1). \quad \text{Q.E.D.}$$

It follows that the number of even functions is $2^{p-1}\,(2^p+1)$.

Let X, Y, Z be any three quadrants, and consider the functions

$$G\,(u+Y+Z),\ G\,(u+Z+X),\ G\,(u+X+Y).$$

If they are all even, or if one is even and two odd, then each is of the same character as the product of the other two. But if they are all odd, or if one is odd and two even, then the fluxion of each is of the same character as the product of the other two. We shall now shew that

$$G'\,(u+X+Y)\,Gu-G\,(u+X+Y)\,G'u$$

may be expressed as the sum of a number of terms such as

$$G\,(u+X+Z),\ G\,(u+Y+Z).$$

Since Gu is an even function, it follows from what has just been said that of the three functions

$$G\,(u+Y+Z),\ G\,(u+Z+X),\ G\,(u+X+Y),$$

either all must be odd, or one odd and two even.

To this end it is important to consider the function which is derived from Gu by adding together the squares of all the terms; namely, we write

$$Fu=\Sigma e\,|\,2\Sigma_i\,(u_i+2\Sigma m_j B_{ij})^2\,|.$$

If now in $G\,(u+X+Z)$ we select the term involving the numbers m, and multiply it by the term in $G\,(u+Y+Z)$ involving the numbers n, the product is

$$e\,|\,\Sigma\,(u+X+Z+2\Sigma mB)^2+\Sigma\,(u+Y+Z+2\Sigma nB)^2\,|.$$

* [Auctoris.]

Now let $\mu_i = m_i + n_i, \ \nu_i = m_i - n_i$;
then we have

$$\mu^2 + \nu^2 = 2(m^2 + n^2), \ \mu\mu' + \nu\nu' = 2(mm' + nn'),$$

and consequently

$$(u + X + Z + 2\Sigma mB)^2 + (u + Y + Z + 2\Sigma nB)^2$$

= twice square of half sum + twice square of half difference

$$= 2(u + \tfrac{1}{2}X + \tfrac{1}{2}Y + Z + \Sigma\mu B)^2 + 2(\tfrac{1}{2}X - \tfrac{1}{2}Y + \Sigma\nu B)^2.$$

In this expression each corresponding pair of μ, ν must be either both even or both odd. We may put separately the B's belonging to the odd ones; let $\mathfrak{B}$ be written for the sum of them, viz. $\mathfrak{B}$ is the sum of any selection of the B's; then $\Sigma\mu B$ may be written $\mathfrak{B} + 2\Sigma m'B$ and $\Sigma\nu B = \mathfrak{B} + 2\Sigma n'B$, where the m' and n' are now independent. Thus we shall have

$$G(u + X + Z)\, G(u + Y + Z)$$
$$= \Sigma F(u + \tfrac{1}{2}X + \tfrac{1}{2}Y + Z + \mathfrak{B})\, F(\tfrac{1}{2}X - \tfrac{1}{2}Y + \mathfrak{B}),$$

where the possible number of terms on the right is the number of different selections $\mathfrak{B}$, viz., 2^p.

We must however investigate the effect of adding half-quadrants to the arguments of the functions F. We have, namely,

$$F(u + \tfrac{1}{2}X) = \Sigma e \,|\, 2\Sigma\, \{u + \tfrac{1}{2}\Sigma aA + \Sigma(2m + \tfrac{1}{2}b) \,.\, B\}^2 \,|$$
$$F(-u + \tfrac{1}{2}X) = F(u - \tfrac{1}{2}X)$$
$$= \Sigma e \,|\, 2\Sigma\, \{u - \tfrac{1}{2}\Sigma aA + \Sigma(2m - \tfrac{1}{2}b) \,.\, B\}^2 \,|$$
$$= \Sigma e \,|\, 2\Sigma\, \{u - \tfrac{1}{2}\Sigma aA + B_x + \Sigma(2m + \tfrac{1}{2}b) \,.\, B\}^2 \,|.$$
$$(B_x = \Sigma bB.)$$

But

$$F(u + B_x + \tfrac{1}{2}X) = \Sigma e \,|\, 2\Sigma\, \{u + \tfrac{1}{2}\Sigma aA + B_x + \Sigma(2m + \tfrac{1}{2}b) \,.\, B\}^2 \,|,$$

the difference of these exponents consists of a term $4\Sigma aAu$, and a term $4\Sigma a \,.\, (4m + b)\, AB = \Sigma a \,.\, (4m + b)\, \pi i$. This therefore gives rise to a factor $e \,|\, \Sigma a \,.\, (4m + b) \,.\, \pi i \,|$ which is ± 1 according as Σab is even or odd, i.e., according as $e^{-2\Sigma aAu}\, G(u + X)$ is even or odd. Thus

$$F(u - \tfrac{1}{2}X) = \pm\, e^{-4\Sigma Aau}\, F(u + B_x + \tfrac{1}{2}X),$$

according to the character of X.

* XXXIX.

THEORY OF MARKS OF MULTIPLE THETA-FUNCTIONS.

[A mark (p) is a symbol $\begin{pmatrix}\alpha, & \beta, & \gamma \dots \\ \alpha', & \beta', & \gamma' \dots\end{pmatrix}$, composed of the $2p$ integers $\alpha, \beta, \gamma \dots \alpha', \beta', \gamma' \dots$, distinguished only as even or odd integers; and the mark is even or odd according as $\alpha\alpha' + \beta\beta' + \gamma\gamma' \dots$ is even or odd. C.]

NUMBER of marks (p) is 2^{2p}.

Number of odd marks ($p+1$) may be calculated from number [of odd marks] p. Let N_p, E_p be numbers of odd and even marks for p; $N_p + E_p = 2^{2p}$. We may prefix to each of these odd marks $\begin{smallmatrix}0\\0\end{smallmatrix}$, $\begin{smallmatrix}0\\1\end{smallmatrix}$ or $\begin{smallmatrix}1\\0\end{smallmatrix}$, and to each of the even marks $\begin{smallmatrix}1\\1\end{smallmatrix}$, to get an odd mark for $p+1$. Therefore

$$N_{p+1} = 3N_p + E_p = 2^{2p} + 2N_p$$

$$= 2^{2p} + 2 \cdot 2^{2(p-1)} + 2^2 \cdot 2^{2(p-2)} + \dots + 2^p = 2^p(2^{p+1} - 1);$$

and consequently $N_p = 2^{p-1}(2^p - 1)$. These numbers are 1, 6, 28, 120, ...

Any mark (p) except 0 may be divided into two odd marks in N_{p-1} ways. For let it begin with $\begin{smallmatrix}a\\b\end{smallmatrix}$. Take any odd mark ($p-1$) for the places after the first, and add it to those places of the given mark. If the result is even, we must divide $\begin{smallmatrix}a\\b\end{smallmatrix}$ into an even and an odd part; if odd, we must divide it into two even parts. The former can be done in one way and one

only, unless $\frac{a}{b}=\frac{0}{0}$; viz. the parts are $\frac{a+1}{b+1}$ and $\frac{1}{1}$. The latter can be done in one way only, viz. $\frac{a}{b}=\frac{0}{0}+\frac{a}{b}$ or $\frac{1}{1}=\frac{1}{0}+\frac{0}{1}$; but here we may attach either of the even parts to either of the odd succeeding parts, and we have consequently *two* divisions of the given mark corresponding to the *two* odd parts. Hence if $\frac{b}{a}$ is not $=\frac{0}{0}$, the number of divisions is exactly N_{p-1}. If it is, we may select any other place of the given mark, which is not $=\frac{0}{0}$.

Thus every mark g, except 0, is the sum of each of N_{p-1} pairs of odd marks. The $2N_{p-1}$ odd marks form the group g. We may include the group 0, consisting of all the odd marks. Then every odd mark occurs in N_p groups.

Every two odd marks l, m are common to $2N_{p-1}$ groups. A group to which l belongs is of the form $l+n$, where n is odd; if m belongs to this, $l+n=m+p$, and therefore $l+m=n+p$, or n, p form a pair of the group $l+m$. There are two groups for each such pair, except l, m; adding to these the group 0, we have $2N_{p-1}$.

To find the odd marks common to two groups, g, h, let $g+h=l+m$, where l, m are odd. Then

$$g+l=h+m, \quad g+m=h+l.$$

Now $(g+l).(g'+l')+(h+l).(h'+l')\equiv gh'+g'h$

$$\equiv gg'+hh'+(g+h)(g'+h').$$

If this $\equiv 1$, then $g+l$, $g+m$ are one even and one odd; therefore of every pair in the group $g+h$, one belongs to the group g and one to the group h; and the odd marks which make up these pairs in g, h are identical. In this case, therefore, every two of the groups g, h, $g+h$ have N_{p-1} marks in common, no two of which form a pair. For this, *one* or *all* of the marks g, h, $g+h$ must be odd. If however $gh'+g'h\equiv 0$, so that *none* or *two* of g, h, $g+h$ are odd, then $g+l$, $g+m$ are both even or

both odd. Now in this case the three groups g, h, $g+h$ contain every odd mark between them. For

$$\begin{aligned}&(g+l)(g'+l')+(h+l)(h'+l')\\ &=gg'+hh'+(g+h)l'+(g'+h')l\\ &=(g+h+l)(g'+h'+l')+gh'+g'h+ll'.\end{aligned}$$

Therefore of $g+l$, $h+l$, $g+h+l$, either *one* is odd or *all* are odd. Let then each group have x marks in common with the other two, and y to itself. Then

$$x+3y=N_p,\quad x+y=2N_{p-1};$$

so that $\qquad y=\frac{1}{2}N_p-N_{p-1}=2^{p-2}(2^p-2^{p-1})=2^{2p-3}$,

$$x=2^{p-1}(2^{p-1}-1)-2^{2p-3}=2^{2p-3}-2^{p-1}=2^{p-1}(2^{p-2}-1)=4N_{p-2}.$$

These common marks are distributed in sets of 4, l, m, n, p, so that

$$g=l+n=m+p,\quad h=l+p=m+n,\quad g+h=l+m=n+p.$$

Relations of two groups.

Two groups g, h stand in one of two possible relations to each other. If they have N_{p-1} odd marks in common, no two of which are paired in either group, they shall be called *near* groups; if they have in common N_{p-2} sets of four marks, each set forming two pairs in each group, they shall be called *far* groups. The sum of two groups is near to or far from both of them, according as the groups are near or far. If they are near, no marks are common to the two groups and their sum; if they are far, the N_{p-2} sets occur in the sum divided into pairs in the only remaining way.

Hence, l, m, n being odd marks, the groups $l+m$, $l+n$ are near if l, which is common to them, does not belong to their sum, or if $l+m+n$ is even; far when $l+m+n$ is odd.

Relations of three groups.

Let f and g be two near groups, and h another group near to f but not equal to $f+g$; then if h be near to g, $f+h$ will be

far from g, and *vice versâ*. The group h cannot contain all the marks common to f and g, for then $f+g$ would have absolutely no marks in common with h, which is impossible. Let then a be a mark common to f and g but not to h, and let $f=a+\alpha_1$, $g=a+\alpha_2$. Then if h is near to g, α_1 and α_2 both occur in h, but not paired, because $\alpha_1+\alpha_2=f+g$, which is not h by hypothesis. If $h=\alpha_1+\beta_2=\alpha_2+\beta_1$, then $a+h=f+\beta_2=g+\beta_1$, and therefore $f+\beta_1=g+\beta_2$, $=b$, suppose. Now b is odd, because either α_2 or β_1 must occur in the group f, and α_2 does not. Hence we have these equations,

$$\begin{aligned} f&=a+\alpha_1 & g&=a+\alpha_2 & h&=\alpha_1+\beta_2 & f+h&=a+\beta_2 \\ &=b+\beta_1 & &=b+\beta_2 & &=\alpha_2+\beta_1 & &=b+\alpha_2, \end{aligned}$$

from which it appears at once that $f+h$ is far from g, as was to be proved.

It follows that one-half of the groups which are near to f are also near to g, the complementary half (counting $f+h$ complementary to h) being far from g. We shall now consider more closely those groups which are near both to f and to g.

We observe that two pairs of the group h are obtained by crossing the constituents from two pairs of the groups f, g having in common the marks a, b. These marks shall be called a *cross-pair* of h; neither of them is contained in h. Since every cross-pair gives rise to a set of four marks common to the far groups g and $f+h$, and *vice versâ*, there are N_{p-2} cross-pairs. Every mark common to f and g either belongs to a cross-pair or occurs in h. Hence there are $N_{p-1}-2N_{p-2}$, $=2^{2p-4}$ marks common to f, g, h.

Relations of four groups.

If j be another group near to f, g and h, it contains one mark and one only out of every cross-pair of h. If it did not contain a or b, it must contain $\alpha_1, \beta_1, \alpha_2, \beta_2$, and therefore be far from h; if it contained both a and b, it must contain none of these four, with the same result. For the same reason h must contain one and one only out of every cross-pair of j. Hence of the marks

common to f, g, there are N_{p-2} occurring neither in h nor j, and N_{p-2} occurring in each but not in the other. Thus $2^{2p-4}-N_{p-2}$ are common to all four; these are also common to $f+g+h+k$.

The system $fghj$ is formed as follows. Given f and g near to one another, we find h near to both; then it has N_{p-2} cross-pairs and 2^{2p-4} marks common with f and g. Let a belong to one of the cross-pairs, and c be one of the common marks; then a group formed to have ac for a cross-pair will be near to h. For it will contain α_1 and α_2 separated from each other, and associated with marks which do not occur in h. Let j be this group; we have seen that it has $2^{2p-4}-N_{p-2}$ marks common with fgh.

Now let a be one of these common marks; then a group k formed to have ad for a cross-pair will be near to h and j. All those N_{p-2} marks which are common to the cross-pairs of i and j will belong separately to the N_{p-2} cross-pairs of k. For if pq be a cross-pair of h, and pr the corresponding one of j, then k cannot have the cross-pair qr, or it would be $=f+g+h+j$, which we do not suppose; hence it must have a cross-pair ps. Then s must occur in the groups h and j. Consequently the number of marks common to $fghjk$ is $2^{2p-4}-2N_{p-2}$. These are also common to $f+g+h+j$, $f+g+j+k$, $f+g+h+k$.

If j be near to f and g and far from h, then of the two marks in every cross-pair of h it must contain neither or both. If it contains a, for example, it cannot contain α_1 or α_2 and therefore not β_1 or β_2, so that it must contain b. Being near to f and g it is far from $f+g$, so that the three groups $f+g$, h, j are far from one another.

We may now find the groups to which three odd marks l, m, n are common. If g be one of these, $g+l$ is common to the groups g, $l+m$, $l+n$. We have therefore to find the marks which are common to the groups $l+m$, $l+n$. If

$$\begin{aligned} 1 &\equiv (l+m)(l'+n')+(l'+m')(l+n), \\ &\equiv ln'+l'n+mn'+m'n+lm'+l'm, \\ &\equiv (l+m+n)(l'+m'+n')+ll'+mm'+nn', \end{aligned}$$

i.e. if $l+m+n$ is even, then there are N_{p-1} marks common to the groups $l+m$, $l+n$; and by adding l to each of these, we obtain the N_{p-1} groups to which l, m, n are common. If however $l+m+n$ is odd, there are $4N_{p-2}$ marks common to the groups $l+m$, $l+n$, and consequently $4N_{p-2}$ groups to which the three marks l, m, n are common. In the former case, no two of the N_{p-1} marks common to $l+m$, $l+n$ are paired in either of them; let $l+g$, $l+h$ be any two of these, then $g+h$ is not equal to $l+m$, or $l+n$, or $m+n$; hence $l+g$, $m+g$, $n+g$ are different from $l+h$, $m+h$, $n+h$. Changing now the notation, let l_1, m_1, n_1 be three odd marks whose sum is even, and let $0, g_2, g_3 \dots g_n$, $(n=N_{p-1})$ be the groups to which they are common. Also let $g_r = l_1 + l_r = m_1 + m_r = n_1 + n_r$; then all the marks l, m, n are different. Moreover the three odd marks l_r, m_r, n_r are common to the N_{p-1} groups $l_r + l_s$, $= m_r + m_s$, $= n_r + n_s$. These three marks are identical, because

$$l_1 + l_r = m_1 + m_r = n_1 + n_r,$$

and $l_1 + l_s = m_1 + m_s = n_1 + n_s$. Hence $l_r + m_r + n_r$ is even.

All the groups to which three odd marks are common shall be called a *set*. Then we have N_{p-1} sets 1, 2, 3 ..., all containing the group 0, and any two of them r, s having also in common the group $(rs) = l_r + l_s =$ etc.

Consider now the case of three odd marks l, m, n, whose sum is odd, and which are therefore common to $4N_{p-2}$ groups. The marks common to 0, $l+m$, $l+n$ arrange themselves in sets of fours, $pqrs$, so that

$$l+m=p+q=r+s, \quad l+n=p+r=q+s.$$

Hence the groups containing l, m, n arrange themselves in sets of fours,

$$\begin{array}{l|l|l|l} f=l+p & g=l+q & h=l+r & k=l+s \\ \quad =m+q & \quad =m+p & \quad =m+s & \quad =m+r \\ \quad =n+r & \quad =n+s & \quad =n+p & \quad =n+q \\ \quad =0+s & \quad =0+r & \quad =0+q & \quad =0+p, \end{array}$$

so that

$$l+m=f+g=h+k, \quad l+n=f+h=g+k, \quad m+n=g+h=f+k.$$

Let $l_1+m_1+n_1+p_1=0$, so that we have to deal with four odd marks whose sum is zero; these will be common to N_{p-2} tetrads of groups $f_\iota, g_\iota, h_\iota, k_\iota$; and if

$$f_\iota = l_1 + l_\iota = m_1 + m_\iota = n_1 + n_\iota = p_1 + p_\iota,$$

we shall have N_{p-2} tetrads l, m, n, p. The order of any tetrad must only be altered by a bifid substitution; such tetrad has therefore four forms; by adding two tetrads of odd marks in various forms we obtain always the same tetrad of groups f, g, h, k. Any such tetrad of odd marks (i) is common to the N_{p-2} tetrads of groups (i, j).

We pass now to the consideration of three groups g, h, k which are not such that $g+h+k=0$, but which are such that every two of them are *contiguous*, i.e. they have N_{p-1} marks in common; and we propose to determine the number of marks common to all three groups. Let a be a mark common to g and h but not to k, so that $g=a+\alpha_1$, $h=a+\alpha_2$; then α_1 and α_2 both occur in k, but not paired, because $\alpha_1+\alpha_2=g+h$, which is not k by hypothesis. Hence $g+h$ and k are non-contiguous groups, since two marks which are paired in $g+h$ occur in k. If $k=\alpha_1+\beta_1=\alpha_2+\beta_2$, then $a+k=g+\beta_1=h+\beta_2$, and therefore $g+\beta_2=h+\beta_1=b$, suppose. Thus $g=b+\beta_2$, $h=b+\beta_1$, so that the six marks $a, b, \alpha_1, \beta_1, \alpha_2, \beta_2$ are distributed as follows:

$$\begin{aligned}
g &= a+\alpha_1 & h &= a+\alpha_2 & k &= \alpha_1+\beta_1 & g+h &= \alpha_1+\alpha_2 \\
&= b+\beta_2 & &= b+\beta_1 & &= \alpha_2+\beta_2 & &= \beta_1+\beta_2.
\end{aligned}$$

Since $g+h$ and k are non-contiguous, there are N_{p-2} such sets of six marks. If c be a mark common to g and h, not belonging to any of these sets, it must also belong to k; and consequently we must have $g=c+\gamma_1$, $h=c+\gamma_2$, $k=c+\gamma_3$. Of such marks c there are $N_{p-1}-2N_{p-2}=N_{p-2}+E_{p-2}=2^{2p-4}$. This is then the number of marks common to three such groups.

Hence four odd marks l, m, n, r, such that $l+m$, $l+n$, $l+r$ are contiguous two and two but their sum not zero, are common to 2^{2p-4} groups; and we have a theory of 2^{2p-4} such sets of four odd marks, any two sets being common to one group besides the group 0.

Let λ be another mark common to the groups $l+m$, $l+n$, $l+r$, and let $l+m=\lambda+\mu$, $l+n=\lambda+\nu$, $l+r=\lambda+\rho$; then λ, μ, ν, ρ form another such set. If g is a group containing l, m, n, r, then $g+l$ belongs to the groups $l+m$, $l+n$, $l+r$, that is, to the groups $\lambda+\mu$, $\lambda+\nu$, $\lambda+\rho$. To find the groups g, then, we must take the marks p belonging to the groups $l+m$, $l+n$, $l+r$ and add them to l. Similarly, to find the groups containing λ, μ, ν, ρ, we must add the same marks p to λ. Suppose two of them p, q are such that $p+l=q+\lambda$, then this group $p+l$ contains all the eight marks $l, m, n, r, \lambda, \mu, \nu, \rho$. So also will the group $p+\lambda$, $=q+l$. For this, we must have $p+q=l+\lambda$; we have therefore to find the number of marks common to the four groups $l+m$, $l+n$, $l+r$, $l+\lambda$. The last is non-contiguous to the first three; we know that

$$l+\lambda=m+\mu=n+\nu=r+\rho.$$

Let $g+h+k=0$, and $g=a+\alpha=b+\beta=c+\gamma$,

$$h=a+\beta=b+\alpha,\quad k=a+b=\alpha+\beta;$$

then if $a+b+c$ is odd, c occurs in the group k; it therefore occurs among the $4N_{p-2}$ marks common to g and k. Thus the marks in g are such that the sum of two from the same tetrad, not paired, and one belonging to no tetrad is always even.

Hence if we take two unpaired marks a, b, and a third c not belonging to the group $a+b$, then a will not belong to the group $b+c$, etc. How many are common to the groups g, $a+b$, $a+c$? If d is common to them, we have

$$g=d+\delta=a+\alpha=b+\beta=c+\gamma,$$

$$d+\epsilon=a+b=\alpha+\beta,\quad d+\theta=a+c=\alpha+\gamma.$$

Common to g, $a+b$ are $4N_{p-2}$ including $ab\alpha\beta$,

" g, $a+c$ " $4N_{p-2}$ " $ac\alpha\gamma$.

Suppose that

$$g=a+\alpha=b+\beta,\ =c+\gamma=d+\delta,\ =e+\epsilon=f+\phi,$$

$$h=a+b=\alpha+\beta,\ =c+d=\gamma+\delta,\ =e+f=\epsilon+\phi,$$

$$k=a+\beta=\alpha+b,\ =c+\delta=\gamma+d,\ =e+\phi=\epsilon+f,$$

a mark may be divided in 2^{2p-1} ways into two marks, namely N_{p-1} ways into two odd marks

$$2^{p-2}(2^{p-1}-1)=2^{2p-3}-2^{p-2};$$

E_{p-1}, $=2^{2p-3}+2^{p-2}$, ways into two even marks; and consequently 2^{2p-2} ways into one odd and one even mark. Thus every mark g can be divided in 2^{2p-2} ways into h and k so that their groups shall have N_{p-2} marks common.

This is the same number as if we were permitted to select the marks to be made common in any way we liked from $2p-2$ of the pairs in g.

Thus, $p=3$, we may divide every mark in 32 ways, and therefore in 16 ways, so that three groups shall have six marks common.

Every [triple] θ-function has 28 zero-values; every value annuls 28 θ-functions.

Every two θ-functions have 12 zero-values in common; every two values annul 12 θ-functions.

The three functions $0, g, h$ have 6 zero-values in common, if

$$gg'+hh'+(g+g')(h+h')\equiv 1,$$

or if $gh'+g'h\equiv 1$ (mod. 2), in this case also the values $0, g, h$ annul 6 θ-functions.

This is true of the functions g, h, k, if

$$(g+h)(g'+k')+(g'+h')(g+k)$$

or
$$hk'+h'k+kg'+k'g+gh'+g'h\equiv 1,$$

or
$$(g+h+k)(g'+h'+k')-gg'-hh'-kk'\equiv 1.$$

If however this quantity is even, the three functions have only 4 zero-values in common.

Six θ-functions having 3 common zeros shall be called a *set*. If the set includes θ_0, the 3 zeros are all odd; whence it appears that their sum must be even. Hence they belong to one or more of Weber's "groups," so that no two form a "pair." Every even mark may be expressed in 56 ways as the sum of three different odd marks; thus the number of sets including θ_0

is 56×36; and generally this is the number of sets including any given mark.

The number of sets including θ_0 and θ_a is $160 = 8 . 20$. For the six pairs in the group (a) give us 20 triads of pairs, and each of these gives eight triads of odd marks constituting a set.

If $0, g, h$; $0, h, k$; $0, k, g$ have each 6 zeros in common, there are 4 zeros common to $0, g, h, k$, unless $g+h+k=0$, when there are none common (Weber).

Suppose then lmn, lnr, lrm to be zeros belonging to three sets which include θ_0. Then

$$0,\ l+m,\ l+n;\quad 0,\ l+n,\ l+r;\quad 0,\ l+r,\ l+m$$

are zeros belonging to three sets which include θ_l. Now

$$l+m+l+n+l+r = l+m+n+r;$$

consequently this must not be zero, if the three sets are to have four θs common. Of these three sets, then, any two can only have in common such θs as belong also to the third.

Let $g = l + l' = m + m' = n + n'$; then the sets lmn, $l'm'n'$ have no θ in common except θ_0 and θ_g. For two groups g, h have 4 or 6 zeros in common, the latter only when no two of them form a pair in either group. Let now h be a mark of the set lmn, so that $h = l + \alpha = m + \beta = n + \gamma$. Then either $\alpha\beta\gamma$ must be different from $l'm'n'$; or, suppose $\alpha = m'$, then $\beta = l'$, and $\gamma = l + m' + n = l' + m + n$. But $l + m' + n$ is necessarily even, so that $m'l'\gamma$ is not a set. Starting then from the set $l_1 l_2 l_3$, let the marks of it be $0, a, b, c, f, g$; then

$$a = l + m,\quad b = l + n,\quad c = l + p,\quad f = l + q,\quad g = l + r;$$

i.e. $$a = l_1 + m_1 = l_2 + m_2 = l_3 + m_3.$$

Any two of the six sets l, m, n, p, q, r have one mark in common besides 0. For (e.g.)

$$p_1 + q_1 = c + f = p_2 + q_2 = p_3 + q_3.$$

$$\begin{array}{lll} g = a + \alpha_1 & h = a + \alpha_2 & k = \alpha_1 + \beta_1 \\ \ \ = b + \beta_2 & \ \ = b + \beta_1 & \ \ = \alpha_2 + \beta_2. \end{array}$$

If g, h, k are contiguous but sum not zero, the same is true of g, $g+h$, $g+k$, etc.

$$\begin{aligned} h+k&=a+\beta_2 & k+g&=a+\beta_1 & g+h&=\alpha_1+\alpha_2 \\ &=b+\alpha_1 & &=b+\alpha_2 & &=\beta_1+\beta_2. \end{aligned}$$

$$\left.\begin{aligned} g+h+k&=\alpha_2+\beta_1 \\ &=a+b \\ &=\alpha_1+\beta_2 \end{aligned}\right\} \text{ is non-contiguous to all six.}$$

If $l+m$, $l+n$, $l+r$ are contiguous two and two but their sum not zero, then $m+n$, $m+r$ are also contiguous. Hence the sum of every three of the four marks l, m, n, r is even.

Now suppose a fourth group f, contiguous to g. If it is non-contiguous to h, then $f+g$ will be contiguous to both of them. Hence one-half of the groups contiguous to g are also contiguous to h. Suppose then that f is contiguous to g and h, then it is non-contiguous to $g+h$.

If the mark a is common to g and h but not to f,

$$g=a+\alpha_1, \quad h=a+\alpha_2,$$

then $f=\alpha_1+\gamma_1=\alpha_2+\gamma_2$, so that $g+\gamma_2=h+\gamma_1$, $=c$, suppose, where c is odd, because either α_2 or γ_2 must occur in the group g, and α_2 does not. Hence we have

$$\begin{aligned} j&=\beta_1+\gamma_2 & g&=a+\alpha_1 & h&=a+\alpha_2 & k&=\alpha_1+\beta_1 & f&=\alpha_1+\gamma_1 \\ &=\beta_2+\gamma_1 & &=b+\beta_2 & &=b+\beta_1 & &=\alpha_2+\beta_2 & &=\alpha_2+\gamma_2 \\ &=a+\alpha_3 & &=c+\gamma_2 & &=c+\gamma_1 & &=c+\gamma_3 & &=b+\beta_3. \\ & & &=d+\delta_2 & &=d+\delta_1 & &=d+\delta_3 & & \end{aligned}$$

If therefore f is contiguous to k, γ_1 and γ_2 do not occur in k, and therefore c does; also b in f. But c cannot occur in f, or b in k.

To make a system of four groups, contiguous two and two, and having common marks; take two contiguous groups

$$g, \ =a+\alpha_1=b+\beta_1=c+\gamma_1, \text{ etc.,}$$

and

$$h, \ =a+\alpha_2=b+\beta_2=c+\gamma_2, \text{ etc.}$$

To these add the group k, $=\alpha_1+\beta_2=\alpha_2+\beta_1=$ etc., which does

not contain a or b. Select two marks, c, d, common to g, h, k, then the fourth group, if any, must be $\gamma_1+\delta_2=\gamma_2+\delta_1$, $=f$, suppose.

$$\begin{array}{lllll} j=\beta_1+\gamma_2 & g=a+\alpha_1 & h=a+\alpha_2 & k=\alpha_1+\beta_2 & f=\gamma_1+\delta_2 \\ \;\;=\beta_2+\gamma_1 & \;\;=b+\beta_1 & \;\;=b+\beta_2 & \;\;=\alpha_2+\beta_1 & \;\;=\gamma_2+\delta_1 \\ \;\;=a\;+\alpha_3 & \;\;=c+\gamma_1 & \;\;=c+\gamma_2 & \;\;=c\;+\gamma_3 & \\ & \;\;=d+\delta_1 & \;\;=d+\delta_2 & \;\;=d\;+\delta_3 & \end{array}$$

$$g+h=c+d+f.$$

f must contain either a or b but not both. Suppose a, then it contains β_1 and β_2.

f is non-contiguous to $h+k=\gamma_2+\gamma_3=\delta_2+\delta_3$.

Since f contains either a or b but not both, this case is the same as the last. f is linked with g, h by N_{p-2} cross-pairs. Out of each cross-pair of k, f must contain *one* mark but not both; thus there are N_{p-2} marks common to g and h which belong to cross-pairs both of k and f, and $2N_{p-2}$ belonging to cross-pairs of k and f separately. Hence the number of marks common to g, h, k, f is $2^{2p-4}-N_{p-2}$.

$$\begin{aligned} j&=\beta_2+\gamma_1=\beta_1+\gamma_2 \\ &=a\;+\alpha_3 \end{aligned}$$

must not contain α_1, α_2, c, b, but must contain a.

$$f+g+h+j+k=0.$$

*XL.

ON THE DOUBLE THETA-FUNCTIONS.

THE θ-series of two variables u_1, u_2 is defined as follows:

$$\theta(u_1, u_2;\ a_{11}, a_{12}, a_{22}) = \sum_{n_1}\sum_{n_2} \epsilon\,|\,n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1 u_1 + 2n_2 u_2\,|\text{*},$$

where the whole numbers n_1, n_2, in respect of which the summation takes place, may have all values from $-\infty$ to $+\infty$. The parameters a_{11}, a_{12}, a_{22} must be such that the real part of $n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22}$ is negative for all values of n_1, n_2. This is necessary and sufficient for the convergency of the series.

The function θ is unaltered if either of the arguments be increased by any multiple of πi; this follows from the fact that $\epsilon\,|\,2\pi i\,| = 1$. This is expressed by saying that u_1, u_2 have independently the period πi; or that taken together they have the two periods πi, 0 and 0, πi.

As the number n_1 may have all integer values, the function will not be altered if we write $n_1 + q_1$ for it (q_1 being any integer) and then give n_1 all integer values. Thus we find

$$\begin{aligned}\theta(u_1, u_2) &= \sum_{n_1}\sum_{n_2} \epsilon\,|\,(n_1+q_1)^2 a_{11} + 2(n_1+q_1) n_2 a_{12} + n_2^2 a_{22} + 2(n_1+q_1) u_1 + 2n_2 u_2\,| \\ &= \epsilon\,|\,q_1^2 a_{11} + 2q_1 u_1\,| \,.\, \sum_{n_1}\sum_{n_2} \epsilon\,|\,n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1(u_1 + q_1 a_{11}) + 2n_2(u_2 + q_1 a_{12})\,| \\ &= \epsilon\,|\,q_1^2 a_{11} + 2q_1 u_1\,|\,\theta(u_1 + q_1 a_{11},\ u_2 + q_2 a_{12});\end{aligned}$$

* [Or as it might be written

$$\Sigma_{n1}\,\Sigma_{n2}\ \text{exp.}\ \{(a_{11}, a_{12}, a_{22} \between n_1, n_2)^2 + 2n_1 u_1 + 2n_2 u_2\}.]$$

or, which is the same thing,

$$\theta(u_1+q_1a_{11},\ u_2+q_1a_{12})=\epsilon\,|-q_1^2a_{11}-2q_1u_1|\,\theta(u_1,\ u_2).$$

In the same manner we may shew that

$$\theta(u_1+q_2a_{12},\ u_2+q_2a_{22})=\epsilon\,|-q_2^2a_{22}-2q_2u_2|\,\theta(u_1,\ u_2).$$

From these equations it appears that when the arguments u_1, u_2 are simultaneously increased by the parameters a_{11}, a_{12} respectively, the function θ becomes multiplied by the exponential factor $\epsilon\,|-a_{11}-2u_1|$; and that when they are simultaneously increased by a_{12}, a_{22} respectively, the θ is multiplied by $\epsilon\,|-a_{22}-2u_2|$. On this account the arguments u_1, u_2 are said to have the *quasi-periods* a_{11}, a_{12} and a_{12}, a_{22}.

The effect of adding a *half*-period to either argument is to change the sign of certain terms in the series. Thus if we write $u_1+\frac{1}{2}\pi i$ for u, each term becomes affected with the factor $\epsilon\,|n_1\pi i|$, that is $(-)^{n_1}$. The series thus arising are conveniently distinguished by the numbers 0 or 1 affixed to the θ, as follows: $\phi(n_1,\ n_2)$ being for shortness written instead of

$$n_1^2a_{11}+2n_1n_2a_{12}+n_2^2a_{22}+2n_1u_1+2n_2u_2,$$

$$\theta(u_1+\tfrac{1}{2}\pi i,\ u_2)=\Sigma\Sigma\,(-)^{n_1}\,\epsilon\,|\phi(n_1,\ n_2)|=\theta^{10}(u_1,\ u_2),$$

$$\theta(u_1,\ u_2+\tfrac{1}{2}\pi i)=\Sigma\Sigma\,(-)^{n_2}\,\epsilon\,|\phi(n_1,\ n_2)|=\theta^{01}(u_1,\ u_2),$$

$$\theta(u_1+\tfrac{1}{2}\pi i,\ u_2+\tfrac{1}{2}\pi i)=\Sigma\Sigma\,(-)^{n_1+n_2}\epsilon\,|\phi(n_1,\ n_2)=\theta^{11}(u_1,\ u_2).$$

We may include these in the formula

$$\begin{aligned}\theta^{\alpha\beta}(u_1+\tfrac{1}{2}\gamma\pi i,\ u_2+\tfrac{1}{2}\delta\pi i)&=\Sigma\Sigma\,(-)^{(\alpha+\gamma)n_1+(\beta+\delta)n_2}\,|\phi(n_1,\ n_2)|\\&=\theta^{\alpha+\gamma,\ \beta+\delta}(u_1,\ u_2),\end{aligned}$$

where, in $\theta^{\alpha\beta}$, we must suppose 0 or 1 substituted for each of the whole numbers α, β according as it is even or odd.

The effect of adding a half quasi-period to the arguments is to add $\frac{1}{2}$ to one of the numbers n_1, n_2 and multiply the θ-function by an exponential factor. The new series thus arising are conveniently distinguished by 0 or 1 *suffixed* to the θ, as follows:

$$\begin{aligned}\theta(u_1+\tfrac{1}{2}a_{11},\ u_2+\tfrac{1}{2}a_{12})&=\epsilon\,|-\tfrac{1}{4}a_{11}-u_1|\,.\,\Sigma\Sigma\epsilon\,|\phi(n_1+\tfrac{1}{2},\ n_2)|\\&=\epsilon\,|-\tfrac{1}{4}a_{11}-u_1|\,.\,\theta_{10}(u_1,\ u_2),\end{aligned}$$

$$\theta\,(u_1+\tfrac{1}{2}a_{12},\ u_2+\tfrac{1}{2}a_{22}) = \epsilon\,|-\tfrac{1}{4}a_{22}-u_2\,|\,.\,\Sigma\Sigma\epsilon\,|\,\phi\,(n_1,\ n_2+\tfrac{1}{2})\,|$$
$$= \epsilon\,|-\tfrac{1}{4}a_{11}-u_1\,|\,.\,\theta_{01}\,(u_1,\ u_2),$$

$$\theta\,(u_1+\tfrac{1}{2}a_{11}+\tfrac{1}{2}a_{12},\ u_2+\tfrac{1}{2}a_{12}+\tfrac{1}{2}a_{22})$$
$$= \epsilon\,|-\tfrac{1}{4}a_{11}-\tfrac{1}{2}a_{12}-\tfrac{1}{4}a_{22}-u_1-u_2\,|\,.\,\Sigma\Sigma\epsilon\,|\,\phi\,(n_1+\tfrac{1}{2},\ n_2+\tfrac{1}{2})\,|$$
$$= \epsilon\,|-\tfrac{1}{4}a_{11}-\tfrac{1}{2}a_{12}-\tfrac{1}{4}a_{22}-u_1-u_2\,|\,.\,\theta_{11}\,(u_1,\ u_2).$$

These cases are included in the formula

$$\theta_{\alpha\beta}\,(u_1+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22})$$
$$= \epsilon\,|-\phi\,(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta)\,|\,.\,\theta_{\alpha+\gamma,\ \beta+\delta}\,(u_1,\ u_2),$$

where, in the suffixes, we must suppose 0 or 1 written for each number according as it is even or odd.

In regard to the periods and quasi-periods of these θ, it is to be remarked that θ^{10}, θ^{01} and θ^{11} have each the period πi for each argument, like θ; but that besides acquiring the exponential factor, θ^{10} changes sign for addition of the quasi-period a_{11}, a_{12}, θ^{01} for addition of a_{12}, a_{22}, and θ^{11} for addition of either. On the other hand, θ_{10}, θ_{01}, and θ_{11} resemble θ in regard to the quasi-periods, but change sign when u_1, u_2, or either of them is increased by πi respectively. A general formula for these cases will be given presently.

If we add both half-periods and half-quasi-periods to the arguments, we obtain nine new θ-series, which together with the preceding are all defined in the following formula:

$$\theta\,(u_1+\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22})$$
$$= \epsilon\,|-\phi\,(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta)\,|\,\Sigma\Sigma\,(-)^{\alpha n_1+\beta n_2}\,\epsilon\,|\,\phi\,(n_1+\tfrac{1}{2}\gamma,\ n_2+\tfrac{1}{2}\delta)\,|$$
$$= \epsilon\,|-\phi\,(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta)\,|\,\theta^{\alpha\beta}_{\gamma\delta}\,(u_1,\ u_2).$$

Here each of the letters α, β, γ, δ stands for either 0 or 1.

This formula may be generalized into the following, which contains the entire theory of the transformation of the θ into one another by addition of half-periods and half-quasi-periods to the arguments.

$$\theta^{ab}_{cd}\,(u_1+\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22})$$
$$= \epsilon\,|\,\tfrac{1}{2}(\alpha c+\beta d)\,\pi i-\phi\,(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta)\,|\,\theta^{a+\alpha,\ b+\beta}_{c+\gamma,\ d+\delta}\,(u_1,\ u_2).$$

Here α, β, γ, δ may be any whole numbers, and the affixes and suffixes are to be understood as before.

Of the sixteen functions θ^{ab}_{cd}, six are odd functions of u_1, u_2, namely those for which $ac+bd$ is an odd number; and these of course vanish for the values $u_1=0$, $u_2=0$. The other ten are even functions, and it is easy by means of the preceding formulæ to assign for each the six pairs of values which make it zero. Namely

$$\theta^{ab}_{cd}\left(\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ \tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22}\right)=0,$$

whenever $\qquad (a+\alpha)(c+\gamma)+(b+\beta)(d+\delta)$

is an odd number.

The Product-Theorem.

If we multiply together term by term two θ-series with different arguments, we shall obtain a quadruply infinite series, the exponent in each term of it being the sum of the exponents in those terms of the θ-series from which it is derived. Thus we shall have

$$\begin{aligned}\theta(u_1, u_2)\,\theta(v_1, v_2) &= \Sigma\Sigma\epsilon\,|\,\phi(m_1, m_2)\,|\,.\,\Sigma\Sigma\epsilon\,|\,\psi(n_1, n_2)\,|\\ &= \Sigma\Sigma\Sigma\Sigma\epsilon\,|\,\phi(m_1, m_2)+\psi(n_1, n_2)\,|,\end{aligned}$$

where

$$\begin{aligned}\phi(m_1, m_2) &= m_1^2a_{11}+2m_1m_2a_{12}+m_2^2a_{22}+2m_1u_1+2m_2u_2,\\ \psi(n_1, n_2) &= n_1^2a_{11}+2n_1n_2a_{12}+n_2^2a_{22}+2n_1v_1+2n_2v_2.\end{aligned}$$

Now if we write

$$\begin{aligned}m_1+n_1=p_1,\ \ m_1-n_1=q_1,\\ m_2+n_2=p_2,\ \ m_2-n_2=q_2,\end{aligned}$$

we shall have

$$\begin{gathered}2(m_1^2+n_1^2)=p_1^2+q_1^2,\ \ 2(m_2^2+n_2^2)=p_2^2+q_2^2,\\ 2(m_1m_2+n_1n_2)=p_1p_2+q_1q_2,\end{gathered}$$

and therefore

$$\begin{aligned}&\phi(m_1, m_2)+\psi(n_1, n_2)\\ &\quad=\tfrac{1}{2}(p_1^2+q_1^2)\,a_{11}+(p_1p_2+q_1q_2)\,a_{12}+\tfrac{1}{2}(p_2^2+q_2^2)\,a_{22}\\ &\qquad+p_1(u_1+v_1)+q_1(u_1-v_1)+p_2(u_2+v_2)+q_2(u_2-v_2)\\ &\quad=\tfrac{1}{2}p_1^2a_{11}+p_1p_2a_{12}+\tfrac{1}{2}p_2^2a_{22}+p_1(u_1+v_1)+p_2(u_2+v_2)\\ &\qquad+\tfrac{1}{2}q_1^2a_{11}+q_1q_2a_{12}+\tfrac{1}{2}q_2^2a_{22}+q_1(u_1-v_1)+q_2(u_2-v_2).\end{aligned}$$

Now the p and q are not all numbers indiscriminately, but p_1 and q_1 must be both odd or both even, being the sum and difference of two integers, and so also p_2 and q_2 must be both odd or both even. In the quadruply infinite series, therefore, there are four kinds of terms; those in which the p, q are all odd, in which they are all even, in which p_1, q_1 are odd and p_2, q_2 even, and in which p_1, q_1 are even and p_2, q_2 odd. We shall sum these separately.

First let all four numbers be even; and let

$$p_1 = 2s_1, \quad p_2 = 2s_2, \quad q_1 = 2t_1, \quad q_2 = 2t_2,$$

then

$$\begin{aligned}\phi(m_1, m_2) + \psi(n_1, n_2) &= 2s_1^2a_{11} + 4s_1s_2a_{12} + 2s_2^2a_{22} + 2s_1(u_1+v_1) + 2s_2(u_2+v_2)\\ &+ 2t_1^2a_{11} + 4t_1t_2a_{12} + 2t_2^2a_{22} + 2t_1(u_1-v_1) + 2t_2(u_2-v_2).\end{aligned}$$

If we sum the exponential of this quantity for all integer values of s and t, we shall clearly obtain

$$\theta(u_1+v_1, u_2+v_2; 2a_{11}, 2a_{12}, 2a_{22}) \,.\, \theta(u_1-v_1, u_2-v_2; 2a_{11}, 2a_{12}, 2a_{22});$$

that is, the product of two θ-series whose parameters are the doubles of those we have been considering. To avoid the trouble of expressing these parameters in every case, it is convenient to denote such θ-series by a capital Θ; thus

$$\Theta(u_1, u_2) = \theta(u_1, u_2; 2a_{11}, 2a_{12}, 2a_{22}).$$

Next let p_1, q_1 be odd, while p_2, q_2 are even; and let

$$p_1 = 2s_1+1, \; p_2 = 2s_2, \; q_1 = 2t_1+1, \; q_2 = 2t_2,$$

then

$$\begin{aligned}\phi(m_1, m_2) + \psi(n_1, n_2) &= 2(s_1+\tfrac{1}{2})^2a_{11} + 4(s_1+\tfrac{1}{2})s_2 \,.\, a_{12} + 2s_2^2a_{22} + 2(s_1+\tfrac{1}{2})(u_1+v_1)\\ &\qquad + 2s_2(u_2+v_2)\\ &+ 2(t_1+\tfrac{1}{2})^2a_{11} + 4(t_1+\tfrac{1}{2})t_2 \,.\, a_{12} + 2t_2^2a_{22} + 2(t_1+\tfrac{1}{2})(u_1-v_1)\\ &\qquad + 2t_2(u_2-v_2),\end{aligned}$$

and the sum of the exponentials of this quantity for all integer values of s and t is clearly

$$\Theta_{10}(u_1+v_1, u_2+v_2)\,\Theta_{10}(u_1-v_1, u_2-v_2).$$

Similarly if p_1, q_1 be even and p_2, q_2 odd, we shall obtain

$$\Theta_{01}(u_1+v_1,\ u_2+v_2)\ \Theta_{01}(u_1-v_1,\ u_2-v_2),$$

and for all four numbers odd,

$$\Theta_{11}(u_1+v_1,\ u_2+v_2)\ \Theta_{11}(u_1-v_1,\ u_2-v_2).$$

The product $\theta(u_1, u_2)$, $\theta(v_1, v_2)$ is thus equal to the sum of these four products. To state the proposition with brevity and clearness, we shall mention only one of the two variables in each case, omitting the suffix; thus $\theta(u)$ will stand for $\theta(u_1, u_2)$, and $\Theta(u+v)$ for $\Theta(u_1+v_1,\ u_2+v_2)$. What we have proved, then, is that

$$\begin{aligned}\theta u.\theta v = &\ \ \Theta\ (u+v)\ \Theta\ (u-v)\\ &+\Theta_{10}(u+v)\ \Theta_{10}(u-v)\\ &+\Theta_{01}(u+v)\ \Theta_{01}(u-v)\\ &+\Theta_{11}(u+v)\ \Theta_{11}(u-v)\\ =&\ \Sigma\Theta_{ab}(u+v)\ \Theta_{ab}(u-v). \qquad (a,\ b=0,\ 1).\end{aligned}$$

From this formula we may, by adding half-periods and half-quasi-periods to the arguments u, v, obtain an expression for any such product as $\theta^{ab}_{cd}\,u\,.\,\theta^{\alpha\beta}_{\gamma\delta}\,v$. The number of such distinct products is 136, including the theorem just stated; the general formula including them all will be subsequently examined. But the case in which α, β, γ, $\delta = a$, b, c, d admits of simple treatment and leads to some important consequences.

First, let us adopt the abbreviation

$\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)$ for

$$\theta\,(u_1+\tfrac{1}{2}a\pi i+\tfrac{1}{2}ca_{11}+\tfrac{1}{2}da_{12},\ u_2+\tfrac{1}{2}b\pi i+\tfrac{1}{2}ca_{12}+\tfrac{1}{2}da_{22}),$$

then we may write

$$\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\ \tfrac{1}{2}d)\,|\ \theta^{ab}_{cd}(u),$$

$$\theta\left(v+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\psi\,(\tfrac{1}{2}c,\ \tfrac{1}{2}d)\,|\ \theta^{ab}_{cd}(v),$$

$$\Theta\left(u+v+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\ \tfrac{1}{2}\delta)\,|\ \Theta^{ab}_{cd}(u+v),$$

where

$$\begin{aligned}
\phi(\tfrac{1}{2}c, \tfrac{1}{2}d) &= \tfrac{1}{4}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + cu_1 + du_2, \\
\psi(\tfrac{1}{2}c, \tfrac{1}{2}d) &= \tfrac{1}{4}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + cv_1 + dv_2, \\
\Phi(\tfrac{1}{2}c, \tfrac{1}{2}d) &= \tfrac{1}{2}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + c(u_1 + v_1) + d(u_2 + v_2) \\
&= \phi(\tfrac{1}{2}c, \tfrac{1}{2}d) + \psi(\tfrac{1}{2}c, \tfrac{1}{2}d).
\end{aligned}$$

Hence by substitution we obtain*

$$\begin{aligned}
&\epsilon\,|-\phi(\tfrac{1}{2}c, \tfrac{1}{2}d) - \psi(\tfrac{1}{2}c, \tfrac{1}{2}d)\,|\,\theta^{ab}_{cd}(u)\,\theta^{ab}_{cd}(v) \\
&\qquad = \Sigma\Theta_{pq}\left(u + v + \begin{matrix}2a, & 2b \\ c & d\end{matrix}\right)\Theta_{pq}(u - v) \\
&\qquad = \Sigma\epsilon\,|\,(ap + bq)\pi i - \Phi(\tfrac{1}{2}c, \tfrac{1}{2}d)\,|\,\Theta_{p+c,\,q+d}(u+v)\,\Theta_{pq}(u-v),
\end{aligned}$$

and consequently

$$\theta^{ab}_{cd}(u)\,\theta^{ab}_{cd}(v) = \Sigma\,(-)^{ap+bq}\,\Theta_{p+c,\,q+d}(u+v)\,\Theta_{pq}(u-v) \qquad (p, q = 0, 1).$$

In this equation write u for v; thus

$$\{\theta^{ab}_{cd}(u)\}^2 = \Sigma\,(-)^{ap+bq}\,\Theta_{p+c,\,q+d}(2u)\,\Theta_{pq}(0).$$

It appears from this result that the square of each of the θ-functions is a linear function of the four quantities

$$\Theta_{00}(2u),\ \Theta_{10}(2u),\ \Theta_{01}(2u),\ \Theta_{11}(2u);$$

the coefficients being the quantities

$$\Theta_{00}(0),\ \Theta_{10}(0),\ \Theta_{01}(0),\ \Theta_{11}(0).$$

It follows therefore that *a linear relation may be found between the squares of any five of the θ-functions.*

In certain cases, however, it may be shewn that such a relation holds between the squares of four of them. Just as each θ vanishes for each of six pairs of values of the arguments, so each such pair of values annuls six θ-functions. Consider four of these six having the same zero, and one other θ which is not annulled by that pair of values. By what we have just proved, there is a linear relation between the squares of these five; but this is impossible unless the coefficient of the fifth be zero, because we can give to the arguments such values that

* On the right-hand side it is to be observed that c, d are written instead of $2c$, $2d$, because of the double parameters of Θ.

the first four functions shall vanish, but not the fifth. It follows therefore that

If four θ-functions vanish for the same pair of values of the variables, their squares are connected by a linear relation.

For example, the four functions

$$\theta(u),\ \theta^{11}(u),\ \theta^{10}_{10}(u),\ \theta^{10}_{01}(u)$$

are all reduced to zero by the values $u = \dfrac{0,\,1}{1,\,1}$, that is to say, $u_1 = \frac{1}{2}a_{11} + \frac{1}{2}a_{12}$, $u_2 = \frac{1}{2}i\pi + \frac{1}{2}a_{12} + \frac{1}{2}a_{22}$. To find the coefficients in the linear relation, let us assume

$$x\theta(u)^2 + y\theta^{11}(u)^2 + z\theta^{10}_{10}(u)^2 + w\theta^{10}_{01}(u)^2 = 0;$$

then we have only to observe that every two of these functions have one other zero in common, so that by giving to the u this pair of values we can find the ratio of two of the coefficients. Thus for

$$u = \frac{10}{11},\ \text{we have} -z\theta_{01}(0)^2 + w\theta_{10}(0)^2 = 0,$$

$$u = \frac{11}{10},\ \quad ,, \quad ,, \quad y\theta_{10}(0)^2 - z\theta^{01}(0)^2 = 0,$$

$$u = \frac{01}{00},\ \quad ,, \quad ,, \quad x\theta^{01}(0)^2 + y\theta^{10}(0)^2 = 0,$$

by successive applications of the formula

$$\theta^{ab}_{cd}\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix} = \epsilon\,|\,\tfrac{1}{2}(\alpha c + \beta d)\pi i - \phi(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta)\,|\,\theta^{a+\alpha,\ b+\beta}_{c+\gamma,\ d+\delta}(0).$$

The result is that

$$-x : y : z : w = \theta^{10}(0)^2 : \theta^{01}(0)^2 : \theta_{10}(0)^2 : \theta_{01}(0)^2,$$

so that the sought relation is

$$-\theta^{10}(0)^2.\theta(u)^2 + \theta^{01}(0)^2.\theta^{11}(u)^2 + \theta_{10}(0)^2.\theta^{10}_{10}(u)^2 + \theta_{01}(0)^2.\theta^{10}_{01}(u)^2 = 0.$$

With a view to further investigation of the group of six functions θ^{11}, θ^{10}_{10}, θ^{10}_{01}, θ^{01}_{10}, θ^{01}_{01}, θ_{11}, whose characteristics are the sums in pairs of the four characteristics $\dfrac{10}{00}, \dfrac{01}{00}, \dfrac{00}{10}, \dfrac{00}{01}$, it will be useful to set down here three other relations which

connect triads of their squares with $\theta(u)^2$. They are obtained in precisely the same manner as the one already written down.

$$-\theta^{10}(0)^2.\theta^{11}(u)^2+\theta^{01}(0)^2.\theta(u)^2-\theta_{10}(0)^2.\theta^{01}_{10}(u)^2-\theta_{01}(0)^2.\theta^{01}_{01}(u)^2=0,$$
$$-\theta^{10}(0)^2.\theta^{10}_{10}(u)^2+\theta^{01}(0)^2.\theta^{01}_{10}(u)^2-\theta_{10}(0)^2.\theta(u)^2+\theta_{01}(0)^2.\theta_{11}(u)^2=0,$$
$$-\theta^{10}(0)^2.\theta^{10}_{01}(u)^2+\theta^{01}(0)^2.\theta^{10}_{01}(u)^2-\theta_{10}(0)^2.\theta_{11}(u)^2+\theta_{01}(0)^2.\theta(u)^2=0.$$

These four equations, it will be observed, are not independent; for if they be multiplied respectively by $\theta^{10}(0)^2$, $\theta^{01}(0)^2$, $\theta_{10}(0)^2$ and $\theta_{01}(0)^2$, the result is $\theta(u)^2$ multiplied by

$$-\theta^{10}(0)^4+\theta^{01}(0)^4-\theta_{10}(0)^4+\theta_{01}(0)^4,$$

which vanishes, as may be proved by writing $u=\begin{matrix}10\\00\end{matrix}$ in the first equation.

Eliminating the constant multipliers, we obtain

$$\begin{vmatrix} -\theta^2, & +\theta^{11^2}, & +\theta^{10^2}_{10}, & +\theta^{10^2}_{01} \\ -\theta^{11^2}, & +\theta^2, & -\theta^{01^2}_{10}, & -\theta^{01^2}_{01} \\ -\theta^{10^2}_{10}, & +\theta^{01^2}_{10}, & -\theta^2, & +\theta_{11}{}^2 \\ -\theta^{10^2}_{01}, & +\theta^{01^2}_{01}, & -\theta_{11}{}^2, & +\theta^2 \end{vmatrix}.$$

[Pages of MS. up to this point are numbered 1 to 15 in Clifford's own handwriting: then comes *the Fluxion-Theorem* not numbered, and some more pages "too incomplete for printing. C."]

The Fluxion-Theorem.

A θ-series may be differentiated in regard to either of the variables, or, as GÖPEL suggested, the question may be left open, if we write $\partial = x\partial_{u_1}+y\partial_{u_2}$, where x and y are quantities to be determined at the end of the investigation. This being so, we find

$$\theta(u).\partial\theta(v)-\theta(v).\partial\theta(u)=2\Sigma\Theta_{pq}(u+v).\partial\Theta_{pq}(u-v)$$
$$(p,q=0,1),$$

by the same process as that which was used for the product-theorem. It is necessary for our subsequent investigation to

extend this theorem so that for $\theta(u)$ we may write $\theta^{ab}_{cd}(u)$. To this end we proceed as follows. We have

$$\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\theta^{ab}_{cd}(u),$$

therefore

$$\partial\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\partial\theta^{ab}_{cd}(u)-\theta^{ab}_{cd}(u)\,\partial\phi\}.$$

Again,

$$\Theta_{pq}\left(u\pm v+\begin{matrix}a & b\\ \tfrac{1}{2}c, & \tfrac{1}{2}d\end{matrix}\right)=\epsilon\,|\,\tfrac{1}{2}(ap+bq)\,\pi i\,|\,\Theta^{ab}_{pq}\,(u\pm v+\tfrac{1}{2}c,\,\tfrac{1}{2}d),$$

therefore

$$\partial\Theta_{pq}\left(u-v+\begin{matrix}a & b\\ \tfrac{1}{2}c, & \tfrac{1}{2}d\end{matrix}\right)=\epsilon\,|\,\tfrac{1}{2}(ap+bq)\,\pi i\,|\,.\,\partial\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

By substitution of these values, the fluxion-theorem becomes when we write in it $u+\begin{matrix}ab\\cd\end{matrix}$ for u,

$$\epsilon\,|-\phi(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\theta^{ab}_{cd}(u)\,.\,\partial\theta(v)-\theta(v)\,\partial\theta^{ab}_{cd}(u)-\theta^{ab}_{cd}(u)\,\theta(v)\,\partial\phi\}$$
$$=2\Sigma\,(-)^{ap+bq}\,\Theta^{ab}_{pq}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)$$
$$(p,\,q=0,\,1).$$

But by the product-theorem we have also

$$\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\theta^{ab}_{cd}(u)\,\theta(v)$$
$$=\Sigma\,(-)^{ap+bq}\,\Theta^{ab}_{pq}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

Moreover

$$2\partial\,.\,\epsilon^{\frac{1}{2}\phi}\,\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)$$
$$=\epsilon^{\frac{1}{2}\phi}\,\{2\partial\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)+\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\phi\},$$

therefore finally

$$\epsilon\,|-\tfrac{1}{2}\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\theta^{ab}_{cd}u\,.\,\partial\theta v-\theta v\,.\,\partial\theta^{ab}_{cd}u\}$$
$$=2\Sigma\,(-)^{ap+bq}\,\Theta^{ab}_{pq}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\,.\,\epsilon^{\frac{1}{2}\phi}\,\Theta^{ab}_{pq}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

*XLI.

MOTION OF A SOLID IN ELLIPTIC SPACE*.

I.

THE *second moment* of a solid body in regard to a plane is the sum obtained by multiplying the mass of each particle into the squared sine of its distance from the plane and adding together the products thus formed. The body being referred to a quadrantal tetrahedron, let the equation of the plane be

$$\Sigma \xi x \equiv \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0,$$

then the coordinates of any particle p of mass P being $p_1 p_2 p_3 p_4$, the second moment of the plane is

$$\int P (\Sigma \xi p)^2 = k^2, \text{ suppose,}$$

it being assumed that $\Sigma p^2 = 1$, $\Sigma \xi^2 = 1$, always. Moreover, as only one body will be here considered, its mass may conveniently be taken as the unit of mass; so that the quantity k is to be called the swing-radius of the body in regard to the plane ξ.

We may write also

$$\begin{aligned} k^2 = (1, 1)\, \xi_1^2 + (2, 2)\, \xi_2^2 + \ldots \\ + 2\,(1, 2)\, \xi_1 \xi_2 + \ldots \end{aligned}$$

where $\quad (1, 1) = \int P p_1^2$, etc.; $(1, 2) = \int P p_1 p_2$, etc.

* [At the Meeting of the *London Mathematical Society* held February 12, 1874, Prof. Clifford gave an account of a paper on "the Free motion of a Solid in Elliptic Space": I take XLI. to be this paper.]

The quantities (1, ·1), (1, 2) are the moments and products of inertia of the body in respect of the coordinate planes.

Those planes whose second moment vanishes envelop a surface of the second order $k^2=0$ (the null-surface), which determines all the dynamical relations of the body. The six axes of this surface (being the edges of the self-conjugate tetrahedron common to it and the absolute) are called the principal axes of the body. We may for all dynamical purposes substitute for the body a system of four particles having appropriate masses placed at the vertices of any tetrahedron self-conjugate to the null-surface.

If two planes at right angles be drawn through any axis, having k_1, k_2 for their swing-radii, then $h^2=k_1^2+k_2^2$ is a constant independent of the orientation of the two planes, and h is called the swing-radius of the body in regard to that axis. Those axes whose swing-radii vanish are therefore such that the tangent planes drawn through them to the null-quadric are at right angles; or the lines are harmonically related to the null-quadric and the absolute. Hence, if the coordinates of an axis are λ_{12}, λ_{23}, etc., we shall have

$$h^2 = \{(1, 1) + (2, 2)\}\lambda_{12}^2 + \text{etc.} + 2(12)(\lambda_{31}\lambda_{23} - \lambda_{14}\lambda_{24}) + \text{etc.},$$

which is easily verified.

It must be observed that we have

$$(1, 1) + (2, 2) + (3, 3) + (4, 4) = 1,$$

which gives three relations among the coefficients in the expression for h^2.

Suppose now that the instantaneous velocity of the body is the twist

$$V = \alpha + \omega\beta = ix + jy + kz + \omega(iu + jv + kw).$$

Here x, y, z are the component rotations about axes through the origin, and u, v, w component velocities along them. Let a, b, c be the swing-radii of the body about the axes i, j, k; then $\sqrt{1-a^2}$, $\sqrt{1-b^2}$, $\sqrt{1-c^2}$ will be the swing-radii about

ωi, ωj, ωk. Let also f^2, g^2, h^2 be the products of inertia in regard to the planes through the origin, l^2, m^2, n^2 the products of inertia of these with the polar of the origin. Supposing for the moment that $xu+yv+zw=0$, the twist becomes a mere rotation, and we can find the swing-radius of its axis. Namely, it is d, where

$$\begin{aligned}-(\alpha+\omega\beta)^2.d^2 &= a^2x^2+b^2y^2+c^2z^2+(1-a^2).u^2+(1-b^2).v^2+(1-c^2).w^2\\ &\quad +f^2(yz-vw)+g^2(zx-wu)+h^2(xy-uv)\\ &\quad +l^2(yw-zv)+m^2(zu-xw)+n^2(xv-yu)=f_2(\alpha+\omega\beta).\end{aligned}$$

Now this expression is evidently twice the kinetic energy of the body, since $T(\alpha+\omega\beta)$ is the angular velocity about the axis. But it is easy to see that the same expression represents twice the kinetic energy in the general case also when $xu+yv+zw$ is not $=0$, and $\alpha+\omega\beta$ is a motor. For, referring it to its axes, we can express it as the sum of two polar rotors; viz.,

$$\begin{aligned}\alpha+\omega\beta &= \tfrac{1}{4}\{\xi U(\alpha+\beta)+\eta U(\alpha-\beta)\}\{T(\alpha+\beta)+T(\alpha-\beta)\}\\ &\quad +\tfrac{1}{4}\{\xi U(\alpha+\beta)-\eta U(\alpha-\beta)\}\{T(\alpha+\beta)-T(\alpha-\beta)\}\\ &= A+\lambda\omega A, \text{ say; where if}\end{aligned}$$

$$A=ix+jy+kz+\omega(iu+jv+kw),$$

then $$\omega A=iu+jv+kw+\omega(ix+jy+kz),$$

and $$f_2(A+\lambda\omega A)=f_2(A)+\lambda^2 f_2(\omega A),$$

because $$xu+yv+zw \doteq 0.$$

But these rotors A and ωA being polar, the velocities due to them are everywhere at right angles; so that the energy of their resultant is the sum of their energies; that is, it is represented by $f_2(A+\lambda\omega A)$.

Now let $\nabla=i\partial_x+j\partial_y+k\partial_z+\omega(i\partial_u+j\partial_v+k\partial_w)=\nabla_1+\omega\nabla_2$, then, T being the kinetic energy, ∇T is a motor which we shall call the *momentum* of the moving body. Namely, it is

$$\begin{aligned}M &= (a^2x+h^2y+g^2z+n^2v-m^2w)\,i\\ &\quad +(h^2x+b^2y+f^2z+l^2w-n^2u)\,j\\ &\quad +(g^2x+f^2y+c^2z+m^2u-l^2v)\,k\\ &\quad +\{(1-a^2).u-h^2y-g^2w-n^2y+m^2z\}\,\omega i\\ &\quad +\{-h^2u+(1-b^2).v-f^2w-l^2z+n^2x\}\,\omega j\\ &\quad +\{-g^2.u-f^2v+(1-b^2).w-m^2x+l^2y\}\,\omega k.\end{aligned}$$

If we write

$$\phi(ix+jy+kz) = \begin{pmatrix} a^2, & h^2, & g^2 \\ h^2, & b^2, & f^2 \\ g^2, & f^2, & c^2 \end{pmatrix}(xyz)(ijk),$$

then the expression for M may be written

$$M = \phi\alpha + \omega(\iota - \phi)\beta + \sigma(\beta - \omega\alpha),$$

where $$\sigma = l^2 i + m^2 j + n^2 k,$$

or $$\begin{aligned} M &= \tfrac{1}{2}(\alpha + \omega\beta) + (\phi - \tfrac{1}{2})(\alpha - \omega\beta) - \omega\sigma(\alpha - \omega\beta) \\ &= \tfrac{1}{2}V + (\phi - \tfrac{1}{2} - \omega\sigma)(\alpha - \omega\beta) \\ &= \tfrac{1}{2}V + \chi(\alpha - \omega\beta), \end{aligned}$$

where $$\chi = \begin{pmatrix} a^2 - \tfrac{1}{2}, & h^2 - \omega n^2, & g^2 + \omega m^2 \\ h^2 + \omega n^2, & b^2 - \tfrac{1}{2}, & f^2 - \omega l^2 \\ g^2 - \omega m^2, & f^2 + \omega l^2, & c^2 - \tfrac{1}{2} \end{pmatrix}.$$

The rate of variation of M is equal to the resultant impressed wrench; now in the case where the impressed forces have a potential P, this resultant wrench is $-\nabla P$. Hence the equation of motion is

$$\dot{M} + \nabla P = 0,$$

wherein M is understood to be expressed as a linear function of V. The kinetic energy T satisfies the equation

$$S.VM + 2T = 0.$$

II.

In the case where there are no forces, P is a constant, and we have

$$\dot{M} = 0,$$

or $$M = \text{a constant motor}.$$

Hence the expression for M as a linear function of V gives us six first integrals of the equation of motion. We have also $S.VM + 2T = 0$, which, now that T and M are constant, gives us a seventh relation among the components of V. This may, however, be deduced from the other six.

In this form the equations are not convenient.

The quantities a^2, b^2, c^2, etc. depend upon the position of the body in regard to the coordinate planes, and are therefore variable. It would be necessary to express them in terms of the position-motor of the body; i.e. the motor which would bring the body from its initial to its final position; then to express the velocity in terms of the variation of this motor (to which it is not equal); and to integrate the resulting equations. We can avoid all this by using moving axes*.

In the first place we have to determine the rates of change of the coordinates of any point p in terms of the twist-velocity $\alpha + \omega\beta$. These are

$$\begin{aligned}
\dot{p}_1 &= - zp_2 + yp_3 + up_4,\\
\dot{p}_2 &= zp_1 - xp_3 + vp_4,\\
\dot{p}_3 &= -yp_1 + xp_2 + wp_4,\\
\dot{p}_4 &= -up_1 - vp_2 - wp_3.
\end{aligned}$$

From these we can find the rates of change of the quantities a^2, etc., namely,

$$\begin{aligned}
\tfrac{1}{2}\frac{d(a^2)}{dt} &= \tfrac{1}{2}\frac{d}{dt}\int P(p_2^2+p_3^2) = \int P(p_2\dot{p}_2 + p_3\dot{p}_3)\\
&= z\int Pp_1p_2 - y\int Pp_1p_3 + v\int Pp_2p_4 + w\int Pp_3p_4\\
&= h^2z - g^2y + m^2v + n^2w,\\
\frac{d(f^2)}{dt} &= \frac{d}{dt}\int Pp_2p_3 = \int P(p_2\dot{p}_3 + \dot{p}_2p_3)\\
&= x\int P(p_2^2 - p_3^2) + z\int Pp_1p_3 - y\int Pp_1p_2\\
&\qquad + v\int Pp_3p_4 + w\int Pp_2p_4\\
&= (c^2 - b^2)x + g^2z - h^2y + n^2v + m^2w,
\end{aligned}$$

* [Cf. Fig. 52.]

$$\frac{d\,(l^2)}{dt} = \frac{d}{dt}\int P p_1 p_4 = \int P\,(p_1 \dot{p}_4 + \dot{p}_1 p_4)$$

$$= u\int P\,(p_4^{\,2} - p_1^{\,2}) + y\int P p_3 p_4 - z\int P p_2 p_4$$

$$- v\int P p_1 p_2 - w\int P p_1 p_3$$

$$= (1 - b^2 - c^2)\,u + n^2 y - m^2 z - h^2 v - g^2 w,$$

and from these the other six may be written down by symmetry.

Now let the moving axes be taken to be the principal axes of the body. Then all the products of inertia vanish, and we have

$$M = a^2 i + b^2 j + c^2 k + \omega\,\{(1 - a^2)\,u + (1 - b^2)\,v + (1 - c^2)\,w\}$$

$$\tfrac{1}{2}\frac{d\,(a^2)}{dt} = 0,$$

$$\frac{d\,(f^2)}{dt} = (c^2 - b^2)\,x,\quad \frac{d\,(g^2)}{dt} = (a^2 - c^2)\,y,\quad \frac{d\,(h^2)}{dt} = (b^2 - a^2)\,z,$$

$$\frac{d\,(l^2)}{dt} = (1 - b^2 - c^2)\,u,\quad \frac{d\,(m^2)}{dt} = (1 - c^2 - a^2)\,v,\quad \frac{d\,(n^2)}{dt} = (1 - a^2 - b^2)\,w.$$

Now by differentiating the coefficients of i and ωi in the value of M on page 380, we find

$$\frac{d}{dt}\,(a^2 x + h^2 y + g^2 z + n^2 v - m^2 w) = 0,$$

$$\frac{d}{dt}\,\{(1 - a^2)\,.\,u - h^2 v - g^2 w - n^2 y + m^2 z\} = 0,$$

that is to say

$$0 = a^2\dot{x} + (b^2 - a^2)\,yz + (a^2 - c^2)\,yz + (1 - a^2 - b^2)\,vw - (1 - c^2 - a^2)\,vw$$

$$= a^2\dot{x} + (b^2 - c^2)\,(yz - vw),$$

and

$$0 = (1 - a^2)\,\dot{u} - (b^2 - a^2)\,zv - (a^2 - c^2)\,yw - (1 - a^2 - b^2)\,yw$$

$$+ (1 - c^2 - a^2)\,zv$$

$$= (1 - a^2)\,\dot{u} + (1 - c^2 - b^2)\,(zv - yw).$$

These equations take the place of Euler's.

From these we get, in the first place,

$$a^2 x\dot{x} + b^2 y\dot{y} + c^2 z\dot{z} = (b^2 - c^2)\, xvw + (c^2 - a^2)\, ywu + (a^2 - b^2)\, zuv,$$

$$\begin{aligned}
&(1 - a^2) \,.\, u\dot{u} + (1 - b^2) \,.\, v\dot{v} + (1 - c^2) \,.\, w\dot{w} \\
= \quad &(-1 + b^2 + c^2)\ (zuv - ywu) \\
&+ (-1 + c^2 + a^2)\ (xvw - zuv) \\
&+ (-1 + a^2 + b^2)\ (ywu - xvw) \\
= \ &(c^2 - b^2)\, xvw + (a^2 - c^2)\, ywu + (b^2 - a^2)\, zuv;
\end{aligned}$$

$$\therefore\ a^2 x\dot{x} + b^2 y\dot{y} + c^2 z\dot{z} + (1 - a^2) \,.\, u\dot{u} + (1 - b^2) \,.\, v\dot{v} + (1 - c^2) \,.\, w\dot{w} = 0,$$

whence $a^2 x^2 + \text{etc.} + (1 - a^2) \,.\, u^2 + \text{etc.} = 2T$, the equation of energy.

*XLII.

[FURTHER NOTE ON BIQUATERNIONS.]

I.

THE two expressions "twice three are six" and "six is the product of two and three," represent two different views of multiplication, although they are written down by the same shorthand formula

$$2 \times 3 = 6.$$

Accordingly they give two different interpretations of this formula.

In the first interpretation 3 is a concrete number of things, say three marbles, while 2 is not a number but an operation, namely the operation of doubling; and we may read the equation "doubling three marbles makes six marbles."

The second interpretation regards 2 and 3 as abstract numbers, and affirms the existence of a third number 6 having a definite relation to them which it is convenient to study, this third number so related being called their product; and various meanings given to the numbers 2 and 3 may lead to various concrete interpretations of the formula. Each of these views of multiplication may be extended to other things besides numbers; I propose at present to consider certain extensions of the first view.

In this we have regarded 2 as a symbol of operation, 3 as a concrete number, and 6 as a concrete number. But we may also regard all three symbols as symbols of operation, and so read the formula "doubling the triple of anything makes the

sextuple of it." Such an equation as $abc = d$ will then always have two meanings:—

1. a times b times c things makes d things;

2. a times b times c times anything makes d times that thing.

That is to say, we may regard the *last* symbol in each term of the equation as either a concrete number or a symbol of operation; but all the others must be regarded as symbols of operation.

To extend this from concrete numbers to *steps* of addition or subtraction is not difficult; but it requires us to give a double meaning to the signs + and −, as well as to all numerical symbols. The first meaning is to indicate the direction of the step; thus + 3 means a step of 3 *forward*, i.e. an addition, and − 3 means a step of 3 *backward*, i.e. a subtraction. But when these symbols are attached to an operation performed upon steps, they mean *retaining* and *reversing* respectively. Thus the equation

$$(-2)\,(+3) = -6$$

has two meanings:—

1. Doubling a step of 3 forward and reversing it makes a step of 6 backward.

2. To triple a step and retain its direction, then to double and reverse it, is the same as to sextuple and reverse it.

These steps of addition and subtraction may be regarded as changes of position, or *vectors*, on a straight line, along which all numbers are supposed to be ranged; and by exchanging numbers for continuous quantities we may deal in this way with all vectors in a straight line. In every equation we may regard the *last* symbol in every term as either a vector or an operation; but all the others must be regarded as operations.

Assuming the law of addition of vectors in a plane, $AB + BC = AC$, we find at once the interpretation of so-called *imaginary* or *impossible* quantities in the operators which convert one vector into another. Thus [Fig. 53] if I operating on

a vector turns it counter-clockwise through a right angle, so that $I \,.\, OA = OA'$, and if

$$a = \frac{OM}{OA},\ b = \frac{MB}{OA'},$$

a and b being ratios of vectors in a line as just previously defined, then

$$OB = OM + MB = a \,.\, OA + b \,.\, OA' = (a + bI)\, OA,$$

and it is clear that $I^2 = -1$. Thus every expression of the form $a + bI$ is the ratio of two vectors.

Every vector in the plane may be represented by $a \,.\, OA + b \,.\, OA'$ by giving to a and b proper values. For shortness we may write $OA = j$, $OA' = k$; then $Ij = k$, $Ik = -j$, and we shall have

$$(a + bI)\,(cj + dk) = (ac - bd)\,j + (ad + bc)\,k.$$

In this way we may have to consider two classes of expressions, those in which the last symbol in every product is a vector, and all the others are ratios of vectors; and those in which all the symbols represent ratios of vectors. But, observing that

$$cj + dk = (c + dI)\,j\,;$$

we may make the useful convention that j is, if convenient, to be understood as written after every term; so that the complex symbol $c + dI$ will now mean *either* a ratio of two vectors as above, *or* the vector $cj + dk$. And then every expression will have a double meaning as before, and we shall have only one kind to deal with.

This artifice amounts to taking a definite vector as the unit, and representing all others by means of their ratios to the unit. The success of the artifice depends on the fact that the product of two such ratios is another ratio of the same kind.

Passing now to vectors in space, we shall find again that the operation which makes one into another is of the form $a + bQ$, where Q turns through a right angle *in the plane of the two vectors.* It will not, therefore, operate on any vector out of that plane; and the variety of these operators Q is the same as the variety of planes, or is doubly infinite. We may represent Q

as a sort of handle or axis of unit length perpendicular to the plane; and the compound operation bQ, which turns through a right angle *and* increases in the ratio $1 : b$, may be represented by an axis of length b. This being so, let Q, R be two such compound operations; there is *one* vector α on which they will both operate, namely, the intersection of their planes. It is found that $Q\alpha + R\alpha$ is a vector at right angles to α, and that if S is the rectangular versor which converts α into $Q\alpha + R\alpha$, so that $S\alpha = Q\alpha + R\alpha$, then the axis of S is got by adding the axes of Q and R as if they were vectors. So we write $S = Q + R$, and we have the equation

$$(Q + R)\,\alpha = Q\alpha + R\alpha.$$

By the law of formation of $Q + R$ it is clear that

$$P + (Q + R) = (P + Q) + R = (P + R) + Q = \text{etc.},$$

this being a rectangular versor whose axis is the vector-sum of the axes of P, Q, R. Accordingly it is called $P + Q + R$; although the equation

$$(P + Q + R)\,\alpha = P\alpha + Q\alpha + R\alpha$$

does not admit of interpretation in general, because there is no vector α which is capable of being operated on by P, Q, and R.

Thus every rectangular versor may be represented by the form $xI + yJ + zK$, where IJK are the three rectangular versors whose axes are the unit-vectors ijk. And we have two kinds of complex quantities to consider; *vectors*, of the form $\rho = ai + bj + ck$, and *quaternions*, of the form $q = w + xI + yJ + zK$. The product of any number of quaternions is itself a quaternion, the units IJK being multiplied by the rules

$$IJ = K = -JI,\ KI = J = -IK,\ JK = I = -KJ,$$
$$I^2 = J^2 = K^2 = -1.$$

But we cannot multiply a vector by a quaternion in general; $q\rho$ will only have a meaning if ρ is perpendicular to the axis of q, or, which is the same thing, if $ax + by + cz = 0$. And even then, although we have the formulæ

$$Ij = k = -Ji,\ Ki = j = -Ik,\ Jk = i = -Kj,$$

we cannot find the value of $q\rho$ by direct multiplication, for the symbols Ii, Jj, Kk are unmeaning. If, however, we assume that they have the *same* value, the result of direct multiplication will come out right whenever $q\rho$ is interpretable.

The artifice, by which in the geometry of two dimensions the two kinds of complex quantities were reduced to one, is not applicable here. For although we may write

$$ai + bj + ck = (a + bK - cJ)\, i,$$

and so represent every vector by its ratio to the unit i, yet it no longer remains true that the product of two such ratios is another ratio of the same sort. But we may attain the desired reduction by a simpler method; viz., by using the symbols ijk in a double sense, as vectors and as versors. Thus in the product $q\rho$, if ρ be regarded as a rectangular-versor, the true value may be obtained by direct multiplication

$$(w + xi + yj + zk)\,(ai + bj + ck),$$

the ijk now standing for what was denoted by IJK. *In certain cases,* namely, when $ax + by + cz = 0$, the expression may have another meaning, and $ai + bj + ck$ may be regarded as a vector. Here the first i being a versor and the second a vector, the assumption $ii = -1$ is unmeaning, but it is without effect on the result. Similarly an expression $pq\rho$ is *always* interpretable if all the symbols are regarded as operations on vectors; it is *sometimes* interpretable when ρ is regarded as a vector, namely, when it is perpendicular to the axis of (pq), provided we make the formal assumption that $i^2 = j^2 = k^2 = -1$ just as in the other interpretation.

Observe that the artifice by which one symbol is made to do duty for two meanings is the same in quaternions, which deal with three dimensions, and in scalars, which deal with one dimension. Namely, the signs + and −, which are originally unit-*vectors,* indicating the direction of a step forward or backward, receive the additional meaning of *versors,* retaining or reversing the direction of a vector; just as the symbols ijk mean vectors originally, and afterwards are made to mean versors too.

But in complex numbers, which deal with two dimensions, the artifice is essentially different; and that which, by a convenient inaccuracy, may be called the *product of two vectors*, has very different geometrical relations to its components. It will be shewn further on that all geometric algebras dealing with an *odd* number of dimensions resemble scalars and quaternions in this respect; while those dealing with an *even* number of dimensions resemble complex numbers.

It is clear that the versors IJK may be represented on great circles of a sphere whose centre is the origin; and if these be regarded as steps on the surface of the sphere, it will be found that the consideration of their ratios leads to the whole theory of quaternions. In this interpretation *vectors* can only be represented by points on the sphere supposed to have definite *weights* attached to them, proportional to the length of the corresponding vectors. Here then we have the geometric algebra of three dimensions interpreted by means of a space of two dimensions which has constant positive curvature, namely the surface of a sphere.

In the same way we may interpret the algebra of a space of four dimensions, which cannot be imaged, by means of a space of three dimensions having constant positive curvature, of which a clear mental picture may be formed.

Consider any vertical line, and a series of horizontal planes cutting it at right angles. In ordinary or Euclidian geometry these planes intersect on the *horizon*, which is a straight line infinitely distant. In the geometry of a space of constant positive curvature, or *elliptic* geometry, the horizon is at a certain finite distance in all directions from the vertical line with which we started; it belongs to that particular line, which is called its *polar*, and is not the same for all vertical lines. Although it appears to be a great circle when viewed from the neighbourhood of its polar, yet if we were to go to it and examine it we should find it straight. Points of it which are in opposite directions from a point on the polar are really identical; and every straight line in this space resembles a circle in being of finite length, so that if we travel far enough along it we shall arrive at our starting point. Every straight line

has a polar line, which is the intersection of all planes at right angles to it.

Let us take a very small circle on a sphere, and suppose it to expand, keeping always the same centre. At the beginning the circle will be concave inside and convex outside; but when the expansion has gone on far enough it will become a great circle of the sphere, which is of the same shape on both sides, or is *straight* so far as the surface of the sphere is concerned. So if in Euclidian space we take a sphere and suppose it to expand, keeping always the same centre, it will continue to be concave inside and convex outside so long as it is finite; but when the radius has become infinite, the inside in one direction is the same as the outside in the opposite direction, opposite points being identical; thus the sphere is of the same shape on both sides, or is a *plane*, viz., the plane at infinity. In elliptic space, just as in geometry on the surface of a sphere, this takes place for a *finite* length of the radius, not for an infinite length; for every point there is a sphere having its centre at that point, which is also a plane. Or, which is the same thing, every point has a polar plane which is the locus of all points situate at a certain distance from it; this distance is called a *quadrant.* So also every plane has a certain point, called its *pole*, which is distant a quadrant from every point in the plane. All lines and planes perpendicular to the plane pass through its pole, and conversely. The polar lines of all lines in the plane pass through its pole, and so do the polar planes of all points in the plane.

When two lines are polars of one another, every point of one is distant a quadrant from every point of the other; hence the polar planes of all points on one pass through the other. Every line which is at right angles to one meets the other, and conversely.

In the *Proceedings of the London Mathematical Society*, [Vol. IV. p. 381—395*] I have given a sketch of a geometric algebra adapted to this elliptic geometry of space, which I have there called Biquaternions.

In the interpretation of quaternions on the surface of a sphere, rectangular versors are represented by quadrants of

great circles. We may represent such a versor accompanied by a tensor, for example xi, by an arc AB measured on the great circle i, so that $\tan AB = x$. This being so, AB differs from a vector in a plane in a most important way; for while a vector in a plane is unaltered by being moved parallel to itself in any direction, AB can only be slid along its great circle, and must not be moved out of it. We shall have to consider similar quantities in the elliptic geometry of three dimensions; namely quantities represented by a length marked off on a certain straight line, which are unaltered when the length is slid along the line but not in any other case. They are as it were *vectors having position.* A vector represents the translation-velocity of a rigid body, which is everywhere the same; these quantities will represent the *rotation-velocity* of a rigid body, which is about a certain definite axis. For this reason I have called them *rotors* (short for *rotators*) by analogy with Hamilton's word *vector.* They are added together according to the law of composition of forces and rotations. That is to say, if a rotor P [fig. 54] along OA added to a rotor Q along OB gives a rotor R along OC making angles α, β with them, then OC is in the plane of OA, OB, and

$$P : Q : R = \sin\beta : \sin\alpha : \sin(\alpha + \beta);$$

this determines the position and magnitude of the resultant. We cannot use the parallelogram construction as in the addition of vectors, for the (elliptic) geometry of the plane AOB is the same as that of the surface of a sphere when opposite points are regarded as identical, and no parallelogram can be drawn on it.

Since any two great circles of a sphere meet one another (in *one* point, according to our present convention) it follows that any two rotors have a single rotor which is their resultant or sum. But in three dimensions this is not the case; if the axes of two rotors do not meet their sum is not equal to any one rotor. We may however find two other rotors which have the same sum, and that in an infinite number of ways. Of these ways one is of the greatest importance, namely, that in

we regard each rotor as representing rotation about its axis, each of these rotations is equivalent to a translation along the other axis. Thus rotation about a vertical line is translation along the horizon, and *vice versa*. Hence the resultant of the two rotations may be regarded as a screw motion about either of the axes. As this describes the most general motion of a rigid body, I have proposed to call the quantity which represents it a *motor*. We shall say then that the sum of two rotors which do not meet is a motor, and that every motor has two axes which are polars of one another.

This being so, let three planes at right angles be drawn through a point O, and let unit rotors along their intersections be denoted by ijk. Then any rotor through the point O is denoted by $ai+bj+ck$. The ratio of two such rotors is a quantity of the form $w+xi+yj+zk$, if we let ijk mean also rectangular versors whose axes are the rotors ijk. In fact we are merely applying the results of quaternions to vectors passing through a fixed point and their ratios.

Let now ω be the operation which converts any rotor into an equal rotor along the polar line of its axis. Then ωi, ωj, ωk [fig. 55] will be rotors along the lines of intersection of the polar plane of O with the three rectangular planes through O. And since (as it is easy to see) any rotor may be resolved into two, one passing through O, and the other lying in the plane PQR, whereof the former is compounded of ijk, and the latter of ωi, ωj, ωk, it follows that the expression for the sum of any number of rotors (i.e. for a motor) is of the form

$$ai+bj+ck+\omega(fi+gj+hk)=\alpha+\omega\beta, \text{ say.}$$

Suppose now that the *versors* ijk are allowed to operate, not only on rotors through O which meet their axes, but on any rotors which meet them at right angles. Then we shall have

$$i(\omega j)=\omega k=\omega ij\,;\; j(\omega k)=\omega i=\omega jk\,;\; k(\omega i)=\omega j=\omega ki,$$

which equations shew that ω is *commutative* with the symbols i, j, k.

versor about the polar line of its axis, we shall have the equations

$$\omega i\,.\,j=\omega k,\ \omega j\,.\,k=\omega i,\ \omega k\,.\,i=\omega j,$$

which shew that the two meanings thus attributed to ω lead to no contradiction. Lastly, we have

$$\omega i\,.\,\omega j=k,\ \omega k\,.\,\omega i=j,\ \omega j\,.\,\omega k=i$$

from which we get $\omega^2=1$.

From this equation we may draw a very important consequence. Writing

$$\xi=\tfrac{1}{2}(1+\omega),\quad \eta=\tfrac{1}{2}(1-\omega),$$

and therefore

$$1=\xi+\eta,\quad \omega=\xi-\eta,$$

we find

$$\xi^2=\tfrac{1}{4}(1+2\omega+\omega^2)=\xi;\quad \eta^2=\tfrac{1}{4}(1+2\omega+\omega^2)=\eta,$$
$$\xi\eta=\tfrac{1}{4}(1-\omega^2)=0.$$

Hence every motor $\alpha+\omega\beta$ may be written in the form

$$(\xi+\eta)\,\alpha+(\xi-\eta)\,\beta \text{ or } \xi\,(\alpha+\beta)+\eta\,(\alpha-\beta).$$

Consider two motors

$$\xi\alpha+\eta\beta,\ \xi\gamma+\eta\delta, \text{ and let } \gamma\alpha^{-1}=p,\ \delta\beta^{-1}=q,$$

so that p and q are known quaternions; then we have

$$(\xi p+\eta q)(\xi\alpha+\eta\beta)=\xi p\alpha+\eta q\beta$$
$$=\xi\gamma+\eta\delta.$$

Thus the ratio of two motors is a quantity of the form $\xi p+\eta q$, or, which is the same thing,

$$s+\omega t\ (\text{if } 2s=p+q,\ t=2p-q),$$

where $p, q, s, t,$ are quaternions. This combination of two quaternions I have called a *Biquaternion**.

* [I am indebted to Dr Spottiswoode for the title I have given to this paper. It is designated as i by Prof. Clifford, but I have not come across any other papers of the series.]

NOTES ON BIQUATERNIONS.

1. Axes of motor $\alpha+\omega\beta$, or $\xi\gamma+\eta\delta$.

If μ be an axis of a motor, the motor is a multiple of the rotor μ together with a multiple of the polar rotor $\omega\mu$. Thus we may write

$$\xi\gamma+\eta\delta=(h+\omega k)\,\mu.$$

Operating by ξ, η, we have

$$\xi\gamma=(h+k)\,\xi\mu,$$

$$\eta\delta=(h-k)\,\eta\mu,$$

whence

$$\mu=\frac{\xi\gamma}{h+k}+\frac{\eta\delta}{h-k}.$$

Thus we have an indeterminate representation of any motor as the sum of two polar motors. But if μ is a *rotor*, we have

$$\frac{T\gamma}{h+k}=\frac{T\delta}{h-k},$$

whence

$$\frac{h}{k}=\frac{T\gamma+T\delta}{T\gamma-T\delta},$$

so that

$$\mu=\xi\,.\,U\gamma+\eta\,.\,U\delta,$$

and

$$\omega\mu=\xi\,.\,U\gamma-\eta\,.\,U\delta.$$

But the component rotors along the axes are

$$(T\gamma\pm T\delta)\,(\xi\,.\,U\gamma\pm\eta\,.\,U\delta).$$

For the sum of these is easily seen to be equal to the original motor $\xi\gamma+\eta\delta$.

* [These "Notes" were apparently written when Prof. Clifford was in the enjoyment of vigorous health, and occupy three pages of MS. pinned together. They are, I should say, intended to be supplementary to his *London Mathematical Society* paper, XX. supra.]

2. Sum of motors of different pitches whose axes meet at right angles.

Let the motors be $(a+\omega b)j$ and $(c+\omega d)k$; the sum is of course

$$aj + ck + \omega(bj + dk)$$

or

$$\tfrac{1}{2}\xi(\overline{a+b}\,.j + \overline{c+d}\,.k) + \tfrac{1}{2}\eta(\overline{a-b}\,.j + \overline{c-d}\,.k).$$

Thus an axis of the sum is

$$\xi\frac{\overline{a+b}\,.j + \overline{c+d}\,.k}{\sqrt{(\overline{a+b}^2 + \overline{c+d}^2)}} + \eta\frac{\overline{a-b}\,.j + \overline{c-d}\,.k}{\sqrt{(\overline{a-b}^2 + \overline{c-d}^2)}}.$$

Now this is a rotor at right angles to i and at an angular distance θ from the origin such that

$$\cos 2\theta = \frac{a^2 - b^2 + c^2 - d^2}{\sqrt{(\overline{a+b}^2 + \overline{c+d}^2)(\overline{a-b}^2 + \overline{c-d}^2)}}$$

$$\tan 2\theta = \frac{2(bc - ad)}{a^2 - b^2 + c^2 - d^2}.$$

*XLIII.

ON THE CLASSIFICATION OF GEOMETRIC ALGEBRAS*.

1806 Argand, *Manière de représenter les quantités imaginaires.*
Buée, *Mém. sur les qu. imag.*
1827 Möbius, *Barycentrischer Calcul.*
1831 Gauss.
1834 Peacock, *Doctrine of Operations in Algebra.*
1843 Hamilton, *Quaternions.*
1844 Grassmann, *Lineale Ausdehnungslehre.*
1845 Saint-Venant, *Multiplication of vectors.*
1848 Kirkman, *Pluquaternions and Homoid Products.*
1853 Cauchy, *Clefs Algébriques.*
1862 Grassmann, *Ausdehnungslehre.*
1870 Peirce, *Linear Associative Algebra.*

In the Barycentric Calculus a point is represented by a complex number which is a linear syzygy of symbols each representing a fixed point; the coefficients are coordinates. By regarding ab the ordinary symbol for a line joining two points, as of the nature of a product, and so *distributive*, we arrive at Grassmann's extensive quantities. For then we must put $aa=0$, i.e. $(a_1\iota_1+a_2\iota_2)^2=0$, which requires $\iota_1\iota_2=-\iota_2\iota_1$, and then $ab=-ba$ always. If there are n independent units, we may consider in such an algebra scalars or quantities of order 0, n of order 1, $\frac{1}{2}n\,.\,(n-1)$ of order 2, etc., 1 of order n; in all 2^n *vids*, to borrow Peirce's term. Every intelligible expression is however homogeneous; if a product of points, the coefficients are determinants.

In the theory of Quaternions the symbols ijk, when used as multipliers, represent not things but operations of turning; thus $i^2=-1$ and not 0. But regarding them as vectors, we may use them to represent the geometry of the plane passing through the ends of ijk, on the principles of the barycentric calculus. Thus $\rho=ix+jy+kz$ will represent the point where it cuts the plane, with a *weight* $x+y+z$. The *Grassmann* algebra will be reproduced if we attend only to the *vector* part of binary products, and the *scalar* part of ternary. Physical considerations however lead us to regard i^2 as a scalar (not zero) even

* [The "forewords" are the abstract which Prof. Clifford communicated to the *London Mathematical Society*, on March 10th, 1876 (*Proceedings*, Vol. VII. p. 135). The paper, which is unfinished, was found amongst his MSS.]

when i is regarded as a vector (not a versor). For these purposes it does not matter whether it is put $=-1$ or $+1$.

I propose here to extend this assumption to the Grassmann representation in general: i.e., I take n units $\iota_1\iota_2 \ldots \iota_n$ such that $\iota_s^2=+1$, and $\iota_r\iota_s=-\iota_s\iota_r$. All products of linear factors must therefore contain either terms of odd order only, or terms of even order only. There can be no term of higher order than the n^{th}, and the whole number of terms is 2^n as with Grassmann; i.e. in both cases we have a linear associative algebra of 2^n units, but the Grassmann algebra is *nilpotent*, and only homogeneous forms occur; while this is *idempotent*, and admits of *odd* forms and *even* forms, which are not in general homogeneous. It is convenient to use *selective* symbols V_0, $V_1 \ldots V_n$ analogous to Hamilton's S and V, for picking out those parts of any expression which are of order $0, 1 \ldots n$ respectively.

The Quaternion symbols satisfy the equation $ijk=-1$, and this together with the assumption $\rho^2=$scalar gives all the laws of their multiplication. If we put $\omega=\iota_1\iota_2 \ldots \iota_n$, this means that in the case $n=3$ we may take ω to be a scalar. There is here a very important distinction between the cases n odd and n even; in the former case ω is commutative with the symbols ι, or $\omega\iota=\iota\omega$, in the latter case $\omega\iota=-\iota\omega$. Hence when n is odd ω acts as a scalar, when n is even it acts as a vector. In putting $\omega=$to a scalar in the former case we are conveniently representing two different things by the same symbols, because they have the same laws of combination. We thus reduce the algebra to 2^{n-1} units when n is odd. When n is even the symbol ω is of even order. In all cases we may if we like consider separately the even units and the odd units; the former make an algebra by themselves, and by restricting ourselves to these we get the same result as by putting $\omega=\pm 1$. Thus, $n=3$, the even algebra gives quaternions at once. When n is even we may still further simplify; for the symbol ω belongs to the even algebra, and this consists of 2^{n-2} terms (i.e. the even terms up to order $n-2$ and half of these) together with the products of these terms by ω, which is commutative with them. Thus $n=4$, the even algebra gives biquaternions, and the general expression is $q+\omega r$ where q, r are quaternions. For n odd, we may represent the odd algebra by the even algebra; this amounts to making $\omega=\pm 1$.

THE extensive quantities of Grassmann, which Hankel has called alternate numbers, namely symbols which have the property of *polar* multiplication $ab=-ba$ and whose square vanishes, $a^2=0$, serve to conveniently represent the projective geometry of n dimensions. In plane geometry, for example, let the symbols ι_1, ι_2, ι_3 represent three points; then

$$a=a_1\iota_1+a_2\iota_2+a_3\iota_3$$

will represent a point which is the centre of inertia of masses a_1, a_2, a_3 placed at the fundamental points respectively. If the

products $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$ be taken to mean the lines joining the fundamental points, then the product

$$\begin{aligned} ab &= (a_1\iota_1 + a_2\iota_2 + a_3\iota_3)(b_1\iota_1 + b_2\iota_2 + b_3\iota_3) \\ &= (a_2b_3 - a_3b_2)\,\iota_2\iota_3 + (a_3b_1 - a_1b_3)\,\iota_3\iota_1 + (a_1b_2 - a_2b_1)\,\iota_1\iota_2 \end{aligned}$$

represents the line joining the points a, b; for the coefficients of the binary products $\iota_2\iota_3$, ... are clearly the coefficients in the equation of that line referred to the fundamental triangle.

In like manner the ternary product

$$abc = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \iota_1\iota_2\iota_3$$

is proportional to the area of the triangle abc, and vanishes when the three points are collinear. And so on.

The system of quaternions differs from this, first in that the squares of the units, instead of being zero, are made equal to -1; and secondly in that the ternary product $\iota_1\iota_2\iota_3$ is made equal to -1. The interpretation is at the same time extended to three dimensions, but with this restriction: that whereas the alternate units represent *any* three points in a plane, and the system deals primarily with projective relations, Hamiltonian units represent three vectors at right angles, and the system is the natural language of metrical geometry and of physics.

I shall now examine the consequence of making, in a system of n alternate numbers $\iota_1, \iota_2, \ldots \iota_n$, the *first* of the modifications just named; namely I shall suppose that the square of each of the units is -1. I shall then enquire whether any assumption can be made which is analogous to the second Hamiltonian law, $\iota_1\iota_2\iota_3 = -1$.

In the system of Grassmann every product of linear factors is homogeneous in the units; namely the product of m linear factors is a linear function of the m-ary products when m is not greater than n, and is zero when $m > n$. But if we make the square of every unit equal to -1, it is clear that the product of m factors (m not greater than n) may contain terms of the

order m, $m-2$, $m-4$, ... $m-2r$, ..., since the order is reduced by 2 for every substitution of -1 for ι^2. Consequently every such product consists of terms which are either all of odd order or all of even order. The same thing is true when m is greater than n, except that the highest order which can occur is either n or $n-1$, according as $m-n$ is even or odd. The products of an even number of the units will therefore form an algebra complete in itself. We have altogether one term of order 0, n of order 1, $\frac{1}{2}n(n-1)$ of order 2, ... one of order n; that is to say $1+n+\frac{1}{2}n(n-1) \ldots +n+1=2^n$ terms. Moreover we have

$$\begin{aligned}&\text{terms of even order} - \text{terms of odd order}\\ &= 1-n+\tfrac{1}{2}n(n-1) - \ldots + (-1)^n = (1-1)^n = 0,\end{aligned}$$

or the number of terms in the even algebra is equal to the number of terms in the odd algebra, namely 2^{n-1}.

Let us now write ω for the product of all the units, $\omega = \iota_1\iota_2 \ldots \iota_n$, and enquire into the value of ω^2. This is

$$\omega^2 = \iota_1\iota_2 \ldots \iota_n\iota_1\iota_2 \ldots \iota_n,$$

which after p interchanges of contiguous symbols can be transformed into

$$\omega^2 = (-)^p\, \iota_1^2\iota_2^2 \ldots \iota_n^2 = (-)^{p+n}.$$

To find the value of p, we observe that it takes $n-1$ changes to bring the second ι_1 between the first ι_1 and ι_2, then $n-2$ changes to bring the second ι_2 between the first ι_2 and ι_3, and so on. Therefore $p+n=1+2+\ldots+n=\frac{1}{2}n(n+1)$; and it follows that $\omega^2=\pm 1$ according as $\frac{1}{2}n(n+1)$ is even or odd, or according as n belongs to the forms $4m$, $4m+3$ or $4m+1$, $4m+2$.

Next let us observe that $\iota_1\omega = -\iota_2\iota_3 \ldots \iota_n$, but $\omega\iota_n = (-)^n\, \iota_2\iota_3 \ldots \iota_n$; that is to say, $\iota_1\omega = \pm\,\omega\iota_1$ according as n is odd or even. Thus the multiplication of ω with the units is polar or commutative according as n is even or odd.

We are thus led to distinguish four classes of geometric algebra, which are characterized by the sign of ω^2 and the nature of the multiplication $\omega\iota$.

Class I. $n \equiv 0 \pmod{4}$. $\omega^2 = 1$, $\omega\iota = -\iota\omega$. The even algebra contains the symbol ω, which is commutative with all even terms. Since every term of order $r > \frac{1}{2}n$ can be expressed as the product of ω into a term of order $n - r$, while the terms of order $\frac{1}{2}n$ divide themselves into complementary pairs, each of which is ω multiplied by the other, this algebra contains 2^{n-2} symbols, together with their products by ω. Thus for $n = 4$, the most general expression in the even algebra is

$$w + x\iota_2\iota_3 + y\iota_3\iota_1 + z\iota_1\iota_2 + \omega\,(w' + x'\iota_2\iota_3 + y'\iota_3\iota_1 + z'\iota_1\iota_2) = q + \omega r,$$

which is precisely what I have elsewhere called a *biquaternion*, because the products $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$, satisfy the laws of the Hamiltonian symbols i, j, k, and therefore the quantities q, r are quaternions in the ordinary sense.

The most general expression in the odd algebra consists also of eight terms, namely it is

$$x\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4 - \omega\,(x'\iota_1 + y'\iota_2 + z'\iota_3 + w\iota_4).$$

The coefficients are purposely so arranged as to bring out the fact that this expression may be derived from the preceding by multiplying it by $\iota_4\omega$. It does not, however, follow that we may always represent the vector $x\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4$ by the proportional quantity of the even algebra

$$x\iota_2\iota_3 + y\iota_3\iota_1 + z\iota_1\iota_2 + w'\omega.$$

For if α, β be any two vectors, the quantity $\iota_4\omega\alpha\,.\,\iota_4\omega\beta$ is not in general proportional to $\alpha\beta$. This would require that the multiplication of α or β with $\iota_4\omega$ should be either polar or commutative; whereas in general it is neither.

Class II. $n \equiv 1 \pmod{4}$ $\omega^2 = -1$, $\omega\iota = \iota\omega$. The odd and even algebras may be included in the same formulæ by putting ω equal to the scalar $\sqrt{-1}$; the case is thus closely analogous to that of $n \equiv 3 \pmod{4}$.

Class III. $n \equiv 2 \pmod{4}$, $\omega^2 = -1$, $\omega\iota = -\iota\omega$. Here ω has clearly the properties of a unit vector, and the system may always be treated as a degenerate case of the next. The general expression of the even algebra may be got from the general expression of the odd algebra by multiplying it by $\omega\iota_n$.

*XLIV.

ON THE THEORY OF SCREWS IN A SPACE OF CONSTANT POSITIVE CURVATURE.

Polar lines.

THE word *line* will be here used to mean *great circle* except where ambiguity might occur. To every line in a space of constant positive curvature corresponds a polar line such that every point on it is distant a quadrant from every point on the line. This is in fact its polar line in regard to the *absolute*, a quadric whose equation referred to a quadrantal tetrahedron is

$$x^2 + y^2 + z^2 + w^2 = 0.$$

Translation = rotation about polar (Klein).

The rotation of a rigid body about any line is a sliding of the body along the polar line; and conversely a *translation* or sliding without twist along any line is a rotation about the polar line. A translation therefore in such a space is not merely a vector having magnitude and direction, but a quantity having also a definite position.

Axes of Screw.

The most general motion of a rigid body is made up of translation along a certain axis and rotation about it; or we may say that it is made of rotations about two polar lines. These factors in the motion may be combined in any order without affecting the result. Let α and α_1 be two polar rect-

angular versors (rotations through a right angle about two polar lines), then the product

$$\alpha^p \alpha_1^q$$

is the most general representation of the motion of a rigid body. This may be regarded either as a twist $p\frac{\pi}{2}$ about a screw whose axis is α and pitch $\frac{q}{p}$, or as a twist $\frac{q\pi}{2}$ about a screw whose axis is α_1 and pitch $\frac{p}{q}$. Thus a screw has two axes which are polars of each other, and its pitch is only completely defined when we have picked out one of these axes for attention.

Rotor Sum.

A *rotor* has magnitude, direction, and position. The rotor AB will equal the rotor CD when they are in the same line, of equal length, and of like sense. Two rotors therefore will not in general meet; and the sum of two rotors will in general be a quantity of the nature of a twist or a wrench, which I call a *motor*. If we suppose a rigid body to have rotation velocities about these rotors proportional to their lengths, the resultant of these will be a twist velocity about the screw which is their sum.

Rotor Ratio = Twist × Tensor.

The *ratio* of two rotors is a twist multiplied by a tensor. For there are two polar lines which cut the two rotors at right angles; hence one vector can be made into the other by rotations about these lines combined with an alteration of length.

Scale of quantities.

We have thus a certain scale of quantities, each of which is obtained by connecting the notion of magnitude with a certain geometrical form. The *ratios* of these quantities are the subject-matter of successive algebras.

Geometrical Form	Quantity	Examples.	Ratio
Sense on straight line	Vector on straight line		Real quantity (±) of algebra
Direction in plane	Vector in plane		Complex ratio $a+bi$
Direction in space	Vector in space	Couple; transl. vel. of rigid body	Quaternion
Axis	Rotor	Force; rotation velocity	Twist
Screw	Motor	System of forces; twist velocity	Biquaternion

It is to be observed that this is not an uniformly ascending series, but that the theoretic order has been broken to meet the requirements of practical application. After Hamilton's quaternion should come the ratio of two vectors in four dimensions; now a *rotor* may be regarded as the *logarithm* of such a ratio *when the vectors are at right angles.*

Ratio of polar rotors.

Let α and β be two rotors; it is clear that the same twist which converts α into β converts also the polar of α into the polar of β. Let that twist which converts any rotor into its polar be called ω; {this is a rectangular twist about any screw of pitch 1 whose axis meets the rotor perpendicularly} then this result may be written

$$\frac{\beta}{\alpha} = \frac{\omega\beta}{\omega\alpha} \equiv q, \text{ suppose.}$$

Polar motor.

The sum of any number of rotors is a *motor;* and it is known that every motor may be expressed as the sum of two polar rotors, that is to say in the form

$$\alpha + k.\omega\alpha \equiv A,$$

where κ is the pitch of the motor in regard to the axis α. The motor

$$k\alpha + \omega\alpha \equiv \omega A$$

will be called the polar of A. In regard to the axis α it has a pitch reciprocal of the former one.

Every motor may be regarded as the sum of two, one having any perfectly arbitrary pitch and the other the pitch reciprocal to it. For the equation

$$\alpha + k\omega\alpha = x\,(\alpha + h\omega\alpha) + y\,(h\alpha + \omega\alpha)$$

gives

$$1 = x + hy,$$
$$k = hx + y,$$

and thence

$$x = \frac{1 - kh}{1 - h^2},\ y = \frac{k - h}{1 - h^2}.$$

Ratio of motors.

If two motors have the same pitch their ratio is a twist. For we have seen above that

$$\beta = q\alpha, \qquad \omega\beta = q\omega\alpha,$$

from which it follows that

$$\beta + k\omega\beta = q\,(\alpha + k\omega\alpha).$$

Let the ratio of $\omega\beta$ to α be called the *polar* twist of q; so that we shall write

$$\omega q = \frac{\omega\beta}{\alpha} = \frac{\beta}{\omega\alpha}.$$

{Observe that q is a *finite* twist, and not a *motor*, or *twist velocity;* so that this must not be confounded with the previous meanings of *polar* and of ω.}

The ratio of any two motors is the sum of two polar twists. Let $\alpha + h\omega\alpha$, $\beta + k\omega\beta$ be the two motors; and let $\frac{\beta}{\alpha} = q$. Then

$$\frac{\beta + k\omega\beta}{\alpha + h\omega\alpha} = \frac{(1 - kh)\,(\beta + h\omega\beta) + (k - h)\,(h\beta + \omega\beta)}{(1 - h^2)\,(\alpha + h\omega\alpha)}$$
$$= \frac{1 - kh}{1 - h^2}\cdot q + \frac{k - h}{1 - h^2}\cdot \omega q.$$

XLV.

REMARKS ON A THEORY OF THE EXPONENTIAL FUNCTION DERIVED FROM THE EQUATION $\frac{du}{dt} = pu$*.

AFTER shewing that a quantity which is equally multiplied in equal times must always increase at a rate proportional to itself, or which is the same thing, satisfy the equation $\frac{du}{dt} = pu$, we may call the ratio p of the growth to the growing quantity, the *intrinsic rate*. If we then define as follows:—$\epsilon^{px}.\,u$ is the result of making u grow at the intrinsic rate p for x seconds. This applies so far to pure quantity only; but if we regard u as

(*a*) a vector on a line, i.e., a signed magnitude,

(*b*) a vector in a plane, i.e., a complex magnitude, or

(*c*) a vector in space,

then p being the ratio of two such vectors is respectively

(*a*) a signed ratio; justifying ϵ^{-px},

(*b*) a complex ratio; from which it becomes obvious that $\epsilon^{i\theta} = \cos\theta + i\sin\theta$,

(*c*) a quaternion; from which we get HAMILTON'S theory of exponential functions of quaternions.

A semijustification of the symbolic form of Taylor's Theorem (independently of the series) is also obtained by this method.

The *series* for $\epsilon^{px}.\,u$ in all cases where it holds good may then be either established by Taylor's theorem, or shewn to be a solution and the only one of the equation $\frac{du}{dt} = pu$ between values 0 and x of t.

* [From *Proceedings of London Mathematical Society*, Vol. IV. No. 47, p. 111.]

XLVI.

NOTES ON VORTEX-MOTION, ON THE TRIPLE GENERATION OF THREE-BAR CURVES, AND ON THE MASS-CENTRE OF AN OCTAHEDRON*.

(i) On Vortex-Motion.

Let σ be the velocity, and ω the rotation, at any point of a moving substance. It is known that $2\omega = V\nabla\sigma$; viz., this is equivalent to the three equations ordinarily written thus:

$$2\xi = \delta_z v - \delta_y w,$$
$$2\eta = \delta_x w - \delta_z u,$$
$$2\zeta = \delta_y u - \delta_x v.$$

If, moreover, k be the *expansion*, or the logarithmic rate of change of the volume, we have $k = -S\nabla\sigma$; viz., this is $\delta_x u + \delta_y v + \delta_z w$. Hence the quaternion q, $= -k + 2\omega$, is simply $\nabla\sigma$. The problem solved by Stokes as a general question of Analysis, and subsequently by Helmholtz for the special case of fluid motion, may be stated as follows:—Given the expansion and the rotation at every point of a moving substance, it is required to find the velocity at every point. In symbols, it being known that $q = \nabla\sigma$, and q being given, it is required to find σ.

The solution of the problem is exhibited in a very simple form if we consider it a little more generally.

A quaternion q is given at every point of space; it is required to find a quaternion r so that $q = \nabla r$. The solution is that $\nabla q = \nabla^2 r$, and therefore r is the potential of ∇q; that is,

$$r_a = \int \frac{\nabla q_b \, . \, dv_b}{D_{ab}},$$

* [From the *Proceedings of the London Mathematical Society*, Vol. IX. No. 125, pp. 26—28.]

where r_a means the value of r at the point a, δv_b means an element of volume at the point b, and D_{ab} the distance between the two points a, b. If r is a pure vector, so that $Sr = 0$, q must satisfy a certain condition, namely, we must have $S\nabla q = 0$ everywhere. This is identically satisfied if $q = -k + 2\omega$, where ω is the rotation at any point of a moving substance; for

$$2S\nabla\omega = S\nabla^2\sigma = 0.$$

(ii) On the Triple Generation of Three-bar Curves.

The theorem on which the triple generation of three-bar curves depends has been stated as follows by Prof. Cayley:—Let the triangles [Fig. 56] *deo*, *fog*, *ohk* be similar, and the figures *adof*, *cgok*, *bhoe* parallelograms. Then the triangle *abc* will be similar to *deo*, &c.

The proof of this is intuitive if we consider the operation which converts *oh* into *ok*. This operation consists in turning through the angle *hok*, and altering the length in the ratio *oh* : *ok*. The same operation converts *eb* into *gc*, *de* into *do*, and therefore into *af*, and *ad* or *fo* into *fg*. Hence *ad*, *de*, *eb* are converted by this operation into *fg*, *af*, *gc*, and therefore the whole line *ab* is converted by it into the whole line *ac*. That is to say, the triangle *abc* is similar to *ohk*, as was to be proved.

If we complete the parallelogram *adel*, this amounts to saying that the broken lines *aleb*, *afgc* are similar to one another.

If one of the three-bar systems is a crossed rhomboid, the other two are kites. This would of course follow from the known fact that the path of the moving point in both these cases is the inverse of a conic. But it is also intuitively obvious as soon as the figure is drawn, and thus supplies an elementary proof that the path *is* the inverse of a conic in the case of a kite, which is not otherwise easy to get.

(iii) On the Mass-centre of an Octahedron.

Let af, bg, ch [Fig. 57] be three finite lines not meeting. By an *Octahedron* I mean the solid whose eight faces are abc, acg, agh, ahb, fbc, fcg, fgh, fhb. If this solid figure be filled with matter of uniform density, its mass-centre may be found by a very simple construction.

The solid is girdled by three skew quadrilaterals $bcgh$, $cahf$, $abfg$. Now the middle points of the sides of any skew quadrilateral are in one plane. Draw, then, three planes bisecting the sides of these quadrilaterals, and let them meet in a point k; which, following Sylvester in a paper to be presently mentioned, I will call the *mass-centre*. Let also m be the mean point of the six vertices a, b, c, f, g, h; it is the mass-centre of the triangle formed by the middle points of af, bg, ch. To find s, the mass-centre of the solid, join km and produce it to s so that $ms = \frac{1}{2}km$.

The proof is that the solid is the sum of the four tetrahedra $afbc$, $afcg$, $afgh$, $afhb$. Now the mass-centre of a tetrahedron is the mean point of its vertices; consequently the line joining the mass-centre of $afbc$ to the middle point of gh is divided by the point m in the ratio 1 : 2. The same is true of the other three tetrahedra and the middle points of hb, bc, cg. Hence the mass-centres of the four tetrahedra are in one plane passing through the point s found by the above construction, and therefore the mass-centre of the whole solid is in this plane. So also it is in the other two planes determined by dividing the solid into tetrahedra having the common edge bg and the common edge ch respectively. Therefore it coincides with the point s.

This remark was suggested to me by Sylvester's construction for the mass-centre of a tetrahedral frustum, of which it is a simple extension. In fact, by making the pairs of faces abh, ahg; acg, cfg; cbf, bhf to be respectively coplanar, we pass at once to that particular case.

XLVII.

GEOMETRICAL THEOREM*.

THE following is a proof by Pure Geometry of the proposition given by Mr Ferrers, in Vol. I. p. 159, as the reciprocal of Prof. Cayley's theorem about rectangular hyperbolæ.

The other corollaries are well known propositions. This form of proof suggests the corresponding propositions in Geometry of three dimensions.

1. *If a series of conics be inscribed in the same quadrilateral, their directors will all have the same radical axis.*

The tangents drawn from any point to a series of conics inscribed in the same quadrilateral form a pencil in involution. If therefore the point be such that any two of the conics subtend right angles at it, all the conics will subtend right angles. But the director of a conic is the locus of intersection of tangents at right angles to each other. Therefore the intersections of any two directors are points on all the others. Q. E. D.

2. *The foci of the quadrilaterals formed by five lines lie on a circle.*

For they all lie on the director of the conic touching the five lines.

3. *The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.*

For the diagonals may be regarded as very thin ellipses inscribed in the quadrilateral.

* [From the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, Vol. III. pp. 31, 2, 1866.]

4. *Every conic through the intersection of two rectangular hyperbolæ is a rectangular hyperbola.*

This is the reciprocal of (1) with regard to either of the points of intersection. This is Prof. Cayley's theorem of Vol. I., p. 77.

5. *The directrix of a parabola passes through the polar centre of every circumscribed triangle.*

This follows from (1) by sending one side of the quadrilateral to infinity. For the circles on the diagonals of the quadrilateral as diameters become then the perpendiculars of the triangle.

6. *The polar centres of the triangles formed by four straight lines lie on the line joining the foci of the quadrilateral.*

This line is the directrix of the inscribed parabola, by (1).

XLVIII.

ON TRIANGULAR SYMMETRY*.

We make the properties of a conic intuitive by studying it under the form of a circle; or those of a quadrilateral, by studying it under the form of a square. This simplification depends upon the projective property of a right angle, viz., that it divides harmonically the chord at infinity of a circle. By means of this property we interpret as general, propositions whose truth we see intuitively through the symmetry of the figure. This kind of symmetry (that of a circle or square) I call the *symmetry of the right angle,* or *rectangular symmetry.*

From the symmetry of two lines we ascend immediately to the symmetry of three lines, or of the equilateral triangle. This is exemplified in the Rhombohedral System of Crystals, just as Rectangular Symmetry is exemplified in the Pyramidal System. I want in this note to call attention to the uses of Triangular Symmetry in presenting general propositions under an intuitive form.

The projective property of an equilateral triangle is this: *it determines on the line at infinity a point-cubic whose Hessian is the circular points.* Given four lines, we may project one of them to infinity so that the other three shall form an equilateral triangle; for we have only to construct the Hessian of the point-cubic determined upon that one by the other three, and then project this Hessian into the circular points. Similar problems are

B. To project at once a conic and triangle into a circle and an equilateral triangle.

* [From *Mathematics from the Educational Times*, Vol. IV. pp. 88, 9.]

C. To project at once two conics and a triangle into two rectangular hyperbolæ and an equilateral triangle.

D. To project at once two triangles into equilateral triangles.

In each of these we have the problem of finding the line which has to be projected to infinity; this problem admits, in the three cases respectively, of 43, 76, and 108 solutions. The *triangle* might of course be replaced by a *cubic*, to be so projected that its asymptotes should form an equilateral triangle; but this case is not particularly interesting. Every cubic may be projected into a perfectly symmetrical form in this way:—the three real inflexions of the cubic lie on a certain straight line, and determine a point-cubic upon it; let the Hessian of this point-cubic be projected into the circular points at infinity. Then the cubic is symmetrically situated in respect of an equilateral asymptotic triangle.

The form in Fig. [58] may be called *inscribed*, that in Fig. [59] *escribed.* The inscribed cubic may have no oval nor double point, and then will lie entirely without the triangle.

Cubics with a crunode cannot be thus symmetrized; but their reciprocals can In fact, every three-cusped quartic can be projected into a hypocycloid of three branches. For, any four points can be projected into any other four points; if then the three cusps and the intersection of the cuspidal tangents be projected respectively into the vertices and centre of an equilateral triangle, the thing is done. This proof is Prof. Cayley's. We learn in this way that, in any three-cusped quartic, the cuspidal tangents and the lines joining the cusps determine on the double tangent two point-cubics, whose common Hessian is the points of contact of that tangent.

Trinodal quartics have four double tangents, which are all real as lines when the nodes are real, but one is ideal (I borrow this convenient expression from Poncelet) or has imaginary contacts. If the three pairs of nodal tangents divide the ideal double tangent in the same anharmonic ratio, the quartic can be projected into a hypotrochoid, the contacts of the ideal

tangent being then the circular points at infinity. The curve has many remarkable properties, which can be recognized at once from the symmetry of the projected figure.

Higher orders of symmetry are special cases. Thus quartic symmetry (as of a regular octogon) requires that the point-quartic shall be an harmonic system, so that its cubinvariant vanishes. Quintic Symmetry (regular pentagon) requires a point-quintic of the form ax^5+fy^5. The conditions that the quintic may be reduced to this form are that the invariants K of the eighth degree and L of the twelfth shall separately vanish (see Professor Sylvester's admirable *Trilogy*, *Phil. Trans.* 1864, p. 619).

The cube and sphere are examples of the Symmetry of the *cubangle*, whose projective property is that it determines on the plane at infinity a conjugate triad of the imaginary circle. One is naturally led to seek for the projective property of a regular tetrahedron. It determines on the plane at infinity four straight lines x, y, z, w; and if we assume the identical relation

$$x+y+z+w\equiv 0,$$

then the equation of the imaginary circle is

$$x^2+y^2+z^2+w^2=0,$$

as is easily shewn by the consideration that each face is an equilateral triangle. Many interesting properties of this conic will be found proved in the solution to Question 1690*.

* [*Mathematics from the Educational Times*, Vol. III. pp. 92—96.]

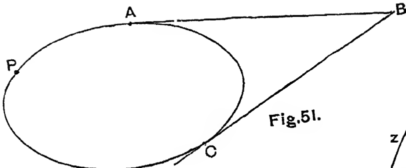


Fig.51.

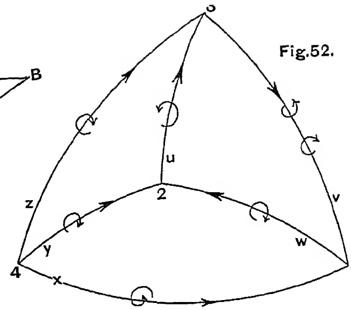


Fig.52.

Fig.53.

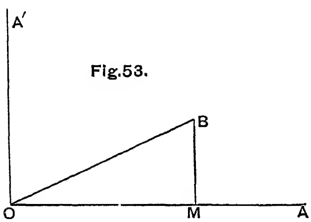
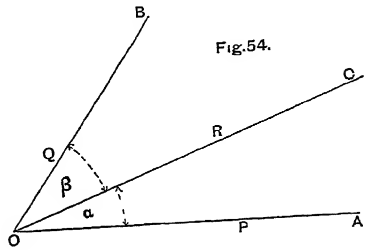


Fig.54.



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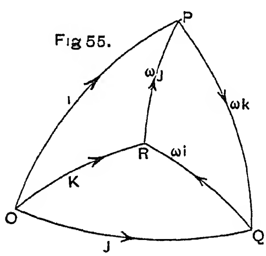


Fig. 56.

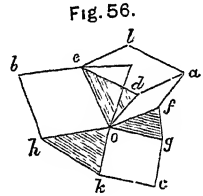


Fig. 57.

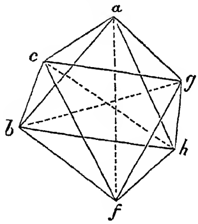


Fig. 58.

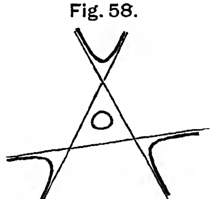
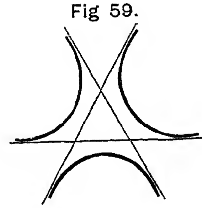


Fig 59.



XLIX.

ON SOME EXTENSIONS OF THE FUNDAMENTAL PROPOSITION IN M. CHASLES'S THEORY OF CHARACTERISTICS*.

I MEAN by the "fundamental proposition" the following, viz.:— "If a variable system of two points on a right line be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily, the second has b positions; then there are $a+b$ points on the right line at which the system of two points coalesces into one point."

This principle has been admirably extended by Dr Salmon to the case of two dimensions, thus:—"If a variable system of two points in a plane be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions, and that p pairs of points, each constituting a position of the system, may be found upon an arbitrary right line; then there are $a+b+p$ points in the plane at which the system of two points coalesces into one point."

The principle admits of further extension in two directions. First, we may consider a system of more than two points; and secondly, we may consider the systems as subject to a less number of relations than is sufficient to determine a single point. We are thus led to the following propositions:—

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point

* [From *Mathematics from the Educational Times*, Vol. v. pp. 49, 50.]

is determined to lie on a surface of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a surface of order b, and so on; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a surface of order $a+b+\ldots$

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a curve of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a curve of order b, and so on, and that when all but the first two points are taken arbitrarily there are on an arbitrary right line p pairs of points each constituting a position of the first two points, and that $q, r \ldots$ are the corresponding numbers for the other pairs of points of the system; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a curve of order $a+b+\ldots+p+q+r+\ldots$

It is not worth while to state the analogous propositions for Geometry of one and two dimensions, or the correlative propositions for lines and planes. I go on to exemplify the application of these propositions.

Let us begin with Mr Thomson's cubic (*Reprint*, Vol. II. p. 57). A conic is inscribed in a triangle so that the normals at the points of contact meet in a point; it is required to find the locus of this point. Consider now a variable system of three points, subject only to this condition; that if perpendiculars be drawn from them respectively to the three sides of a triangle, a conic may be drawn touching the sides of the triangle at the feet of those perpendiculars. Then, if we take two of the points arbitrarily, we determine two of the points of contact of the inscribed conic; that is we determine the conic itself uniquely, and therefore the third point of contact; and the normal at this point is therefore the locus of the third point of the system. That is to say, we have a variable system of three points so related that when any two of the points are taken arbitrarily, the locus of the third is a straight line;

consequently there are points in the plane at which the system of three points coalesces into one point (that is, where the three normals meet in a point), and the locus of such points is a curve of order $(1 + 1 + 1 =) 3$.

To complicate the question, let us suppose that a conic is drawn to touch three given *conics*, so that the normals at the points of contact may meet in a point. Here, as before, we take our variable system of three points, one on each of the three normals. Take two of the points arbitrarily; from each of these we can draw four normals to the corresponding conic. Pairing these together, we have 16 pairs of points of contact. Now when we have given two tangents and their points of contact, the number of conics of the system which can be drawn to touch a given conic is 4. By determining two points of our variable system we have, therefore, determined 64 conics; on the third *given* conic, these determine 64 points of contact, and the normals through these may be held to constitute a curve of the 64th order. Thus we have a variable system of three points so related that when any two of them are taken arbitrarily, the third is determined to lie on a curve of the 64th order; consequently the locus of those points at which the system coalesces into one point, or the three normals meet in a point, is a curve of the order $(3 \times 64) = 192$. More generally, if we substitute for the variable conic a curve of order m, class n, and deficiency D, or say a curve of *species* (m, n, D), and for the three fixed conics $\frac{1}{2}(m + n - D + 2)$ curves of orders $m_1, m_2, \ldots$ and of classes $n_1, n_2, \ldots$, then the corresponding locus will be of the order $3\phi(m, n, D).(m_1+n_1)(m_2+n_2)\ldots$ where $\phi(m, n, D)$ is the number of curves of species (m, n, D) which can be drawn through $(m + n - D + 1)$ points, or touching $(m + n - D + 1)$ lines.

For another example, let us find the locus of those points the feet of the perpendiculars from which to four lines or planes in space are coplanar. In both these cases the locus comes out primarily of the fourth order; but the plane at infinity is evidently a part of the locus, the remainder of which is thus of

the third order. In both cases the envelope of the plane through the feet of the perpendiculars is of the fourth class, and touches the plane at infinity. I conjecture that the imaginary circle is a curve of contact.

If a conicoid be drawn to touch five straight lines, so that the normal planes at the points of contact meet in a point, the locus of this point is of the tenth order. And so on *ad libitum.*

L*.

INSTRUMENTS USED IN MEASUREMENT.

By *Measurement*, for scientific purposes, is meant the measurement of *quantities*. In each special subject there are quantities to be measured; and these are very various, as may be seen from the following list of those belonging to geometry and dynamics.

Geometrical Quantities.

Lengths
Areas
Volumes
Angles (plane and solid)
Curvatures (plane and solid)
Strains (elongation, torsion, shear).

Circumstances of Motion.	*Properties of Bodies.*
Time	Mass
Velocity	Weight
Momentum	Density
Acceleration	Specific gravity
Force	Elasticity (of form and volume)
Work	Viscosity
Horse-power	Diffusion
Temperature	Surface tension
Heat	Specific heat.

* [*Handbook to the Special Loan Collection of Scientific Apparatus*, 1876, L. pp. 55—59, LI. pp. 60—77. The object of L., LI. and other introductory Notices in the "Handbook" was to "make the Exhibition as useful and interesting as possible."]

Notwithstanding the very different characters of these quantities, they are all measured by reducing them to the same kind of quantity, and estimating that in the same way. Every quantity is measured by finding a *length* proportional to the quantity, and then measuring this length. This will, perhaps, be better understood if we consider one or two examples.

The measurement of *angles* occurs in a very large majority of scientific instruments. It is always effected by measuring the *length of an arc* upon a graduated circle; the circumference of this circle being divided not into inches or centimeters, but into degrees and parts of a degree—that is, into aliquot parts of the whole circumference.

As a step towards their final measurement, some quantities, of which work is a good instance, are represented in the form of *areas;* and there seems reason to believe that this method is likely to be extended. Instruments for measuring areas are called Planimeters; and one of the simplest of these is Amsler's, consisting of two rods jointed together, the end of one being fixed and that of the other being made to run round the area which is to be measured. The second rod rests on a wheel, which turns as the rod moves; and it is proved by geometry that the area is proportional to the distance through which the wheel turns. Thus the measurement of an area is reduced to the measurement of a length.

Volumes are measured in various ways, but all depending on the same principle. Quantities of earth excavated for engineering purposes are estimated by a rough determination of the shape of the cavity, and the measurement of its *dimensions,* namely, certain lengths belonging to it. The contents of a vessel are sometimes gauged in the same way; but the more accurate method is to fill it with liquid and then pour the liquid into a cylinder of known section, when the quantity is measured by the height of the liquid in the cylinder, that is, by a length. The volumes of irregular solids are also measured by immersing them in liquid contained in a uniform cylinder, and observing the height to which the liquid rises; that is, by measuring a length. An apparatus for this purpose is called a

Stereometer. The liquid must be so chosen that no chemical action takes place between it and the solid immersed, and that it wets the solid so that no air bubbles adhere to the surface. Thus mercury is used in the case of metals by the Standards Department.

Time is measured for ordinary purposes by the length of the arc traced out by a moving hand on a circular clock-face. For astronomical purposes it is sometimes measured by counting the ticks of a clock which beats seconds, and estimating mentally the fractions of a second; and in cases where the period of an oscillation has to be found, it is determined by counting the number of oscillations in a time sufficient to make the number considerable, and then dividing that time by the number. But by far the most accurate way of measuring time is by means of the line traced by a pencil on a sheet of paper rolled round a revolving cylinder, or a spot of light moving on a sensitive surface. If the pencil is made to move along the length of the cylinder so as to indicate what is happening as time goes along, the time of each event will be found when the cylinder is unrolled by measuring the distance of the mark recording it from the end of the unrolled sheet, provided that the rate at which the cylinder goes round is known. In this way Helmholtz measured the rate of transmission of nerve-disturbance.

A very common case of the measurement of *force* is the barometer, which measures the pressure of the atmosphere per square inch of surface. This is determined by finding the height of the column of mercury which it will support (mercurial barometer), or the strain which it causes in a box from which the air has been taken out (aneroid barometer). The height in the former case may be measured directly, or it may first be converted into the quantity of turning of a needle, and then read off as length of arc on a graduated circle; in the latter case the strain is always indicated by a needle turning on a graduated circle.

The *mass*, and (what is proportional to it) the *weight*, of different bodies at the same place, are measured by means of a balance; and at first sight this mode of measurement seems different from those which we have hitherto considered. For we

put the body to be weighed in one scale, and then put known weights into the other until equilibrium is obtained or the scale turns, and then we count the weights. But in a steelyard the weight is determined directly by means of a length; and in a balance which is accurate enough for scientific purposes, both methods are employed. We get as near as we can with the weights, and then the remainder is measured by a small rider of wire which is moved along the beam, and which determines the weight by its position; that is, by the measurement of a length.

For the measurement of weight in different places a spring-balance has to be used, and the weight is determined by the alteration it produces in the length of the spring; or else the length of the seconds pendulum is measured, from which the force of gravity on a given mass can be calculated. This last is an example of a very common and useful mode of measuring forces called into play by displacement or strain; namely, by measuring the period of the oscillations which they produce.

It seems unnecessary to consider any further examples, as all other quantities are measured by means of some simple geometrical or dynamical quantity which is proportional to them; as temperature by the height of mercury in a thermometer, heat by the quantity of ice it will melt (the volume of the resulting water), electric resistance by the length of a standard wire which has an equivalent resistance. It only remains to show how, when a length has been found proportional to the quantity to be measured, this length itself is measured.

For rough purposes, as for example in measuring the length of a room with a foot-rule, we apply the rule end on end, and count the number of times. For the piece left, we should apply the rule to it and count the number of inches. Or if we wanted a length expressed roughly for scientific purposes, we should describe it in metres or centimetres. But if it has to be expressed with greater accuracy, it must be described in hundredth, or thousandth, or millionth parts of a millimetre; and this is still done by comparing it with a scale.

But in order to estimate a length in terms of these very small quantities, it must be *magnified;* and this is done in three ways. First, geometrically, by what is called a vernier scale. This is a movable scale, which gains on the fixed one by one-tenth in each division. To measure any part of a division, we find how many divisions it takes the vernier to gain so much as that part; this is how many tenths the part is. The quantity to be measured is here geometrically multiplied by ten. Next, optically, by looking at the length and scale with a microscope or telescope. Third, mechanically, by a screw with a disc on its head, on which there is a graduated rim, called a micrometer screw. If the pitch of the screw is one-tenth and the radius of the disc ten times that of the screw, the motion is multiplied by one hundred. The two latter modes are combined together in an instrument called a micrometer-microscope. Another mechanical multiplier is a mirror which turns round and reflects light on a screen at some distance, as in Thomson's reflecting galvanometer.

Properly speaking, however, any description of a length by counting of standard lengths is imperfect and merely approximate. The true way of indicating a length is to draw a straight line which represents it on a fixed scale. And this is done by means of self-recording instruments, which measure lengths from time to time on a cylinder in the manner described above. It is only by this graphical representation of quantities that the laws of their variation become manifest, and that higher branch of measurement becomes possible which determines the nature of the connection between two simultaneously varying quantities.

LI.

INSTRUMENTS ILLUSTRATING KINEMATICS, STATICS, AND DYNAMICS.

Science of Motion.

GEOMETRY teaches us about the sizes, the shapes, and the distances of things; to know sizes and distances we have to measure *lengths*, and to know shapes we have to measure *angles*. The science of *Motion*, which is the subject of the present sketch, tells us about the changes in these sizes, shapes, and distances which take place from time to time. A body is said to move when it changes its place or position; that is to say, when it changes its distance from surrounding objects. And when the parts of a body move relatively to one another, *i.e.* when they alter their distance from one another, the body changes in size, or shape, or both. All these changes are considered in the science of motion.

Kinematics.

It is divided into two parts; the accurate description of motion, and the investigation of the circumstances under which particular motions take place. The description of motion may again be divided into two parts, namely, that which tells us *what* changes of position take place, and that which tells us *when* and *how fast* they take place. We might, for example, describe the motion of the hands of a clock, and say that they turn round on their axes at the centre of the clock-face in such a way that the minute-hand always moves twelve times as much as the hour-hand; this is the first part of the description of the motion. We might go on to say that when the clock is going correctly this motion takes place uniformly, so that the minute-

hand goes round once in each hour; and this would be the second part of the description. The first part is what was called Kinematics by Ampère; it tells us how the motions of the different parts of a machine depend on each other in consequence of the machinery which connects them. This is clearly an application of geometry alone, and requires no more measurements than those which belong to geometry, namely, measurements of lines and angles. But the name Kinematics is now conveniently made to include the second part also of the description of motion—when and how fast it takes place. This requires in addition the measurement of *time*, with which geometry has nothing to do. The word Kinematic is derived from the Greek *kinēma*, "motion;" and will therefore serve equally well to bear the restricted sense given it by Ampère, and the more comprehensive sense in which it is now used. And since the principles of this science are those which guide the construction not only of scientific apparatus, but of all instruments and machines, it may be advisable to describe in some detail the chief topics with which it deals.

Dynamics.

That part of the science which tells us about the circumstances under which particular motions take place is called *Dynamics*. It is found that the change of motion in a body depends on the position and state of surrounding bodies, according to certain simple laws; when considered as so depending on surrounding bodies, the rate of change in the quantity of motion is called *force*. Hence the name Dynamic, from the Greek *dynamis*, "force." The word *force* is here used in a technical sense, peculiar to the science of motion; the connection of this meaning with the meaning which the word has in ordinary discourse will be explained further on.

Statics and Kinetics.

Dynamics are again divided into two branches; the study of those circumstances in which it is possible for a body to remain at rest is called Statics, and the study of the circum-

stances of actual motion is called Kinetics. The simplest part of Statics, the doctrine of the Lever, was successfully studied before any other part of the science of motion, namely, by Archimedes, who proved that when a lever with unequal arms is balanced by weights at the ends of it, these weights are inversely proportional to the arms. But no real progress could be made in determining the conditions of rest, until the laws of actual motion had been studied.

Translation of Rigid Bodies.

Returning, then, to the description of motion, or Kinematics, we must first of all classify the different changes of position, of size, and of shape, with which we have to deal. We call a body *rigid* when it changes only its position, and not its size or shape, during the time in which we consider it. It is probable that every actual body is constantly undergoing slight changes of size and shape, even when we cannot perceive them; but in Kinematics, as in most other matters, there is a great convenience in talking about only one thing at a time. So we first of all investigate changes of position on the assumption that there are no changes of size and shape; or, in technical phrase, we treat of the motion of rigid bodies. Here an important distinction is made between motion in which the body merely travels from one place to another, and motion in which it also turns round. Thus the wheels of a locomotive engine not only travel along the line, but are constantly turning round; while the coupling-bar which joins two wheels on the same side remains always horizontal, though its changes of position are considerably complicated. A change of place in which there is no rotation is called a *translation*. In a rotation the different parts of the body are moving different ways, but in a translation all parts move in the same way. Consequently, in describing a translation we need only specify the motion of any one particle of the moving body; where by a *particle* is meant a piece of matter so small that there is no need to take account of the differences between its parts, which may therefore be treated for purposes of calculation as a point.

We are thus brought down to the very simple problem of describing the motion of a point. Of this there are certain cases which have received a great deal of attention on account of their frequent occurrence in nature; such as Parabolic Motion, Simple Harmonic Motion, Elliptic Motion. We propose to say a few words in explanation of each of these.

Parabolic Motion.

The motion of a *projectile*, that is to say, of a body thrown in any direction and falling under the influence of gravity, was investigated by Galileo; and this is the first problem of Kinetics that was ever solved. We must confine ourselves here to a description of the motion, without considering the way in which it depends on the circumstance of the presence of the earth at a certain distance from the moving body. Galileo found that the path of such a body, or the curve which it traces out, is a parabola; a curve which may be described as the shadow of a circle cast on a horizontal table by a candle which is just level with the highest point of the circle.

It is convenient to consider separately the vertical and the horizontal motion, for in accordance with a law subsequently stated in a general form by Newton, these two take place in complete independence of one another. So far as its horizontal motion is concerned, the projectile moves uniformly, as if it were sliding on perfectly smooth ice; and, so far as its vertical motion is concerned, it moves as if it were falling down straight. The nature of this vertical motion may be described in two ways, each of which implies the other. First, a falling body moves faster and faster as it goes down; and the rate at which it is going at any moment is strictly proportional to the number of seconds which has elapsed since it started. Thus its downward velocity is continually being added to at a uniform rate. Secondly, the whole distance fallen from the starting-point is proportional to the *square* of the number of seconds elapsed; thus, in three seconds a body will fall nine times as far as it will fall in one second. The latter of these statements was experimentally proved by Galileo; not, however, in the case of bodies falling vertically, which move too quickly for the

time to be conveniently measured, but in the case of bodies falling down inclined planes, the law of which he at first assumed, and afterwards proved to be identical with that of the other. The former statement, that the velocity increases uniformly, is directly tested by an apparatus known as Attwood's machine, consisting essentially of a pulley, over which a string is hung with equal weights attached to its ends. A small bar of metal is laid on one of the weights, which begins to descend and pull the other one up; after a measured time the bar is lifted off, and then, both sides pulling equally, the motion goes on at the rate which had been acquired at that instant. The distance travelled in one second is then measured, and gives the velocity; this is found to be proportional to the time of falling with the bar on.

The second statement, that the space passed over is proportional to the square of the number of seconds elapsed, is verified by Morin's machine, which consists of a vertical cylinder which revolves uniformly while a body falling down at the side marks it with a pencil. The curve thus described is a record of the distance the body had fallen at every moment of time.

Fluxions.

This investigation of Galileo's was in more than one aspect the foundation of dynamical science; but not the least important of these aspects is the proof that either of the two ways of stating the law of falling bodies involves the other. Given that the distance fallen is proportional to the square of the time, to show that the velocity is proportional to the time itself; this is a particular case of the problem. Given where a body is at every instant, to find how fast it is going at every instant. The solution of this problem was given by Newton's Method of Fluxions. When a quantity changes from time to time, its *rate* of change is called the *fluxion* of the quantity. In the case of a moving body the quantity to be considered is the distance which the body has travelled; the fluxion of this distance is the rate at which the body is going. Newton's method solves the problem, Given *how big* a quantity is at

any time, to find its fluxion at any time. The method has been called on the Continent, and lately also in England, the Differential Calculus; because the difference between two values of the varying quantity is mentioned in one of the processes that may be used for calculating its fluxion. The inverse problem, Given that the velocity is proportional to the time elapsed, to find the distance fallen, is a particular case of the general problem, Given how fast a body is going at every instant, to find where it is at any instant; or, Given the fluxion of a quantity, to find the quantity itself. The answer to this is given by Newton's Inverse Method of Fluxions; which is also called the Integral Calculus, because in one of the processes which may be used for calculating the quantity, it is regarded as a whole (integer) made up of a number of small parts. The method of Fluxions, then, or Differential and Integral Calculus, takes its start from Galileo's study of parabolic motion.

Harmonic Motion.

The ancients, regarding the circle as the most perfect of figures, believed that circular motion was not only *simple*, that is, not made up by putting together other motions, but also *perfect*, in the sense that when once set up in perfect bodies it would maintain itself without external interference. The moderns, who know nothing about perfection except as something to be aimed at, but never reached, in practical work, have been forced to reject both of these doctrines. The second of them, indeed, belongs to Kinetics, and will again be mentioned under that head. But as a matter of Kinematics it has been found necessary to treat the uniform motion of a point round a circle as compounded of two oscillations. To take again the example of a clock, the extreme point of the minute-hand describes a circle uniformly; but if we consider separately its vertical position and its horizontal position, we shall see that it not only oscillates up and down, but at the same time swings from side to side, each in the same period of one hour. If we suppose a button to move up and down in a slit between the figures XII. and VI., in such a way as to be always at the

same height at the end of the minute-hand, this button will have only one of the two oscillations which are combined in the motion of that point; and the other oscillation would be exhibited by a button constrained to move in a similar manner between the figures III. and IX., so as always to be either vertically above or vertically below the extreme point of the minute-hand. The laws of these two motions are identical, but they are so timed, that each is at its extreme position when the other is crossing the centre. An oscillation of this kind is called a *simple harmonic motion;* the name is due to Sir William Thomson, and was given on account of the intimate connection between the laws of such motions and the theory of vibrating strings. Indeed, the harmonic motion, simple or compound, is the most universal of all forms; it is exemplified not only in the motion of every particle of a vibrating solid, such as the string of a piano or violin, a tuning-fork, or the membrane of a drum, but in those minute excursions of particles of air which carry sound from one place to another, in the waves and tides of the sea, and in the amazingly rapid tremor of the luminiferous ether which, in its varying action on different bodies, makes itself known as light or radiant heat or chemical action. Simple harmonic motions differ from one another in three respects; in the extent or *amplitude* of the swing, which is measured by the distance from the middle point to either extreme; in the *period* or interval of time between two successive passages through an extreme position; and in the time of starting, or *epoch,* as it is called, which is named by saying what particular stage of the vibration was being executed at a certain instant of time. One of the most astonishing and fruitful theorems of mathematical science is this; that every *periodic* motion whatever, that is to say, every motion which exactly repeats itself again and again at definite intervals of time, is a compound of simple harmonic motions, whose periods are successively smaller and smaller aliquot parts of the original period, and whose amplitudes (after a certain number of them) are less and less as their periods are more rapid. The "harmonic" tones of a string, which are always heard along with the fundamental tone, are a particular case of

these constituents. The theorem was given by Fourier in connection with the flow of heat, but its applications are innumerable, and extend over the whole range of physical science.

The laws of combination of harmonic motions have been illustrated by some ingenious apparatus of Messrs Tisley and Spiller, and by a machine invented by Mr Donkin; but the most important practical application of these laws is to be found in Sir W. Thomson's Tidal Clock, and in a more elaborate machine which draws curves predicting the height of the tide at a given port for all times of the day and night with as much accuracy as can be obtained by direct observation. One special combination is worthy of notice. The union of a vertical vibration with a horizontal one of half the period gives rise to that figure of 8 which M. Marey has observed by his beautiful methods in the motion of the tip of a bird's or insect's wing.

Elliptic Motion.

The motion of the sun and moon relative to the earth was at first described by a combination of circular motions; and this was the immortal achievement of the Greek astronomers Hipparchus and Ptolemy. Indeed, in so far as these motions are periodic, it follows from Fourier's theorem mentioned above that this mode of description is mathematically sufficient to represent them; and astronomical tables are to this day calculated by a method which practically comes to the same thing. But this representation is not the simplest that can be found; it requires theoretically an infinite number of component motions, and gives no information about the way in which these are connected with one another. We owe to Kepler the accurate and complete description of planetary or elliptic motion. His investigations applied in the first instance to the orbit of the planet Mars about the sun, but it was found true of the orbits of all planets about the sun, and of the moon about the earth. The path of the moving body in each of these motions, is an ellipse, or oval shadow of a circle, a curve having various properties in relation to two internal points or foci, which replace as it were the one centre of a circle. In the

case of the ellipse described by a planet, the sun is in one of these foci; in the case of the moon, the earth is one focus. So much for the geometrical description of the motion. Kepler further observed that a line drawn from the sun to a planet, or from the earth to the moon, and supposed to move round with the moving body, would sweep out equal areas in equal times. These two laws, called Kepler's first and second laws, complete the kinematic description of elliptic motion; but to obtain formulæ fit for computation, it was necessary to calculate from these laws the various harmonic components of the motion to and from the sun, and round it; this calculation has much occupied the attention of mathematicians.

The laws of rotatory motion of rigid bodies are somewhat difficult to describe without mathematical symbols, but they are thoroughly known. Examples of them are given by the apparatus called a gyroscope, and the motion of the earth; and an application of the former to prove the nature of the latter, made by Foucault, is one of the most beautiful experiments belonging entirely to dynamics.

Rotation.

Next in simplicity after the *translation* of a rigid body, come two kinds of motion which are at first sight very different, but between which a closer observation discovers very striking analogies. These are the motion of rotation about a fixed point, and the motion of sliding on a fixed plane. The first of these is most easily produced in practice by what is well known as a ball-and-socket joint; that is to say, a body ending in a portion of a spherical surface which can move about in a spherical cavity of the same size. The centre of the spherical surface is then a fixed point, and the motion is reduced to the sliding of one sphere inside another. In the same way, if we consider, for instance, the motion of a flat-iron on an ironing-board, we may see that this is not a pure translation, for the iron is frequently turned round as well as carried about; but the motion may be described as the sliding of one plane upon another. Thus in each case the matter to be studied is the sliding of one surface on another which it exactly fits. For

two surfaces to fit one another exactly, in all positions, they must be either both spheres of the same size, or both planes; and the latter case is really included under the former, for a plane may be regarded as a sphere whose radius has increased without limit. Thus, if a piece of ice be made to slide about on the frozen surface of a perfectly smooth pond, it is really rotating about a fixed point at the centre of the earth; for the frozen surface may be regarded as part of an enormous sphere, having that point for centre. And yet the motion cannot be practically distinguished from that of sliding on a plane.

In this latter case it is found that, excepting in the case of a pure translation, there is at every instant a certain point which is at rest, and about which as a centre the body is turning. This point is called the instantaneous centre of rotation; it travels about as the motion goes on, but at any instant its position is perfectly definite. From this fact follows a very important consequence; namely, that every possible motion of a plane sliding on a plane may be produced by the *rolling* of a curve in one plane upon a curve in the other. The point of contact of the two curves at any instant is the instantaneous centre at that instant. The problems to be considered in this subject are thus of two kinds: Given the curves of rolling to find the path described by any point of the moving plane; and, Given the paths described by *two* points of the moving plane (enough to determine the motion) to find the curves of rolling and the paths of all other points. An important case of the first problem is that in which one circle rolls on another, either inside or outside; the curves described by points in the moving plane are used for the teeth of wheels. To the second problem belongs the valuable and now rapidly increasing theory of *link-work,* which, starting from the wonderful discovery of an exact parallel motion by M. Peaucellier, has received an immense and most unexpected development at the hands of Professor Sylvester, Mr Hart, and Mr A. B. Kempe.

Passing now to the spherical form of this motion, we find that the instantaneous centre of rotation (which is clearly equivalent to an instantaneous axis perpendicular to the plane) is replaced by an instantaneous axis passing through the com-

mon centre of the moving spheres. In the same way the rolling of one curve on another in the plane is replaced by the rolling of one *cone* upon another, the two cones having a common vertex at the same centre.

Analogous theorems have been proved for the most general motion of a rigid body. It was shown by M. Chasles that this is always similar to the motion of a corkscrew descending into a cork; that is to say, there is always a rotation about a certain instantaneous axis, combined with translation along this axis. The amount of translation per unit of rotation is called the *pitch* of the screw. The instantaneous screw moves about as the motion goes on, but at any given instant it is perfectly definite in position and pitch. And any motion whatever of a rigid body may be produced by the rolling and sliding of one surface on another, both surfaces being produced by the motion of straight lines. This crowning theorem in the geometry of motion is due to Professor Cayley. The laws of combination of screw motions have been investigated by Dr Ball.

Thus, proceeding gradually from the more simple to the more complex, we have been able to describe every change in the position of a body. It remains only to describe changes of size and shape. Of these there are three kinds, but they are all included under the same name—*strains*. We may have, first, a change of size without any change of shape, a uniform dilatation or contraction of the whole body in all directions, such as happens to a sphere of metal when it is heated or cooled. Next, we may have an elongation or contraction in one direction only, all lines of this body pointing in this direction being increased or diminished in the same ratio; such as would happen to a rod six feet long and an inch square, if it were stretched to seven feet long, still remaining an inch square. Thirdly, we may have a change of shape produced by the sliding of layers over one another, a mode of deformation which is easily produced in a pack of cards; this is called a *shear*. By appropriate combinations of these three, every change of size and shape may be produced; or we may even leave out the second element, and produce any strain whatever by a dilatation or contraction, and two shears.

Dynamics.

We have already said that the change of motion of a body depends upon the position and state of surrounding bodies. To make this intelligible it will be necessary to notice a certain property of the three kinds of motion of a point which we described.

The combination of velocities may be understood from the case of a body carried in any sort of cart or vehicle in which it moves about. The whole velocity of the body is then compounded of the velocity of the vehicle and of its velocity relative to the vehicle. Thus, if a man walks across a railway carriage his whole velocity is compounded of the velocity of the railway carriage and of the velocity with which he walks across.

When the velocity of a body is changed by adding to it a velocity in the same direction or in the opposite direction, it is only altered in amount; but when a transverse velocity is compounded with it, a change of direction is produced. Thus, if a man walks fore and aft on a steamboat, he only travels a little faster or slower; but if he walks across from one side to the other, he slightly changes the direction in which he is moving.

Now, in the parabolic motion of a projectile, we found that while the horizontal velocity continues unchanged, the vertical velocity increases at a uniform rate. Such a body is having a downwards velocity continually poured into it, as it were. This gradual change of the velocity is called *acceleration;* we may say that the acceleration of a projectile is always the same, and is directed vertically downwards.

In a simple harmonic motion it is found that the acceleration is directed towards the centre, and is always proportional to the distance from it. In the case of elliptic motion it was proved by Newton that the acceleration is directed towards the focus, and is inversely proportional to the square of the distance from it.

Let us now consider the circumstances under which these motions take place. To produce a simple harmonic motion we may take a piece of elastic string, whose length is equal to the

height of a smooth table; then fasten one end of the string to a bullet and the other end to the floor, having passed it through a hole in the table, so that the bullet just rests on the top of the hole when the string is unstretched. If the bullet be now pulled away from the hole so that the string is stretched, and then let go, it will oscillate to and fro on either side of the hole with a simple harmonic motion. The acceleration (or rate of change of velocity) is here proportional to the distance from the hole; that is, to the *amount of elongation of the string.* It is directed towards the hole; that is, in the direction of this elongation. In the case of the moon moving round the earth, the acceleration is directed towards the earth, and is inversely proportional to the square of the distance from the earth.

In both these cases, then, the change of velocity depends upon surrounding circumstances; but in the case of the bullet, this circumstance is the strained condition of an adjoining body, namely, the elastic string; while in the case of the moon the circumstance is the position of a distant body, namely, the earth. The motion of a projectile turns out to be only a special case of the motion of the moon; for the parabola which it describes may be regarded as one end of a very long ellipse, whose other end goes round the earth's centre.

There is a remarkable difference between the two cases. The swing of the bullet depends upon its size; a large bullet will oscillate more slowly than a small one. This leads us to modify the rule. If a large bullet is equivalent to two small ones, then when it is going at the same rate it must contain twice as much motion as one of the small ones; or, as we now say, with the same velocity it has twice the *momentum.* Now the change of momentum is found to be the same for all bullets, when the momentum is reckoned as proportional to the quantity of matter in the bullet as well as to the velocity. The quantity of matter in a body is called its *mass;* for bodies of the same substance it is, of course, simply the quantity of that substance; but for bodies of different substances it is so reckoned as to make the rule hold good. The rule for this case may then be stated thus; the change of momentum of a body (that is, the change of velocity multiplied by the mass), depends

on the state of strain of adjoining bodies. Regarded as so depending, this change of momentum is called the *pressure* or *tension* of the adjoining body, according to the nature of the strain; both of these are included in the name *stress,* introduced by Rankine.

But in the case of projectiles, the acceleration is found to be the same for all bodies at the same place; and this rule holds good in all cases of planetary motion. So that it seems as if the change of velocity, and not the change of momentum, depended upon the position of distant bodies. But this case is brought under the same rule as the other by supposing that the mass of the moving body is to be reckoned among the "circumstances." The change of momentum is in this case called the attraction of gravitation, and we say that the attraction is proportional to the mass of the attracted body. And this way of representing the facts is borne out by the electrical and magnetic attractions and repulsions, where the change of momentum depends on the position and state of the attracting thing, and upon the electric charge or the induced magnetism of the attracted thing.

Force, then, is of two kinds; the stress of a strained adjoining body, and the attraction or repulsion of a distant body. Attempts have been made with more or less success to explain each of these by means of the other. In common discourse the word "force" means muscular effort exerted by the human frame. In this case the part of the human body which is in contact with the object to be moved is in a state of strain, and the force, dynamically considered, is of the first kind. But this state of strain is preceded and followed by nervous discharges, which are accompanied by the sensations of effort and of muscular strain; a complication of circumstances which does not occur in the action of inanimate bodies. What is common to the two cases is, that the change of momentum depends on the strain.

Having thus explained the law of Force, which is the foundation of Dynamics, we may consider the remaining laws of motion. It is convenient to state them first for particles, or bodies so small that we need take account only of their position.

Every particle, then, has a rate of change of momentum due to the position or state of every other particle, whether adjoining it or distant from it. These are compounded together by the law of composition of velocities, and the result of the whole is the actual change of momentum of the particle. This statement, and the law of Force stated above, amount together to Newton's first and second laws of motion. His third law is, that the change of momentum in one particle, due to the position or state of another, is equal and opposite to the change of momentum in the other, due to the position or state of the first.

By the help of these laws D'Alembert showed how the motion of rigid bodies, or systems of particles, might be dealt with. It appears from his method that two stresses, acting on a rigid body, may be equivalent, in their effect on the body as a whole, to a single stress, whose direction and position will be totally independent of the shape and nature of the body considered. The law of combination of stresses acting on a system of particles is, in fact, the same as the law of combination of velocities, so far as regards the motion of the system as a whole. This beautiful but somewhat complex result of Dynamics has been used in some text-books as the independent foundation of Statics, under the name of the parallelogram of forces; a singular inversion of the historical order and of the methods of the great writers.

When the result of all the circumstances surrounding a body is that there is no change of momentum, the body is said to be in equilibrium. In this case, if the body is at rest, it will remain so; and on this account the study of such conditions is called Statics. In dealing with the statics of rigid bodies, we have only to examine those cases in which the resultant of the external stresses and attractions acting on the body amounts to nothing. But the most important part of statics is that which finds the stresses acting in the interior of bodies between contiguous parts of them; for upon this depends the determination of the requisite strength of structures which have to bear given loads. It is found that the way in which the stress due to a given strain depends on the strain, varies according to the physical nature of the body; for bodies, however, which are

not crystalline or fibrous, but which have the same properties in all directions, there are two quantities which, if known, will enable us always to calculate the stress due to a given strain. These are, the elasticity of volume, or resistance to change of size; and the rigidity, elasticity of figure, or resistance to change of shape. Problems relating to the interior state of bodies are far more difficult than those which regard them as rigid. Thus, if a beam is supported at its two ends, it is very easy to find the portion of its weight which is borne by each support; but the determination of the state of stress in the interior is a problem of great complexity.

There is one theorem of kinetics which must be mentioned here. If we multiply half the momentum of every particle of a body by its velocity, and add all the results together, we shall get what is called the kinetic energy of the body. When the body is moved from one position to another, if we multiply each force acting on it—whether attraction or stress—by the distance moved in the direction opposite to the force, and add the results, we shall get what is called the work done against the forces during the change of position. It does not at all depend on the rate at which the change is made, but only on the two positions. If a body moves, and loses kinetic energy, it does an amount of work equal to the kinetic energy lost. If it gains kinetic energy, an amount of work equal to this gain must be done to take it back from the new position to the old one. The amount of work which must be done to take a body from a certain standard position to the position which it has at present is called the potential energy of the body. The theorem may be stated in this form; the sum of the potential and kinetic energies is always the same, provided the surrounding circumstances do not alter. Hence the theorem is called the Conservation of Energy. It is one fact out of many that may be deduced from the equations of motion; it is not sufficient to determine the motion of a body, but it is exceedingly useful as giving a general result in cases where it might be difficult or undesirable to investigate all the particulars; and it is especially applicable to machines, the important question in regard to which is the amount of work which they can do.

It will have been seen that the science of motion depends on a few fundamental principles which are easily verified, and consists almost entirely of mathematical deductions and calculations based on those principles. It is no longer therefore an experimental science in the same sense as those are in which the fundamental facts are still being discovered. The apparatus connected with it may be conveniently classified under three heads :

(*a*) Apparatus for illustrating theorems or solving problems of kinematics, such as those mentioned above for compounding harmonic motions. There is reason to hope for great extension of our powers in this direction.

(*b*) Apparatus for measuring the dynamical quantities, such as weight, work, and the elasticities of different substances. These are more fully classified under Measurements.

(*c*) Apparatus designed for purposes belonging to other sciences, but illustrating by its structure and functions the results of kinematics or dynamics. In this class the remainder of the collection is included.

APPENDIX.

[Among Clifford's Papers three *Cahiers* have been found relating to the theory of Elliptic functions. The most complete of these is entitled "Algebraic Introduction to Elliptic Functions," the second is "A Tract on Elliptic Functions;" the third is without any title. In all three, the elliptic functions are treated according to what may be termed the second method of Jacobi; viz. the properties of the theta functions are first investigated and the properties of the elliptic functions are deduced from them. The three cahiers appear to have been intended either as notes for a course of Lectures on Elliptic functions, or as drafts for a treatise. They contain no new results, and perhaps no original methods of investigation. But as the mode of treatment adopted by Clifford is not employed in any English treatise, and as information respecting it would have to be sought by the student in scattered original memoirs, it has been thought advisable to print, in this collection, nearly the whole of the "Algebraic Introduction," and one section of the "Tract." It is unnecessary to say that Clifford is never a mere copyist, and that these fragments possess an independent value of their own, even when they relate to elementary parts of the subject. H. J. S. S.]

ALGEBRAIC INTRODUCTION TO ELLIPTIC FUNCTIONS.

I.

Definitions and Elementary Properties of the Theta Functions.

The geometric series $a+ar+ar^2+\ldots$ may be easily expressed as a sum of exponentials. If $\log a=\alpha$, $\log r=\beta$, the series in fact becomes

$$e^{\alpha}+e^{\alpha+\beta}+e^{\alpha+2\beta}+\ldots$$

in which the general term is $e^{\alpha+n\beta}$; and we may then conveniently write it $\Sigma e^{\alpha+n\beta}$. Here the exponent of the n^{th} term is of the first order in n. It seems natural to extend the conception of this series by considering exponents of the second and higher orders in n, which lead to the series

$$\Sigma e^{\alpha+n\beta+n^2\gamma},\ \Sigma e^{\alpha+n\beta+n^2\gamma+n^3\delta},\ \text{etc.}$$

We shall in fact now occupy ourselves with the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ which is called a Theta-series.

All these series may be regarded as extending in both directions; i.e. n may have all integer values both positive and negative; but there is an important difference between the cases in which the exponent is of an odd order in n and those in which it is of even order. The geometric series for example may be written

$$\ldots+e^{\alpha-3\beta}+e^{\alpha-2\beta}+e^{\alpha-\beta}+e^{\alpha}+e^{\alpha+\beta}+e^{\alpha+2\beta}+e^{\alpha+3\beta}+\ldots$$

but, as is well known, the series is always convergent towards one end and divergent towards the other, unless $\beta=0$. But the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ is always convergent towards both ends or divergent towards both ends, according as the real part of γ is negative or positive. And a similar thing is true of all those series in which the exponent is of even order in n.

For simplicity we shall suppose $\alpha=0$, which is equivalent to dividing the whole series by e^{α}. We shall also now use the symbols $2u$, a instead of β and γ. This being so, *the series*

$$\sum_{n=-\infty}^{n=+\infty} e^{n^2a+2nu} = \ldots + e^{9a-6u} + e^{4a-4u} + e^{a-2u} + 1 + e^{a+2u} + e^{4a+4u} + e^{9a+6u} + \ldots$$
$$= 1 + e^{a}(e^{2u}+e^{-2u}) + e^{4a}(e^{4u}+e^{-4u}) + e^{9a}(e^{6u}+e^{-6u}) + \ldots$$

will be called $\Theta(u, a)$.

Another form, differing from this in being multiplied by an exponential, is sometimes useful; namely

$$G(U, A) = \sum_{n=-\infty}^{n=+\infty} e^{(nA+U)^2} = e^{U^2} \cdot \Sigma e^{n^2A^2+2nAU} = e^{U^2} \cdot \Theta(AU, A^2).$$

Since $e^{2\pi i}=1$, we shall leave unaltered every term in the series $\Theta(u, a)$ if we increase or diminish u by any multiple of πi; that is to say,

$$\Theta(u+p\pi i, a) = \Theta(u, a).$$

Again, in the summation Σe^{n^2a+2nu} the whole number n takes all integer values + and −; we shall therefore get exactly the same series if we write $n+q$ instead of n. That is

$$\Theta(u, a) = \Sigma e^{(n+q)^2a+2(n+q)u} = e^{q^2a+2qu} \cdot \Sigma e^{n^2a+2n(u+qa)}.$$

But the Σ here is precisely $\Theta(u+qa)$. Hence we have

$$\Theta(u, a) = e^{q^2a+2qu} \cdot \Theta(u+qa, a) \text{ or } \Theta(u+qa, a) = e^{-q^2a-2qu} \cdot \Theta(u, a).$$

These results may be stated as follows. If in the function $\Theta(u, a)$ we increase the argument u by any multiple of πi, the function is unaltered; but if we increase the argument by any multiple qa of a, it is altered by being multiplied into the factor e^{-q^2a-2qu}. The function Θ is accordingly said to have the *period* πi and the *quasi-period* a.

Similar properties belong to the function $G(u, a) = \Sigma e^{(na+u)^2}$. It is obvious that to increase u by qa is the same thing as to increase n by q, and therefore this operation leaves the function unaltered, or

$$G(u+qa, a) = G(u, a),$$

and the function G has the period a. But it also has a quasi-period; for

$$G(u+b, a) = \Sigma e^{(na+u+b)^2} = e^{2ub+b^2} \cdot \Sigma e^{2nab}\, e^{(na+u)^2}$$
$$= e^{2ub+b^2} \cdot G(u, a),$$

provided that $e^{2ab}=1$, or, which is the same thing, that ab is a multiple of πi. We shall in future write b for $\frac{\pi i}{a}$, and we then have the theorem

$$G(u+pb, a) = e^{2upb+p^2b^2} \cdot G(u, a),$$

or the function G has the quasi-period b.

Other forms of these functions are obtained by adding to the argument either half the period, or half the quasi-period, or both. For the Θ function we have by adding half the period πi

$$\Theta(u+\tfrac{1}{2}\pi i, a) = 1 - e^{a}(e^{2u}+e^{-2u}) + e^{4a}(e^{4u}+e^{-4u}) - e^{9a}(e^{6u}+e^{-6u}) + \text{etc.}$$

the terms being alternately + and −, instead of all + as before. This is distinguished as $\Theta'(u, a)$; viz., we have $\Theta(u+\frac{1}{2}\pi\iota, a)=\Theta'(u, a)$ and moreover

$$\Theta'(u+\tfrac{1}{2}\pi i, a)=\Theta(u, a).$$

Next, adding half the quasi-period, we find

$$\Theta(u+\tfrac{1}{2}a, a)=\Sigma e^{n^2a+2nu+na}=e^{-u-\frac{a}{4}}\Sigma e^{(n+\frac{1}{2})^2a+2(n+\frac{1}{2})u}$$

$$=e^{-u-\frac{a}{4}}\{e^{\frac{1}{4}a}(e^{u}+e^{-u})+e^{\frac{9}{4}a}(e^{3u}+e^{-3u})+e^{\frac{25}{4}a}(e^{5u}+e^{-5u})+\ldots\}.$$

The series in the brackets is distinguished as $\Theta_1(u, a)$, so that we have

$$\Theta(u+\tfrac{1}{2}a, a)=e^{-u-\frac{a}{4}}\Theta_1(u, a),\quad \Theta_1(u+\tfrac{1}{2}a, a)=e^{-u-\frac{a}{4}}\Theta(u, a).$$

We now calculate the result of adding $\frac{1}{2}\pi i+\frac{1}{2}a$ to u. It is

$$\Theta(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a)=e^{-u-\frac{a}{4}}\{e^{\frac{1}{4}a}(e^{u}-e^{-u})-e^{\frac{9}{4}a}(e^{3u}-e^{-3u})+e^{\frac{25}{4}a}(e^{5u}-e^{-5u})-\ldots\}$$

(since $e^{\frac{1}{2}\pi i}=i$, $e^{-\frac{1}{2}\pi i}=-i$). The series in the brackets is distinguished as $\Theta_1'(u, a)$; thus

$$\Theta(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a)=e^{-u-\frac{a}{4}}\Theta_1'(u, a),\quad \Theta_1'(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a)=ie^{-u-\frac{a}{4}}\Theta(u, a).$$

It will be observed that this last function vanishes when $u=0$ and changes sign with u.

For most physical applications it is convenient to convert these series into a trigonometrical form, by the substitution of $+ix$ for u. These then regarded as functions of x will be denoted by a small ϑ instead of a large one. Namely we shall write (q standing for e^a),

$$\vartheta(x, a)=\Theta(\iota x, a)=1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\ldots$$

$$\vartheta'(x, a)=\Theta'(ix, a)=1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots$$

$$\vartheta_1(x, a)=\Theta_1(ix, a)=2q^{\frac{1}{4}}\cos x+2q^{\frac{9}{4}}\cos 3x+2q^{\frac{25}{4}}\cos 5x+\ldots$$

$$i\vartheta_1'(x, a)=\Theta_1'(ix, a)=i\{2q^{\frac{1}{4}}\sin x-2q^{\frac{9}{4}}\sin 3x+2q^{\frac{25}{4}}\sin 5x-\ldots\}.$$

The three latter functions differ from ϑ in the matter of the period and quasi-period. ϑ has the period π and the quasi-period ia; all have the period 2π and are multiplied by an exponential factor when the argument is increased by $2\iota a$. But ϑ_1 and ϑ_1' change sign when the argument is increased by π, and ϑ' and ϑ_1' when it is increased by ia. These changes are indicated in the following table: (m, n any two positive integers),

$$\vartheta(x+m\pi+na\iota)=\qquad e^{-n^2a+2nxi}\vartheta(x),$$

$$\vartheta'(x+m\pi+nai)=(-)^n\quad e^{-n^2a+2nx\iota}\vartheta'(x),$$

$$\vartheta_1(x+m\pi+na\iota)=(-)^m\quad e^{-n^2a+2nxi}\vartheta_1(x),$$

$$\vartheta_1'(x+m\pi+na\iota)=(-)^{m+n}e^{-n^2a+2nxi}\vartheta_1'(x).$$

The quantities $\frac{1}{2}\pi$, $\frac{1}{2}ai$, $\frac{1}{2}\pi+\frac{1}{2}ai$, are conveniently called *quadrants*. When we increase the arguments of the ϑ by quadrants, they pass into one another according to the following table:—

$\vartheta(x+\frac{1}{2}\pi)=\vartheta' x$	$\vartheta(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta_1 x$	$\vartheta(x+\frac{1}{2}\pi+\frac{1}{2}ai)=\ e^{ix-\frac{a}{4}}i\vartheta_1' x$
$\vartheta'(x+\frac{1}{2}\pi)=\vartheta x$	$\vartheta'(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.i\vartheta_1' x$	$\vartheta'(x+\frac{1}{2}\pi+\frac{1}{2}ai)=\ e^{ix-\frac{a}{4}}\vartheta_1 x$
$\vartheta_1(x+\frac{1}{2}\pi)=i.i\vartheta_1' x$	$\vartheta_1(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta x$	$\vartheta_1(x+\frac{1}{2}\pi+\frac{1}{2}ai)=i.e^{ix-\frac{a}{4}}\vartheta' x$
$i\vartheta_1'(x+\frac{1}{2}\pi)=i.\vartheta_1 x$	$i\vartheta_1'(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta' x$	$i\vartheta_1'(x+\frac{1}{2}\pi+\frac{1}{2}ai)=i.e^{ix-\frac{a}{4}}\vartheta x$

where it will be observed that the factor i has been restored to ϑ_1' for the sake of symmetry; the rule is then that the addition of $\frac{1}{2}\pi$ multiplies ϑ_1 and $i\vartheta_1'$ by i (besides the transformation) and the addition of $\frac{1}{2}ai$ multiplies all the ϑ by $e^{ix-\frac{a}{4}}$.

Putting these results together we obtain the following table:—(Königsberger)

Increase of argument	ϑ	ϑ'	ϑ_1	$i\vartheta_1'$	Exponential factor.
$m\pi+nai$	ϑ	$(-)^n\vartheta'$	$(-)^m\vartheta_1$	$(-)^{m+n}i\vartheta_1'$	$e^{-n^2a+2nxi}$
$(m+\frac{1}{2})\pi+nai$	$(-)^n\vartheta'$	ϑ	$i(-)^{m+n}i\vartheta_1'$	$i(-)^m\vartheta_1$	
$m\pi+(n+\frac{1}{2})ai$	ϑ_1	$(-)^n i\vartheta_1'$	$(-)^m\vartheta$	$(-)^{m+n}\vartheta'$	$e^{-(n+\frac{1}{2})^2a+2(n+\frac{1}{2})xi}$
$(m+\frac{1}{2})\pi+(n+\frac{1}{2})ai$	$(-)^n i\vartheta_1'$	ϑ_1	$i(-)^{m+n}\vartheta'$	$i(-)^{m+n}\vartheta$	

II.

Product of two Theta Functions—Differential formulæ—Introduction of the Elliptic functions.

We shall now return for simplicity to the exponential form Θ, and establish two theorems of great importance.

To prove that

(i) $\Theta u.\Theta v=\Theta(u+v,\,2a)\,\Theta(u-v,\,2a)+\Theta_1(u+v,\,2a)\,\Theta_1(u-v,\,2a)$,

we have $\Theta u=\Sigma e^{n^2a+2nu}$, $\Theta v=\Sigma e^{m^2a+2mv}$, where n, m take all integer values. Let us multiply these series together term by term; the result will be

$$\Sigma\Sigma e^{(m^2+n^2)\,a+2nu+2mv},$$

where a double summation has to be effected, namely in regard to m and in regard to n. But now if we write $m+n=\mu$, $m-n=\nu$, this gives

$$2\,(m^2+n^2)=\mu^2+\nu^2,$$

and we find

$$\Theta u.\Theta v=\Sigma\Sigma e^{\frac{1}{2}(\mu^2+\nu^2)\,a+\mu\,(v+u)+\nu\,(v-u)},$$

where μ, ν must not take all values independently, but must be either both even or both odd, because they are the sum and difference of two numbers. Putting

them first equal to $2p$, $2q$, and then to $2p+1$, $2q+1$ we obtain two parts of the sum, which are respectively

$$\Sigma\Sigma e^{2(p^2+q^2)\,a+2p\,(v+u)+2q\,(v-u)}=\Sigma e^{2p^2a+2p\,(v+u)}\times\Sigma e^{2q^2a+2q\,(v-u)}$$

and

$$\Sigma\Sigma e^{2\,(\overline{p+\frac{1}{2}}^2+\overline{q+\frac{1}{2}}^2)\,a+2\overline{p+\frac{1}{2}}\,(v+u)+2\overline{q+\frac{1}{2}}\,(v-u)}$$
$$=\Sigma e^{2\overline{p+\frac{1}{2}}^2a+2\overline{p+\frac{1}{2}}\,\overline{v+u}}\times\Sigma e^{2\overline{q+\frac{1}{2}}^2a+2\overline{q+\frac{1}{2}}\,\overline{v-u}}.$$

But the products on the right are simply

$$\Theta\,(u+v,\ 2a)\,\Theta\,(u-v,\ 2a)\text{ and }\Theta_1\,(u+v,\ 2a)\,\Theta_1\,(u-v,\ 2a)$$

and therefore

$$\Theta u\,.\,\Theta v=\Theta\,(u+v,\ 2a)\,\Theta\,(u-v,\ 2a)+\Theta_1\,(u+v,\ 2a)\,\Theta_1\,(u-v,\ 2a)$$

as was to be proved.

We shall apply this theorem to obtain formulæ relating to the squares and the products of the Θ. Writing $v=u$, and then $u=0$, we have

$$\Theta^2u=\Theta\,(2u,\ 2a)\,\Theta\,(0,\ 2a)+\Theta_1\,(2u,\ 2a)\,\Theta_1\,(0, 2a)\text{ and }\Theta^2 0=\Theta^2\,(0,\ 2a)+\Theta_1^2\,(0,\ 2a),$$

and now adding to u successively $\frac{1}{2}\pi i$, $\frac{1}{2}a$, $\frac{1}{2}\pi i+\frac{1}{2}a$, we get

$$\Theta'^2u=\Theta\,(2u,\ 2a)\,\Theta\,(0,\ 2a)-\Theta_1\,(2u,\ 2a)\,\Theta_1\,(0,\ 2a),\qquad \Theta'^2 0=\Theta^2\,(0,\ 2a)-\Theta_1^2\,(0,\ 2a),$$
$$\Theta_1^2u=\Theta_1\,(2u,\ 2a)\,\Theta\,(0,\ 2a)+\Theta\,(2u,\ 2a)\,\Theta_1\,(0,\ 2a),\qquad \Theta_1^2 0=2\Theta\,(0,\ 2a)\,\Theta_1\,(0,\ 2a),$$
$$\Theta_1'^2u=\Theta_1\,(2u,\ 2a)\,\Theta\,(0,\ 2a)-\Theta\,(2u,\ 2a)\,\Theta_1\,(0,\ 2a).$$

From these we obtain the following important equations (Θ_1^2 written for $\Theta_1^2\,(0)$, etc.)

$$\Theta_1^2\,.\,\Theta'^2u+\Theta^2\,.\,\Theta_1'^2u=\Theta'^2\,.\,\Theta_1^2u,$$
$$\Theta_1^2\,.\,\Theta_1'^2u+\Theta^2\,.\,\Theta'^2u=\Theta'^2\,.\,\Theta^2u,$$
$$\Theta'^2\,.\,\Theta'^2u+\Theta_1^2\,.\,\Theta_1^2u=\Theta^2\,.\,\Theta^2u.$$

And, either by putting $u=0$ in the last, or directly from the equation on the right, we get

$$\Theta'^4+\Theta_1^4=\Theta^4.$$

It is convenient to write k for $\dfrac{\Theta_1^2}{\Theta^2}$; if $k^2+k'^2=1$, the last equation shews us that $k'=\dfrac{\Theta'^2}{\Theta^2}$.

To obtain expressions for the products of the Θ, substitute $u+\frac{1}{2}\pi i$ for v in the formula above, thus we find

$$\Theta u\,.\,\Theta' u=\Theta'\,(2u,\ 2a)\,\Theta'\,(0,\ 2a),\ \text{ since }\Theta_1'\,(0,\ 2a)=0.$$

Next put $v=u+\frac{1}{2}a$; then

$$e^{-u-\frac{a}{4}}\Theta u\,.\,\Theta_1 u=\Theta\,(2u+\tfrac{1}{2}a,\ 2a)\,\Theta\,(\tfrac{1}{2}a,\ 2a)+\Theta_1\,(2u+\tfrac{1}{2}a,\ 2a)\,\Theta_1\,(\tfrac{1}{2}a,\ 2a).$$

Lastly put $v=u+\frac{1}{2}i\pi+\frac{1}{2}a$; thus

$$e^{-u-\frac{a}{4}}\Theta u\,.\,\Theta_1' u=\Theta'\,(2u+\tfrac{1}{2}a,\ 2a)\,\Theta'\,(\tfrac{1}{2}a,\ 2a)-\Theta_1'\,(2u+\tfrac{1}{2}a,\ 2a)\,\Theta_1'\,(\tfrac{1}{2}a,\ 2a).$$

From these three formulæ we may derive three others by increasing the argument in each by $\frac{1}{2}a$ or $\frac{1}{2}\pi i$. Thus we obtain

$$\Theta_1 u \,.\, \Theta_1' u = \Theta_1' (2u, 2a)\, \Theta' (0, 2a),$$

$$e^{-\dot{u}-\frac{a}{4}} \Theta' u \,.\, \Theta_1' u = \Theta (2u + \tfrac{1}{2}a, 2a)\, \Theta (\tfrac{1}{2}a, 2a) - \Theta_1 (2u + \tfrac{1}{2}a, 2a)\, \Theta_1 (\tfrac{1}{2}a, 2a),$$

$$e^{-\dot{u}-\frac{a}{4}} \Theta' u \,.\, \Theta_1 u = \Theta' (2u + \tfrac{1}{2}a, 2a)\, \Theta' (\tfrac{1}{2}a, 2a) + \Theta_1' (2u + \tfrac{1}{2}a, 2a)\, \Theta_1' (\tfrac{1}{2}a, 2a).$$

But we have clearly $\Theta (\frac{1}{2}a, 2a) = \Theta_1 (\frac{1}{2}a, 2a)$ and $\Theta' (\frac{1}{2}a, 2a) = -\Theta_1' (\frac{1}{2}a, 2a)$; whence, putting $u=0$ in the first, second and sixth of the formulæ just written down, we get $\Theta\Theta' = \Theta'^2 (0, 2a)$; $\Theta\Theta_1 = 2e^{\frac{1}{4}a} . \Theta^2 (\frac{1}{2}a, 2a)$; $\Theta'\Theta_1 = 2e^{\frac{1}{4}a} . \Theta'^2 (\frac{1}{2}a, 2a)$.

Precisely as the formula for $\Theta u . \Theta v$ was proved, the following may be demonstrated :—

(i) $\Theta u . \dot{\Theta} v - \dot{\Theta} u . \Theta v = 2\dot{\Theta} (v-u, 2a)\, \Theta (v+u, 2a) + 2\dot{\Theta}_1 (v-u, 2a)\, \Theta_1 (v+u, 2a)$
and from this, by giving special values to u and v, we may derive the following:

$$\dot{\Theta} u \,.\, \Theta' u - \Theta u \,.\, \dot{\Theta}' u = 2\dot{\Theta}_1' (0, 2a)\, \Theta_1' (2u, 2a), \ = \frac{2\dot{\Theta}_1' (0, 2a)}{\Theta' (0, 2a)} \Theta_1 u \,.\, \Theta_1' u.$$

For the transformation of this formula it is necessary to consider the values of the fluxions of Θ for special values of the argument. We have clearly

$$\dot{\Theta} (u + \tfrac{1}{2}\pi i) = \dot{\Theta}' u,\ \ \Theta' (u + \tfrac{1}{2}\pi i) = \dot{\Theta} u,\ \ \dot{\Theta}_1 (u + \tfrac{1}{2}\pi i) = i \,.\, \dot{\Theta}_1' (u),\ \ \dot{\Theta}_1' (u + \tfrac{1}{2}\pi i) = i \,.\, \dot{\Theta}' u,$$

and therefore $\dot{\Theta}_1 (\frac{1}{2}\pi i) = i . \dot{\Theta}_1'$, the others all vanishing for $u=0$. But from

$$\Theta (u + \tfrac{1}{2}a) = e^{-\dot{u}-\frac{a}{4}} \Theta_1 u$$

we get
$$\dot{\Theta} (u + \tfrac{1}{2}a) = -e^{-\dot{u}-\frac{a}{4}} \Theta_1 u + e^{-\dot{u}-\frac{a}{4}} \dot{\Theta}_1 u,$$

and therefore
$$\dot{\Theta} (\tfrac{1}{2}a) = -e^{-\frac{a}{4}} \Theta_1 + e^{-\frac{a}{4}} \dot{\Theta}_1 = e^{-\frac{a}{4}} (\dot{\Theta}_1 - \Theta_1).$$

To avoid such complications it will be convenient to use the symbol $\partial_0 f(u)$ instead of $\dot{f}(0)$; viz. ∂_0 means the result of putting $u=0$ in the fluxion of the function which follows it. This being so, we have

$$\dot{\Theta} (u + \tfrac{1}{2}a)\, \Theta' u - \dot{\Theta}' u \,.\, \Theta (u + \tfrac{1}{2}a) = e^{-\dot{u}-\frac{a}{4}} \{\dot{\Theta}_1 u \,.\, \Theta u - \Theta' u \,.\, \Theta_1 u - \Theta' u \,.\, \Theta_1 u\}.$$

But by the formula it is also equal to

$$2\dot{\Theta}' (\tfrac{1}{2}a, 2a)\, \Theta' (2u + \tfrac{1}{2}a, 2a) + 2\dot{\Theta}_1' (\tfrac{1}{2}a, 2a)\, \Theta_1' (2u + \tfrac{1}{2}a, 2a).$$

Moreover we have

$$e^{-\dot{u}-\frac{a}{4}} \Theta' u \,.\, \Theta_1 u = \Theta' (\tfrac{1}{2}a, 2a)\, \Theta' (2u + \tfrac{1}{2}a, 2a) + \Theta_1' (\tfrac{1}{2}a, 2a)\, \Theta_1' (2u + \tfrac{1}{2}a, 2a).$$

Now

$$\Theta' (u + \tfrac{1}{2}a, 2a) = e^{-u}\, \Theta_1' (u - \tfrac{1}{2}a, 2a) \text{ or } e^{\frac{1}{2}u} \Theta' (u + \tfrac{1}{2}a, 2a) = e^{-\frac{1}{2}u}\, \Theta_1' (u - \tfrac{1}{2}a, 2a);$$

$$\therefore\ \dot{\Theta}' (u + \tfrac{1}{2}a, 2a) = -e^{-\dot{u}} \Theta_1' (u - \tfrac{1}{2}a, 2a) + e^{-\dot{u}} \dot{\Theta}_1' (u - \tfrac{1}{2}a, 2a);$$

$$\therefore\ \dot{\Theta}' (\tfrac{1}{2}a, 2a) = \dot{\Theta}_1' (-\tfrac{1}{2}a, 2a) - \Theta_1' (-\tfrac{1}{2}a, 2a) = \dot{\Theta}_1' (\tfrac{1}{2}a, 2a) + \Theta_1' (\tfrac{1}{2}a, 2a);$$

$$\therefore \text{ also } \Theta' (\tfrac{1}{2}a, 2a) + 2\dot{\Theta}' (\tfrac{1}{2}a, 2a) = \Theta_1' (\tfrac{1}{2}a, 2a) + 2\dot{\Theta}_1' (\tfrac{1}{2}a, 2a) = 2\partial_0 \,.\, e^{\frac{1}{2}\dot{u}} \Theta' (u + \tfrac{1}{2}a, 2a).$$

Hence finally

$$e^{-\dot{u}-\frac{a}{4}}\{\dot{\Theta}_1 u.\Theta' u-\dot{\Theta}' u.\Theta_1 u\}=2\partial_0 e^{\frac{1}{2}\dot{u}}\Theta'(u+\tfrac{1}{2}a,2a)\{\Theta'(2u+\tfrac{1}{2}a,2a)+\Theta_1'(2u+\tfrac{1}{2}a,2a)\},$$

$$\therefore\ \dot{\Theta}_1 u.\Theta' u-\dot{\Theta}' u.\Theta_1 u=\frac{2\partial_0 e^{\frac{1}{2}\dot{u}}\Theta'(u+\frac{1}{2}a,2a)}{\Theta'(\frac{1}{2}a,2a)}\Theta u.\Theta_1' u.$$

Similarly, we find by putting $u+\frac{1}{2}\pi i$, $v+\frac{1}{2}\pi i$ in the original formula

$$\Theta' u.\dot{\Theta}' v-\dot{\Theta}' u.\Theta' v=2\dot{\Theta}(v-u,2a)\,\Theta(v+u,2a)-2\dot{\Theta}_1(v-u,2a)\,\Theta_1(v+u,2a).$$

Now write $v=u+\frac{1}{2}a$; we have in the first place

$$\Theta' u.\dot{\Theta}'(u+\tfrac{1}{2}a)-\dot{\Theta}' u.\Theta'(u+\tfrac{1}{2}a)=e^{-\dot{u}-\frac{a}{4}}\{\Theta' u.\dot{\Theta}_1' u-\dot{\Theta}' u.\Theta_1' u-\Theta' u.\Theta_1' u\},$$

but also $\quad =2\dot{\Theta}(\frac{1}{2}a,2a)\,\Theta(2u+\frac{1}{2}a,2a)-2\dot{\Theta}_1(\frac{1}{2}a,2a)\,\Theta_1(2u+\frac{1}{2}a,2a).$

Moreover we have

$$e^{-\dot{u}-\frac{a}{4}}\Theta' u.\Theta_1' u=\Theta(\tfrac{1}{2}a,2a)\,\Theta(2u+\tfrac{1}{2}a,2a)-\Theta_1(\tfrac{1}{2}a,2a)\,\Theta_1(2u+\tfrac{1}{2}a,2a).$$

Now

$$\Theta(u+\tfrac{1}{2}a,2a)=e^{-\dot{u}}\Theta_1(u-\tfrac{1}{2}a,2a)\ \text{ or }\ e^{\frac{1}{2}\dot{u}}\Theta(u+\tfrac{1}{2}a,2a)=e^{-\frac{1}{2}\dot{u}}\Theta_1(u-\tfrac{1}{2}a,2a),$$

whence

$$\tfrac{1}{2}e^{\frac{1}{2}\dot{u}}\Theta(u+\tfrac{1}{2}a,2a)+e^{\frac{1}{2}\dot{u}}\dot{\Theta}(u+\tfrac{1}{2}a,2a)=-\tfrac{1}{2}e^{-\frac{1}{2}\dot{a}}\Theta_1(u-\tfrac{1}{2}a,2a)+e^{-\frac{1}{2}\dot{u}}\dot{\Theta}_1(u-\tfrac{1}{2}a,2a),$$

or $(u=0)$

$$\Theta(\tfrac{1}{2}a,2a)+2\dot{\Theta}(\tfrac{1}{2}a,2a)=-\Theta_1(\tfrac{1}{2}a,2a)-2\dot{\Theta}_1(\tfrac{1}{2}a,2a)=\partial_0 e^{\frac{1}{2}\dot{u}}\Theta(u+\tfrac{1}{2}a,2a).$$

Hence finally

$$e^{-\dot{u}-\frac{a}{4}}\{\Theta' u.\dot{\Theta}_1' u-\dot{\Theta}' u.\Theta_1' u\}=\partial_0 e^{\frac{1}{2}\dot{u}}\Theta(u+\tfrac{1}{2}a,2a)\{\Theta(2u+\tfrac{1}{2}a,2a)+\Theta_1(2u+\tfrac{1}{2}a,2a)\},$$

and $\quad \Theta' u.\dot{\Theta}_1' u-\dot{\Theta}' u.\Theta_1' u=\dfrac{\partial_0 e^{\frac{1}{2}\dot{u}}\Theta(u+\frac{1}{2}a,2a)}{\Theta(\frac{1}{2}a,2a)}.\Theta u.\Theta_1 u.$

The three formulæ thus arrived at may be written as follows:—

$$\begin{aligned}\Theta' u.\dot{\Theta} u-\dot{\Theta}' u.\Theta u&=\alpha.\Theta_1 u.\Theta_1' u,\\ \Theta' u.\dot{\Theta}_1 u-\dot{\Theta}' u.\Theta_1 u&=\beta.\Theta_1' u.\Theta u,\\ \Theta' u.\dot{\Theta}_1' u-\dot{\Theta}' u.\Theta_1' u&=\gamma.\Theta u.\Theta_1 u,\end{aligned}$$

and the quantities α, β, γ may now be determined in terms of Θ, Θ', Θ_1 and $\dot{\Theta}_1'$ by substituting particular values of u. Thus putting $u=\frac{1}{2}a+\frac{1}{2}\pi i$, $\frac{1}{2}\pi i$, and 0 successively, we find

$$\begin{aligned}\Theta_1\dot{\Theta}_1'&=-\alpha\Theta\Theta',\\ \Theta\dot{\Theta}_1'&=\beta\Theta'\Theta_1,\\ \Theta'\dot{\Theta}_1'&=\gamma\Theta\Theta_1.\end{aligned}$$

If we now write

$$fu=\frac{\Theta}{\Theta_1}.\frac{\Theta_1' u}{\Theta' u},\quad gu=\frac{\Theta'}{\Theta_1}.\frac{\Theta_1 u}{\Theta' u},\quad hu=\frac{\Theta'}{\Theta}.\frac{\Theta u}{\Theta' u},$$

(the multipliers in the two latter being so chosen as to make them $=1$ for $u=0$, and in the former to simplify the following formulæ, which are derived from those on p. 447, viz.:—

$$1+fu^2=gu^2, \quad 1+k^2fu^2=hu^2, \quad k'^2+k^2gu^2=hu^2),$$

then we have for the fluxions of these functions

$$\dot{f}u=\lambda gu\,.\,hu, \quad \dot{g}u=\lambda hu\,.\,fu, \quad \dot{h}u=k^2\lambda fu\,.\,gu, \quad \text{where } \lambda=\frac{\Theta\dot{\Theta}_1'}{\Theta_1\Theta'}.$$

By substitution from the three equations connecting the squares of f, g, h we obtain

$$\dot{f}u=\lambda\sqrt{(1+fu^2)(1+k^2fu^2)}, \qquad \lambda u=\int_0^f \frac{df}{\sqrt{(1+f^2)(1+k^2f^2)}},$$

$$\dot{g}u=\lambda\sqrt{(gu^2-1)(k'^2+k^2gu^2)}, \qquad \lambda u=\int_1^g \frac{dg}{\sqrt{(g^2-1)(k'^2+k^2g^2)}},$$

$$\dot{h}u=\lambda\sqrt{(1+h^2)(k'^2+h^2)}, \qquad \lambda u=\int_1^h \frac{dh}{\sqrt{(1+h^2)(k'^2+h^2)}},$$

when $u=\frac{1}{2}\pi i$, $fu=i$; writing ix for f in the integral,

$$\tfrac{1}{2}\pi\lambda=\int_0^1 \frac{dx}{\sqrt{(1-x^4)(1-k^2x^2)}}=K, \qquad \lambda=\frac{2K}{\pi}.$$

To suit the trigonometric form, we may write $if(u)=z(u)$; then $z(\frac{1}{2}\pi)=1$, which is the same thing as saying that

$$\frac{\lambda\pi}{2}=\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=K; \text{ then } \lambda=\frac{2K}{\pi},$$

which gives λ as an explicit function of k.

In passing now to the trigonometric form, we shall divide the argument by λ, because $\dot{f}\left(\frac{x}{\lambda}\right)$ becomes 1 when $x=0$. Writing then $u=\frac{ix}{\lambda}=\frac{i\pi x}{2K}$, we shall have in Jacobi's notation,

$$f\left(\frac{i\pi x}{2K}\right)=i\sin\operatorname{am} x=\frac{i}{\sqrt{k}}\,\frac{\vartheta_1'\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)},$$

$$g\left(\frac{i\pi x}{2K}\right)=\cos\operatorname{am} x=\sqrt{\frac{k'}{k}}\,\frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)},$$

$$h\left(\frac{i\pi x}{2K}\right)=\Delta\operatorname{am} x=\sqrt{k'}\,\frac{\vartheta\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)}.$$

III.

The Addition-Theorem.

Starting from the formula

$$\theta u \,.\, \theta v = \theta\overline{u+v,\,2a} \,.\, \theta\overline{u-v,\,2a} + \theta_1\overline{u+v,\,2a} \,.\, \theta_1\overline{u-v,\,2a} \quad\ldots\ldots(1),$$

we derive by writing $u+\frac{1}{2}a$ for u and $v+\frac{1}{2}a+\frac{1}{2}\iota\pi$ for v,

$$\theta_1 u \,.\, \theta_1' v = \theta_1'\overline{u+v,\,2a} \,.\, \theta'\overline{u-v,\,2a} - \theta'\overline{u+v,\,2a} \,.\, \theta_1'\overline{u-v,\,2a} \quad\ldots\ldots(2),$$

and so $\theta_1' u \,.\, \theta_1 v = \theta_1'\overline{u+v,\,2a} \,.\, \theta'\overline{u-v,\,2a} + \theta'\overline{u+v,\,2a} \,.\, \theta_1'\overline{u-v,\,2a} \quad\ldots\ldots(3).$

Write $v=0$ in this formula; then

$$\theta_1' u \,.\, \theta_1 = 2\theta'\overline{u,\,2a} \,.\, \theta_1'\overline{u,\,2a} \quad\ldots\ldots(4).$$

So from $\theta_1 u \,.\, \theta_1 v = \theta_1\overline{u+v,\,2a} \,.\, \theta\overline{u-v,\,2a} + \theta\overline{u+v,\,2a} \,.\, \theta_1\overline{u-v,\,2a} \quad\ldots\ldots(5),$

we get $\theta_1 u \,.\, \theta_1 = 2\theta_1\overline{u,\,2a} \,.\, \theta\overline{u,\,2a}$, and $\theta_1^2 = 2\theta_1(0,\,2a)\,.\,\theta(0,\,2a) \quad\ldots\ldots(6,\,7).$

Substituting in (2), (3), we get

$$\theta_1'\overline{u+v,\,2a} \,.\, \theta'\overline{u-v,\,2a} - \theta'\overline{u+v,\,2a} \,.\, \theta_1'\overline{u-v,\,2a} = \theta_1 u \,.\, \theta_1' v$$

$$= \frac{4\theta_1\overline{u,\,2a} \,.\, \theta\overline{u,\,2a} \,.\, \theta'\overline{v,\,2a} \,.\, \theta_1'\overline{v,\,2a}}{(\theta_1^2=)\,2\theta_1\overline{0,\,2a} \,.\, \theta\overline{0,\,2a}},$$

or omitting the $2a$ throughout

$$\theta \,.\, \theta_1 (\theta_1'\overline{u+v} \,.\, \theta'\overline{u-v} - \theta'\overline{u+v} \,.\, \theta_1'\overline{u-v}) = 2\theta_1 u \,.\, \theta u \,.\, \theta' v \,.\, \theta_1' v,$$

$$\theta \,.\, \theta_1 (\theta_1'\overline{u+v} \,.\, \theta'\overline{u-v} + \theta'\overline{u+v} \,.\, \theta_1'\overline{u-v}) = 2\theta_1 v \,.\, \theta v \,.\, \theta' u \,.\, \theta_1' u.$$

Divide throughout by $\theta_1^2 \,.\, \theta'\overline{u+v} \,.\, \theta'\overline{u-v}$; thus

$$f(u+v) + f(u-v) = \frac{2\theta_1 v \,.\, \theta v \,.\, \theta' u \,.\, \theta_1' u}{\theta_1^2 \,.\, \theta'\overline{u+v} \,.\, \theta'\overline{u-v}} = 2fu \,.\, gv \,.\, hv\, \frac{\theta' u^2 \,.\, \theta' v^2}{\theta'^2 \,.\, \theta'\overline{u+v} \,.\, \theta'\overline{u-v}} \quad\ldots\ldots(11).$$

To find the value of the dexter side we proceed as follows. We have

$$\theta' x \,.\, \theta y = \theta'\overline{x+y,\,2a} \,.\, \theta'\overline{x-y,\,2a} - \theta_1'\overline{x+y,\,2a} \,.\, \theta_1'\overline{x-y,\,2a} \quad\ldots\ldots(12),$$

$$\theta x \,.\, \theta' y = \theta'\overline{x+y,\,2a} \,.\, \theta'\overline{x-y,\,2a} + \theta_1'\overline{x+y,\,2a} \,.\, \theta_1'\overline{x-y,\,2a} \quad\ldots\ldots(13),$$

therefore $\theta x \,.\, \theta' x = \theta'\overline{2x,\,2a} \,.\, \theta'\overline{0,\,2a} \quad\ldots\ldots(14).$

Multiply together (12) and (13), taking account of (14) on the left, and omit the $2a$ throughout; thus

$$\theta'^2 \,.\, \theta' 2x \,.\, \theta' 2y = \theta'\overline{x+y}^2 \,.\, \theta'\overline{x-y}^2 - \theta_1'\overline{x+y}^2 \,.\, \theta_1'\overline{x-y}^2,$$

or writing $2x=u+v,\ 2y=u-v,\ x+y=u,\ x-y=v,$

$$\theta'^2 \,.\, \theta'\overline{u+v} \,.\, \theta'\overline{u-v} = \theta' u^2 \,.\, \theta' v^2 - \theta_1' u^2 \,.\, \theta_1' v^2,$$

and therefore $$\frac{\theta'^2 . \theta'\overline{u+v} . \theta'\overline{u-v}}{\theta' u^2 . \theta' v^2} = 1 - k^2 fu^2 . fv^2 ;$$

whence $$f(u+v) + f(u-v) = \frac{2fu . gv . hv}{1 - k^2 fu^2 . fv^2} ;$$

and by interchanging u with v

$$f(u+v) - f(u-v) = \frac{2fv . gu . hu}{1 - k^2 fu^2 . fv^2} ;$$

therefore $$f(u \pm v) = \frac{fu . gv . hv \pm fv . gu . hu}{1 - k^2 fu^2 . fv^2} .$$

IV.

Elliptic Functions of the second kind.

By easy calculation we may establish the theorem

$$\ddot{\theta} u . \theta v - \dot{\theta} u . \dot{\theta} v = 2 \begin{cases} \dot{\Theta}\overline{u+v} . \dot{\Theta}\overline{u-v} + \Theta\overline{u+v} . \ddot{\Theta}\overline{u-v} \\ + \dot{\Theta}_1\overline{u+v} . \dot{\Theta}_1\overline{u-v} + \Theta_1\overline{u+v} . \dot{\Theta}_1\overline{u-v}, \end{cases}$$

and thence putting $u=v$, and observing that $\dot{\Theta} = \dot{\Theta}_1 = 0$,

$$\ddot{\theta} u . \theta u - \dot{\theta} u^2 = 2 \{\ddot{\Theta} . \Theta(2u) + \ddot{\Theta}_1 . \Theta_1(2u)\},$$

$$\ddot{\theta}' u . \theta' u - \dot{\theta}' u^2 = 2 \{\dot{\Theta} . \Theta(2u) - \Theta_1 . \Theta_1(2u)\}.$$

We are therefore entitled to assume

$$\dot{\theta}' u . \theta' u - \dot{\theta}' u^2 = \alpha \theta u^2 + \beta \theta' u^2,$$

and in fact, writing successively $u = \frac{1}{2}a$ and $u = \frac{1}{2}\pi i + \frac{1}{2}a$, we find

$$- \dot{\theta}_1'^2 = \alpha \theta_1^2, \qquad \ddot{\theta}_1 . \theta_1 = \beta \theta_1^2,$$

so that $$\theta_1^2 (\dot{\theta}' u . \theta' u - \dot{\theta}' u^2) = - \dot{\theta}_1'^2 . \theta u^2 + \dot{\theta}_1 . \theta_1 . \theta' u^2 ;$$

therefore

$$\partial_u \frac{\dot{\theta}' u}{\theta' u} = - \frac{\dot{\theta}_1'^2}{\theta_1^2} \frac{\theta u^2}{\theta' u^2} + \frac{\ddot{\theta}_1}{\theta_1} = \lambda^2 h u^2 + \mu \qquad \left(\lambda = i \frac{\theta \dot{\theta}_1'}{\theta_1 \theta'}, \quad \mu = \frac{\dot{\theta}_1}{\theta_1} .\right)$$

Suppose then that $\dfrac{\dot{\theta}' u}{\theta' u} = \lambda Z(\lambda u)$; we shall have

$$\partial_x Z(x) = \mathrm{dn}^2 x + \frac{\mu}{\lambda^2}, \quad \partial_y Z(\mathrm{sn}^{-1} y) = \sqrt{\frac{1 - k^2 y^2}{1 - y^2}} + \frac{\mu}{\lambda^2} \frac{1}{\sqrt{(1 - y^2 . 1 - k^2 y^2)}},$$

integrating from 0 to 1, since $Z(K) = 0$, we have $0 = E + \dfrac{\mu K}{\lambda^2}$,

$$\therefore\ Z(x) = \int_0^x \mathrm{dn}^2\, x dx - \frac{E}{K} x, \text{ (Jacobi's definition).}$$

Whence also $$\int_0^y \sqrt{\frac{1 - k^2 y^2}{1 - y^2}}\, dy = Z(\mathrm{sn}^{-1} y) + \frac{E}{K} \mathrm{sn}^{-1} y.$$

V.

Product of Four Theta functions. Smith's reconstruction of Jacobi's method.

Consider four integers n_1, n_2, n_3, n_4, and let $n_1+n_2+n_3+n_4=s$, $\nu_1=s-2n_1$, $\nu_2=s-2n_2$, $\nu_3=s-2n_3$, $\nu_4=s-2n_4$. Then $\nu_1+\nu_2+\nu_3+\nu_4=2s$, and $2s-2\nu_1=4n_1$, etc.

Multiply now together the four θ-series.

$$\theta x=\Sigma e\,|\,n_1^2a+2n_1x\,|,\ \theta y=\Sigma e\,|\,n_2^2a+2n_2y\,|,$$
$$\theta z=\Sigma e\,|\,n_3^2a+2n_3z\,|,\ \theta w=\Sigma e\,|\,n_4^2a+2n_4w\,|,$$

and we get a quadruply infinite series in which the exponent of the general term is

$$a\Sigma n^2+2\,(n_1x+n_2y+n_3z+n_4w).$$

Now $\Sigma\nu^2=4\Sigma n^2$, and if we write $\sigma=x+y+z+w$,

then $$2\,(n_1x+n_2y+n_3z+n_4w)=\nu_1\xi+\nu_2\eta+\nu_3\zeta+\nu_4\omega,$$

where $$2\xi=\sigma-2x,\ 2\eta=\sigma-2y,\ 2\zeta=\sigma-2z,\ 2\omega=\sigma-2w.$$

So the exponent becomes

$$\tfrac{1}{4}a\Sigma\nu^2+(\nu_1\xi+\nu_2\eta+\nu_3\zeta+\nu_4\omega).$$

Here if s is odd, the ν are all odd, and if s is even, they are all even. Thus the ν and $\frac{1}{2}\Sigma\nu$ must be either all odd or all even together.

First let them be even, and $=2p_1$, $2p_2$, $2p_3$, $2p_4$ respectively. Then

$$p_1+p_2+p_3+p_4$$

must be even. Substituting, the exponent becomes

$$a\Sigma p^2+2\,(p_1\xi+p_2\eta+p_3\zeta+p_4\omega),$$

and this has to be summed under the condition that Σp is even. Call the exponent P, twice the sum required is

$$\Sigma^4 e\,|\,P\,|+\Sigma^4\,(-)^{\Sigma p}e\,|\,P\,|,\ \text{ or }\ \theta\xi\,.\,\theta\eta\,.\,\theta\zeta\,.\,\theta\omega+\theta'\xi\,.\,\theta'\eta\,.\,\theta'\zeta\,.\,\theta'\omega.$$

Next let the ν be odd and equal to $2p_h+1$ $(h=1, 2, 3, 4)$. Then $\frac{1}{2}\Sigma\nu=\Sigma p+2$ must be odd, or Σp must be odd. Hence if P is the exponent

$$a\Sigma\,(p+\tfrac{1}{2})^2+2(\overline{p_1+\tfrac{1}{2}}\xi+\ldots),$$

twice the sum is

$$\Sigma^4 e\,|\,P\,|-\Sigma^4\,(-)^{\Sigma p}e\,|\,P\,|,\ \text{ or }\ \theta_1\xi\,.\,\theta_1\eta\,.\,\theta_1\zeta\,.\,\theta_1\omega-\theta_1'\xi\,.\,\theta_1'\eta\,.\,\theta_1'\zeta\,.\,\theta_1'\omega.$$

Hence the Theorem

$$2\theta x\,.\,\theta y\,.\,\theta z\,.\,\theta w=\theta\xi\,.\,\theta\eta\,.\,\theta\zeta\,.\,\theta\omega+\theta'\xi\,.\,\theta'\eta\,.\,\theta'\zeta\,.\,\theta'\omega+\theta_1\xi\,.\,\theta_1\eta\,.\,\theta_1\zeta\,.\,\theta_1\omega-\theta_1'\xi\,.\,\theta_1'\eta\,.\,\theta_1'\zeta\,.\,\theta_1'\omega$$
$$=\Sigma\,(-)^{\alpha\beta}\,\theta^{\alpha}_{\beta}\,(\xi)\,.\,\theta^{\alpha}_{\beta}\,(\eta)\,.\,\theta^{\alpha}_{\beta}\,(\zeta)\,.\,\theta^{\alpha}_{\beta}\,(\omega).$$

To generalize this, add to the arguments $x\ y\ z\ w$ the quadrants $\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}$

respectively whose sum shall be $\frac{p}{q}$, (p, q must be even); and let $\frac{2\alpha}{2\beta}=\frac{p-2\alpha}{q-2\beta}$; then we shall find that $\xi\ \eta\ \zeta\ \omega$ will be increased by $\frac{\alpha_1}{\beta_1}$ etc.; and so

$$2\theta^{\alpha_1}_{b_1}(x).\theta^{\alpha_2}_{b_2}(y).\theta^{\alpha_3}_{b_3}(z).\theta^{\alpha_4}_{b_4}(w)=\Sigma\,(-)^{g\,(f+\frac{1}{2}p)}\,\theta^{\alpha_1+f}_{\beta_1+g}(\xi).\theta^{\alpha_2+f}_{\beta_2+g}(\eta).\theta^{\alpha_3+f}_{\beta_3+g}(\zeta).\theta^{\alpha_4+f}_{\beta_4+g}(\omega),$$
$$(f,\ g=0,\ 1.$$

Make $x\ y\ z\ w$ all $=0$ in the original theorem; then

$$\theta^4=\theta_1{}^4+\theta'^4.$$

Next write for them $x\ x\ 0\ 0$, and $\begin{matrix} a & a & 0 & 0 \\ b & b & 0 & 0\end{matrix}$ for the a and b. Then

$$\theta^a_b x^2\,.\,\theta^2=\theta^{a+1}_b x^2\,.\,\theta'^2+(-)^a\theta^a_{b+1}x^2\,.\,\theta_1{}^2,$$

that is

$$\begin{array}{l} \theta x^2\ .\ \theta^2=\theta' x^2\ .\ \theta'^2+\theta_1 x^2\ .\ \theta_1{}^2 \\ \theta' x^2\ .\ \theta^2=\theta x^2\ .\ \theta'^2-\theta_1' x^2\,.\,\theta_1{}^2 \\ \theta_1 x^2\ .\ \theta^2=\theta_1' x^2\,.\,\theta'^2+\theta x^2\ .\ \theta_1{}^2 \\ \theta_1' x^2\,.\,\theta^2=\theta_1 x^2\ .\ \theta'^2-\theta' x^2\ .\ \theta_1{}^2 \end{array} \quad \text{or} \quad \left\{ \begin{array}{l} hu^2=k'^2+k^2gu^2, \\ 1=hu^2-k^2fu^2, \\ gu^2=k'^2fu^2+k^2hu^2, \\ fu^2=gu^2-1. \end{array}\right.$$

Next writing for the $x,\ x-y,\ x+y,\ 0,\ 0$ we find for the $\xi,\ y,\ -y,\ x,\ x$, and we get such formulæ as

$$\theta\overline{x-y}\,.\,\theta_1\overline{x+y}\,.\,\theta\,.\,\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1 x\,.\,\theta x+\theta_1' y\,.\,\theta' y\,.\,\theta_1' x\,.\,\theta' x.$$

Differentiate in regard to y and then put $y=0$; we get

$$\theta\theta_1\,(\theta x\,.\,\dot\theta_1 x-\dot\theta x\,.\,\theta_1 x)=+\,\dot\theta_1'\theta\,.\,\theta_1' x\,.\,\theta' x,\ \text{etc.}$$

The important formulæ of this kind are the three

$$\theta'\overline{x-y}\,.\,\theta_1\overline{x+y}\ .\ \theta'\theta_1=\theta' y\,.\,\theta_1 y\,.\,\theta' x\,.\,\theta_1 x-\theta y\,.\,\theta_1' y\,.\,\theta x\,.\,\theta_1' x,$$

$$\theta'\overline{x-y}\,.\,\theta_1'\overline{x+y}\,.\,\theta\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1' x\,.\,\theta' x+\theta_1' y\,.\,\theta' y\,.\,\theta_1 x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta\overline{x+y}\ .\ \theta'\theta=\theta' y\,.\,\theta y\,.\,\theta' x\,.\,\theta x-\theta_1' y\,.\,\theta_1 y\,.\,\theta_1' x\,.\,\theta_1 x\,;$$

and the fluxional equations are

$$\left.\begin{array}{l} (\dot\theta_1' x\,.\,\theta' x-\theta_1' x\,.\,\dot\theta' x)\ \theta\theta_1\ =\ \dot\theta_1'\theta'\,.\,\theta x\,.\,\theta_1 x \\ (\dot\theta_1 x\,.\,\theta' x-\theta_1 x\,.\,\dot\theta' x)\ \theta'\theta_1=\dot\theta_1'\theta\,.\,\theta_1' x\,.\,\theta x \\ (\dot\theta x\,.\,\theta' x-\theta x\,.\,\dot\theta' x)\ \theta\theta'\ =\dot\theta_1'\theta_1\,.\,\theta_1 x\,.\,\theta_1' x \end{array}\right\}\ \text{giving}\ \begin{array}{l} \dot fu=\lambda gu\,.\,hu=\lambda\sqrt{\overline{1+f^2}\,.\,\overline{1+k^2f^2}} \\ \dot gu=\lambda hu\,.\,fu=\lambda\sqrt{\overline{k'^2+k^2g^2}\,.\,\overline{g^2-1}} \\ \dot hu=\lambda k^2 fu\,.\,gu=\lambda\sqrt{\overline{h^2-1}\,.\,\overline{h^2-k'^2}} \end{array}$$

whence λ is given as a function of k by $\frac{1}{2}\pi\lambda=\int_0^1\frac{dx}{\sqrt{\overline{1-x^2}\,.\,\overline{1-k^2x^2}}}=K.$

In the circular form $f\left(\frac{ix}{\lambda}\right)=i\,\text{sn}\,x,\ g\left(\frac{ix}{\lambda}\right)=\text{cn}\,x,\ h\left(\frac{ix}{\lambda}\right)=\text{dn}\,x$; and then

$$\left.\begin{array}{l} \dot{\text{sn}}\,x=\text{cn}\,x\,\text{dn}\,x=\sqrt{(1-\text{sn}^2 x)(1-k^2\,\text{sn}^2 x)} \\ \dot{\text{cn}}\,x=-\,\text{dn}\,x\,\text{sn}\,x=-\sqrt{(1-\text{cn}^2 x)\,(k'^2+k^2\,\text{cn}^2 x)} \\ \dot{\text{dn}}\,x=-\,k^2\,\text{sn}\,x\,\text{cn}\,x=-\sqrt{(1-\text{dn}^2 x)\,(\text{dn}^2 x-k'^2)} \end{array}\right\}\ \dot{\text{tn}}\,x=\sqrt{(1+\text{tn}^2 x)\,(1+k'^2\,\text{tn}^2 x)}.$$

The Addition-Theorem.

We have found expressions for the product of two different θ functions of $x+y$, $x-y$; the following value for $\theta^a_b(x+y)\,\theta^a_b(x-y)$ is used subsequently for functions of the second kind, as well as here for the addition-theorem. Writing for the xab the values $\begin{matrix} x-y & x+y & 0 & 0 \\ a & a & 1 & 1 \\ b & b & 0 & 0 \end{matrix}$ we have for the $\xi\alpha\beta$, $\begin{matrix} y & -y & x & x \\ 1 & 1 & a & a \\ 0 & 0 & b & b \end{matrix}$ and so

$$2\theta^a_b(x+y)\,.\,\theta^a_b(x-y)\,\theta'^2$$
$$=\theta'y^2\,.\,\theta^a_b x^2+\theta y^2\,.\,\theta^{a+1}_b x^2+(-)^a\theta_1'y^2\,.\,\theta^a_{b+1}x^2-(-)^a\theta_1 y^2\,.\,\theta^{a+1}_{b+1}x^2 \ldots\ldots (B).$$

But now writing for the xab, $\begin{matrix} x-y & x+y & 0 & 0 \\ a & a & 1 & 1 \\ b+1 & b+1 & 1 & 1 \end{matrix}$ we get for the $\xi\alpha\beta$, $\begin{matrix} y & -y & x & x \\ 1 & 1 & a & a \\ 1 & 1 & b+1 & b+1 \end{matrix}$ so that

$$0=+\theta_1'y^2\,.\,\theta^a_{b+1}x^2+\theta_1 y^2\,.\,\theta^{a+1}_{b+1}x^2+(-)^a\theta'y^2\,.\,\theta^a_b x^2-(-)^a\theta y^2\,.\,\theta^{a+1}_b x^2 \ldots\ldots (B');$$

then $B\pm(-)^a B'$ gives

$$\theta^a_b(x+y)\,.\,\theta^a_b(x-y)\,\theta'^2=\theta'y^2\,.\,\theta^a_b x^2+(-)^a\theta_1'y^2\,.\,\theta^a_{b+1}x^2$$
$$=\theta y^2\,.\,\theta^{a+1}_b x^2-(-)^a\theta_1 y^2\,.\,\theta^{a+1}_{b+1}x^2.$$

For example, writing a, $b=1$, 0 we have the important formula

$$\theta'(x+y)\,.\,\theta'(x-y)\,\theta'^2=\theta'y^2\,.\,\theta'x^2-\theta_1'y^2\,.\,\theta_1'x^2,\ \text{or}\ \frac{\theta'\overline{x+y}\,.\,\theta'\overline{x-y}\,.\,\theta'^2}{\theta'x^2\,.\,\theta'y^2}=1-k^2fx^2\,.\,fy^2.$$

Dividing by this equation the three formulæ

$$\theta'\overline{x-y}\,.\,\theta_1'\overline{x+y}\,.\,\theta\,.\,\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1'x\,.\,\theta'x+\theta_1'y\,.\,\theta'y\,.\,\theta_1 x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta_1\overline{x+y}\,.\,\theta_1\,.\,\theta'=\theta_1 y\,.\,\theta'y\,.\,\theta_1 x\,.\,\theta'x+\theta_1'y\,.\,\theta y\,.\,\theta_1'x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta\overline{x+y}\,.\,\theta'\,.\,\theta=\theta'y\,.\,\theta y\,.\,\theta'x\,.\,\theta x+\theta_1'y\,.\,\theta_1 y\,.\,\theta_1'x\,.\,\theta_1 x,$$

we find from the first

$$\frac{\theta_1'\overline{x+y}}{\theta'\overline{x+y}}\,.\,\frac{\theta\theta_1}{\theta'^2}=\frac{\theta_1 y\,.\,\theta y\,.\,\theta_1'x\,.\,\theta'x+\theta_1'y\,.\,\theta'y\,.\,\theta_1 x\,.\,\theta x}{\theta'x^2\,.\,\theta'y^2-\theta_1'x^2\,.\,\theta_1'y^2},$$

or

$$f(x+y)=\frac{fx\,.\,gy\,.\,hy+fy\,.\,gx\,.\,hx}{1-k^2fx^2\,.\,fy^2},$$

and so from the second and third

$$g(x+y)=\frac{gx\,.\,gy+fx\,.\,fy\,.\,hx\,.\,hy}{1-k^2fx^2\,.\,fy^2},$$

$$h(x+y)=\frac{hx\,.\,hy+k^2fx\,.\,fy\,.\,gx\,.\,gy}{1-k^2fx^2\,.\,fy^2}.$$

Functions of the Second Kind.

Differentiate the equation

$$\theta'(x+y)\,.\,\theta'(x-y)\,.\,\theta'^2=\theta'y^2\,.\,\theta'x^2-\theta_1'y^2\,.\,\theta_1'x^2$$

twice with respect to y; we obtain successively

$$(\dot\theta'\overline{x+y}\,.\,\theta'\overline{x-y}-\theta'\overline{x+y}\,.\,\dot\theta'\overline{x-y})\,\theta'^2=2\theta'y\,.\,\dot\theta'y\,.\,\theta'x^2-2\theta_1'y\,.\,\dot\theta_1'y\,.\,\theta_1'x^2,$$

$$(\ddot\theta'\overline{x+y}\,.\,\theta'\overline{x-y}+\theta'\overline{x+y}\,.\,\ddot\theta'\overline{x-y}-2\dot\theta'\overline{x+y}\,.\,\dot\theta'\overline{x-y})\,\theta'^2$$
$$=2\,(\theta'y\,.\,\ddot\theta'y+\dot\theta'y^2)\,\theta'x^2-2\,(\theta_1'y\,.\,\ddot\theta_1'y+\dot\theta_1'y^2)\,\theta_1'x^2.$$

In this put $y=0$; then

$$(\ddot\theta'x\,.\,\theta'x-\dot\theta'x^2)\,\theta'^2=\theta'\ddot\theta'\,.\,\theta'x^2-\dot\theta_1'^2\,.\,\theta_1'x^2,$$

$$\text{or }\partial_x\frac{\dot\theta'x}{\theta'x}=\frac{\ddot\theta'}{\theta'}-\frac{\dot\theta_1'^2}{\theta'^2}\,.\,\frac{\theta_1'x^2}{\theta'x^2}=\mu-\lambda^2k^2fx^2,\text{ if }\mu=\frac{\ddot\theta'}{\theta'},\ \lambda=\frac{\dot\theta_1'\theta}{\theta'\theta_1}.$$

Herein writing $x=\dfrac{iu}{\lambda}$, we have $\partial_u=\dfrac{i}{\lambda}\partial_x$, and

$$\partial_u\frac{\partial_u\theta'\left(\frac{iu}{\lambda}\right)}{\theta'\left(\frac{iu}{\lambda}\right)}=-\frac{\mu}{\lambda^2}-k^2\,\mathrm{sn}^2\,u.$$

Integrating from 0 to u, we get

$$\partial_u\log\theta'\left(\frac{iu}{\lambda}\right)=-\frac{\mu}{\lambda^2}u-\int_0^u k^2\,\mathrm{sn}^2\,udu.$$

If $y=\mathrm{sn}\,u$, $dy=\sqrt{(1-y^2)(1-k^2y^2)}\,.\,du$, and we have

$$\int_0^u k^2\,\mathrm{sn}^2\,udu=\int_0^y\frac{k^2y^2dy}{\sqrt{Y}}=\int_0^y\frac{dy}{\sqrt{Y}}-\int_0^y\frac{\overline{1-k^2y^2}\,.\,dy}{\sqrt{Y}},$$
$$=u-\int_0^y\sqrt{\frac{1-k^2y^2}{1-y^2}},\,dy.$$

Put here $y=1$, $u=K$, $\dfrac{iu}{\lambda}=\tfrac12 i\pi$, then since $\dot\theta'(\tfrac12 i\pi)=0$, we have $\left(\lambda=\dfrac{2K}{\pi}\right)$

$$0=-\frac{\mu}{\lambda^2}K-K+E,\text{ or }-\frac{\mu}{\lambda^2}=1-\frac{E}{K},$$

if $$E=\int_0^1\sqrt{\frac{1-k^2y^2}{1-y^2}}\,dy=\int_0^{\frac{\pi}{2}}\sqrt{(1-k^2\sin^2\phi)}\,.\,d\phi,$$

$$\int_0^y\sqrt{\frac{1-k^2y^2}{1-y^2}}\,dy=u-\int_0^u k^2\,\mathrm{sn}^2\,udu=\frac{E}{K}u+\partial_u\log\theta'\left(\frac{iu}{\lambda}\right)$$
$$=\frac{E}{K}\,\mathrm{sn}^{-1}y+\frac{i\pi}{2K}\frac{\dot\theta'\left(\frac{i\pi\,\mathrm{sn}^{-1}y}{2K}\right)}{\theta'\left(\frac{i\pi\,\mathrm{sn}^{-1}y}{2K}\right)}.$$

Functions of the Third Kind.

In the equation

$$\frac{\theta'\overline{x+y}\,.\,\theta'\overline{x-y}\,.\,\theta'^2}{\theta'x^2\,.\,\theta'y^2}=1-k^2fx^2\,.\,fy^2$$

take logarithmic fluxion in respect of y; this is

$$\partial_y\log\theta'\overline{x+y}-\partial_y\log\theta'\overline{x-y}-2\partial_y\log\theta'y=\frac{-2k^2fx^2\,.\,\lambda gy\,.\,hy\,.\,fy}{1-k^2fx^2\,.\,fy^2}.$$

Integrate in respect of x from 0 to x; thus

$$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}}+\frac{\dot\theta'y}{\theta'y}x=\lambda fy\,.\,gy\,.\,hy\int_0^x\frac{k^2fx^2dx}{1-k^2fy^2\,.\,fx^2};$$

$$\therefore\quad\tfrac{1}{2}\log\frac{\theta'\left(\dfrac{i\pi}{2K}\overline{u-v}\right)}{\theta'\left(\dfrac{i\pi}{2K}\overline{u+v}\right)}+u\frac{\dot\theta'\left(\dfrac{i\pi v}{2K}\right)}{\theta'\left(\dfrac{i\pi v}{2K}\right)}=\int_0^u\frac{\operatorname{sn}v\operatorname{cn}v\operatorname{dn}v\,k^2\operatorname{sn}^2u\,du}{1-k^2\operatorname{sn}^2v\operatorname{sn}^2u}.$$

VI.

Abelian form of the Addition-Theorem.

Consider the curve

$$y^2=x(1-x)(1-k^2x)$$

If we write $\qquad x=\operatorname{sn}^2u,$

then $\qquad y=\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u,$

and every point on the curve may be denoted by its parameter u.

Let us now cut the curve by the straight line

$$\xi x+\eta y=1;$$

the abscissæ of the point of contact are given by the equation

$$0=(1-\xi x)^2-\eta^2x(1-x)(1-k^2x),\ \text{say}\ \phi(x)=0.$$

If we vary ξ, η in this equation, we shall also vary the roots of it. Let x now signify any one of the roots, then we shall have

$$\phi'xdx+2x(1-\xi x)\,d\xi-2\eta x(1-x)(1-k^2x)\,d\eta=0.$$

Now since $\phi x=0$, $\eta\sqrt{x(1-x)(1-k^2x)}=1-\xi x$,

$$\therefore\quad\frac{dx}{\sqrt{x(1-x)(1-k^2x)}}+2\,\frac{\eta d\xi-(1-\xi x)\,\eta d\eta}{\phi'x}=0.$$

Sum this equation for the three roots $x_1\,x_2\,x_3=\operatorname{sn}^2u$, sn^2v, sn^2w; the sum of the second terms vanishes, and the first terms are $2du\ 2dv\ 2dw$. Hence inte-

grating from $\xi=0$ $\eta=0$, which gives $x_1\ x_2\ x_3=\infty$ and therefore $u=v=w=iK$, we have $u+v+w=3iK$ when the three points are in a line; or if $u-iK=u_1$, $v-iK=v_1$, $w-iK=w_1$, then $u+v+w=0$.

But eliminating $\xi\ \eta$ between the three equations

$$\begin{aligned}&1-\xi x_1-\eta\sqrt{(x_1\,.\,\overline{1-x_1}\,.\,\overline{1-k^2x_1})}=0\\&1-\xi x_2-\eta\sqrt{(x_2\,.\,\overline{1-x_2}\,.\,\overline{1-k^2x_2})}=0\\&1-\xi x_3-\eta\sqrt{(x_3\,.\,\overline{1-x_3}\,.\,\overline{1-k^2x_3})}=0\end{aligned}\quad\text{whence}\quad\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}=0,$$

and substituting $\operatorname{sn}^2u\ \operatorname{sn}^2v\ \operatorname{sn}^2w$ for $x_1\ x_2\ x_3$, we get the addition-theorem in the form

$$\begin{vmatrix}1 & \operatorname{sn}^2u & \operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\\ 1 & \operatorname{sn}^2v & \operatorname{sn}v\operatorname{cn}v\operatorname{dn}v\\ 1 & \operatorname{sn}^2w & \operatorname{sn}w\operatorname{cn}w\operatorname{dn}w\end{vmatrix}=0$$

when $u+v+w\equiv iK'$ (mod. $2K$, $2iK'$).

But observing that

$$\operatorname{sn}(u+iK')=\frac{1}{k\operatorname{sn}u},\quad \operatorname{cn}(u+iK')=-i\frac{\operatorname{dn}u}{k\operatorname{sn}u},\quad \operatorname{dn}(u+iK')=-i\frac{\operatorname{cn}u}{\operatorname{sn}u},$$

we have

$$x=\operatorname{sn}^2(u_1+iK')=\frac{1}{k^2\operatorname{sn}^2u}$$

$$y=\operatorname{sn}(u_1+iK')\,.\operatorname{cn}(u_1+iK')\operatorname{dn}(u_1+iK')=-\frac{\operatorname{cn}u\operatorname{dn}u}{k^2\operatorname{sn}^3u},$$

and the addition-theorem becomes

$$\begin{vmatrix}\operatorname{sn}u & \operatorname{sn}^3u & \operatorname{cn}u\operatorname{dn}u\\ \operatorname{sn}v & \operatorname{sn}^3v & \operatorname{cn}v\operatorname{dn}v\\ \operatorname{sn}w & \operatorname{sn}^3w & \operatorname{cn}w\operatorname{dn}w\end{vmatrix}=0$$

when $u+v+w\equiv 0$.

To verify this observe that

$$\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v-\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u=\operatorname{sn}(u-v)\,(1-k^2\operatorname{sn}^2u\operatorname{sn}^2v).$$

We have therefore to prove that

$$\begin{aligned}&\operatorname{sn}^3u\,.\operatorname{sn}\overline{v-w}+\operatorname{sn}^3v\,.\operatorname{sn}\overline{w-u}+\operatorname{sn}^3w\,.\operatorname{sn}\overline{u-v}\\&\qquad=k^2\operatorname{sn}^2u\operatorname{sn}^2v\operatorname{sn}^2w\,(\operatorname{sn}u\,.\operatorname{sn}\overline{v-w}+\operatorname{sn}v\,.\operatorname{sn}\overline{w-u}+\operatorname{sn}w\operatorname{sn}\overline{u-v}).\end{aligned}$$

Examples of use of the elliptic parameter.

1. The inflexions of the cubic (cusps of 3rd class curve) are given by $3u\equiv 0$ (mod. $2K$, $2iK'$) or $u=\frac{2}{3}aK+\frac{2}{3}a'iK'$, ($a$, $a'=0, 1, 2$). Hence there are nine of them, and the line joining any two passes through a third. If v is point of contact of tangent from an inflexion, $2v+u\equiv 0$, and thus the points of contact of the *three* tangents from u are in one straight line.

2. If six points $a\ b\ c\ f\ g\ h$ lie on a conic, $a+b+c+f+g+h=0$. This may be proved in the same way as the corresponding property for a straight line, or

thus: if $a+f+x=0$, $b+g+y=0$, $c+h+z=0$, we know that $x+y+z=0$, and therefore ..(Clebsch). Hence if a conic have six-pointic contact at v, $6v\equiv 0$; this shews that the v are points of contact of tangents from an inflexion, and there are 27 of them.

3. The points of contact of four tangents from u are $-\frac{1}{2}u$, $-\frac{1}{2}u+K$, $-\frac{1}{2}u+iK'$, $-\frac{1}{2}u+K+iK'$. Hence the line joining two of them meets the line joining the other two on the cubic. [Theory of *corresponding points.*]

4. Grassmann's construction; $xaA \,.\, xbB \,.\, xcC=0$ is equation to a cubic circumscribing the triangles abc, ABC and passing through the three intersections bcA, caB, abC of their corresponding sides. For BC, CA, AB write α, β, γ;

then $$b+c\equiv\beta+\gamma,\; c+a\equiv\gamma+\alpha,\; a+b\equiv\alpha+\beta,$$

whence we find $$a-\alpha\equiv b-\beta\equiv c-\gamma\equiv K \text{ or } iK' \text{ or } K+iK'.$$

Thus given abc there is one triangle $\alpha\beta\gamma$ on the same branch.

VII.

The Theta functions expressed as infinite products.

To expand the product of n factors

$$\Pi^n(1+r^sx)=(1+x)\,(1+rx)\,(1+r^2x)\ldots(1+r^{n-1}x)$$

in powers of x, assume that the expansion is

$$P=1+p_1x+p_2x^2+\ldots+p_nx^n.$$

Now if in the product we change x into rx, it becomes multiplied by $1+r^nx$ and divided by $1+x$. But if in P we change x into rx, it becomes

$$Q=1+p_1rx+p_2r^2x^2+\ldots+p_nr^nx^n.$$

Consequently we must have $P\,.\,(1+r^nx)=Q\,.\,(1+x)$. But

$$P\,(1+r^nx)=1+(p_1+r^n)\,x+(p_2+r^np_1)\,x^2+\ldots+(p_n+r^np_{n-1})\,x^n,$$
$$Q\,(1+x)=1+(p_1r+1)\,x+(p_2r^2+p_1r)x^2+\ldots+(p_nr^n+p_{n-1}r^{n-1})\,x^n.$$

Equating coefficients of like powers, we find

$$p_1+r^n=p_1r+1, \quad \text{or } p_1=\frac{1-r^n}{1-r},$$

$$p_2+r^np_1=p_2r^2+p_1r, \text{ or } p_2=p_1\,.\,\frac{1-r^{n-1}}{1-r^2}\,.\,r,$$

$$p_3+r^np_2=p_3r^3+p_2r^2, \text{ or } p_3=p_2\,.\,\frac{1-r^{n-2}}{1-r^3}\,.\,r^2,$$

etc. = etc.

$$p_n+r^np_{n-1}=p_nr^n+p_{n-1}r^{n-1}, \text{ or } p_n=p_{n-1}\,.\,\frac{1-r}{1-r^n}\,.\,r^{n-1}.$$

Let the product $(1-r)(1-r^2)\dots(1-r^n)$ be called $\mathfrak{R}(n)$. Then our result may be written

$$\frac{\Pi^n(y-r^s x)}{\mathfrak{R}(n)}=\frac{y^n}{\mathfrak{R}(n)}+\frac{y^{n-1}}{\mathfrak{R}(n-1)}\cdot\frac{x}{\mathfrak{R}(1)}\cdot r^0+\frac{y^{n-2}}{\mathfrak{R}(n-2)}\cdot\frac{x^2}{\mathfrak{R}(2)}\cdot r^1$$
$$+\dots+\frac{x^n}{\mathfrak{R}(n)}r^{\frac{1}{2}n(n-1)}.$$

As r approaches the limit 1, the fraction $\frac{1-r^p}{1-r^q}$ approaches the limit $\frac{p}{q}$. Hence the series just obtained passes into the binomial theorem.

To expand the product of the two factorials

$$\Pi^n(1+xr^{2s-1})=(1+xr)(1+xr^3)(1+xr^5)+\dots(1+xr^{2n-1})$$

$$\Pi^n(1+x^{-1}r^{2s-1})=(1+x^{-1}r)(1+x^{-1}r^3)(1+x^{-1}r^5)\dots(1+x^{-1}r^{2n-1})$$

in positive and negative powers of x, we proceed as follows. Let each factor $1+x^{-1}r^{2s-1}$ of the second be replaced by $x^{-1}r^{2s-1}(1+xr^{1-2s})$, to which it is equal; then the factorial becomes

$$\Pi^n(1+x^{-1}r^{2s-1})=x^{-n}r^{nn}(1+xr^{-1})(1+xr^{-3})(1+xr^{-5})\dots(1+xr^{1-2n}),$$

and in this form it is seen to be a continuation of the former factorial backwards, for negative powers of r. To put them both together therefore we must begin with the last factor of the second, $(1+xr^{1-2n})$. Let this be called $1+y$, then $x^{-n}r^{nn}=y^{-n}r^{n(1-n)}$, and we have

$$\Pi^n.(1+xr^{2s-1})(1+x^{-1}r^{2s-1})=y^{-n}r^{n(1-n)}(1+y)(1+yr^2)(1+yr^4)\dots(1+yr^{2(n-1)})$$
$$=y^{-n}r^{n(1-n)}\Pi^{2n}(1+yr^{2s}).$$

But by our previous result

$$\frac{\Pi^{2n}(1+yr^{2s})}{\mathfrak{R}^2[2n]}=\frac{1}{\mathfrak{R}^2[2n]}+\frac{1}{\mathfrak{R}^2[2n-1]}\cdot\frac{y}{\mathfrak{R}^2[1]}$$
$$+\frac{1}{\mathfrak{R}^2[2n-2]}\cdot\frac{y^2}{\mathfrak{R}^2[2]}r^2+\dots+\frac{y^{2n}}{\mathfrak{R}^2[2n]}r^{2n(2n-1)};$$

$$\therefore\frac{\Pi^n(1+xr^{2s-1})(1+x^{-1}r^{2s-1})}{\mathfrak{R}^2[2n]}=\frac{x^{-n}}{\mathfrak{R}^2[2n]}r^{nn}+\frac{x^{1-n}}{\mathfrak{R}^2[2n-1]\mathfrak{R}^2[1]}\cdot r^{(n-1)^2}$$
$$+\frac{x^{2-n}}{\mathfrak{R}^2[2n-2]\mathfrak{R}^2[2]}r^{(n-2)^2}+\dots+\frac{1}{\mathfrak{R}^2[n].\mathfrak{R}^2[n]}+\dots+\frac{x^{n-1}}{\mathfrak{R}^2[2n-1]\mathfrak{R}^2[1]}\cdot r^{(n-1)^2}$$
$$+\frac{x^n}{\mathfrak{R}^2[2n]}rn^2;$$

therefore

$$\frac{\mathfrak{R}^2[n].\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}\Pi^n(1+xr^{2s-1})\Pi^n(1+x^{-1}r^{2s-1})=1+r(x+x^{-1}).\frac{1-r^{2n}}{1-r^{2n+2}}$$
$$+r^4(x^2+x^{-2}).\frac{\overline{1-r^{2n}}.\overline{1-r^{2n-2}}}{1-r^{2n+2}.1-r^{2n+4}}+\dots$$
$$+r^{ss}(x^s+x^{-s})..\frac{1-r^{2n}.1-r^{2n-2}\dots1-r^{2n-2s+2}}{1-r^{2n+2}.1-r^{2n+4}\dots1-r^{2n+2s}}\dots+r^{nn}(x^n+x^{-n})\frac{\mathfrak{R}^2[n]\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}.$$

Suppose r less than 1, and let n increase indefinitely. Then

$\frac{\mathfrak{R}^2[n]\,\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]} = \frac{1-r^2 . 1-r^4 . . 1-r^{2n}}{1-r^{2n+2} . 1-r^{2n+4} \ldots 1-r^{4n}}$; now each of the factors $\frac{1-r^{2s}}{1-r^{2n+s}}$ approaches the value $1-r^{2s}$; therefore the limiting value of the expression is $\mathfrak{R}^2[\infty]$. On the right hand side the factors $\frac{1-r^{2n-2s}}{1-r^{2n+2s+2}}$ approach the limit 1; therefore we have

$$\mathfrak{R}^2[\infty]\,\Pi_0^\infty\,(1+xr^{2s+1})\,(1+x^{-1}r^{2s+1}) = 1 + r\,(x+x^{-1}) + r^4\,(x^2+x^{-2}) + r^9\,(x^3+x^{-3}) + \text{etc.}$$

Write q for $\mathfrak{R}$ and $e^{2i\phi}$ for x; the formula becomes

$$q^2[\infty] . \Pi_0^\infty\,(1+2q^{2s+1}\cos 2\phi + q^{4s+2}) = 1 + 2q\cos 2\phi + 2q^4\cos 4\phi + 2q^9\cos 6\phi + \ldots$$
$$= \theta\,(y, \log q) \text{ if } \phi = iy.$$

Either putting $\phi + \frac{\pi}{2}$ for ϕ or changing the sign of q, we have

$$q^2[\infty] . \Pi_0^\infty\,(1-2q^{2s-1}\cos 2\phi + q^{4s-2}) = 1 - 2q\cos 2\phi + 2q^4\cos 4\phi - 2q^9\cos 6\phi + \ldots$$
$$= \theta'\,(y, \log q).$$

In the original formula write rx for x; it becomes

$$\mathfrak{R}^2[\infty] . \Pi_1^\infty\,(1+xr^{2s})\,(1+x^{-1}r^{2s-2}) = 1 + r^2x + x^{-1} + r^6x^2 + r^2x^{-2} + r^{12}x^3 + r^6x^{-3} + \ldots$$
$$= x^{-\frac{1}{2}}r^{-\frac{1}{4}}\{r^{\frac{1}{4}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + r^{\frac{9}{4}}(x^{\frac{3}{2}} + x^{-\frac{3}{2}}) + \ldots\}.$$

Here also write q for $\mathfrak{R}$ and $e^{2i\phi}$ for x, then we find

$$q^2[\infty] . 2q^{\frac{1}{4}}\cos\phi\,\Pi_0^\infty\,(1+2q^{2s+2}\cos 2\phi + q^{4s+4})$$
$$= 2q^{\frac{1}{4}}\cos\phi + 2q^{\frac{9}{4}}\cos 3\phi + 2q^{\frac{25}{4}}\cos 5\phi + \ldots$$

In this if we write $\phi + \frac{\pi}{2}$ for ϕ, it becomes

$$q^2[\infty] . 2q^{\frac{1}{4}}\sin\phi\,\Pi_0^\infty\,(1-2q^{2s+2}\cos 2\phi + q^{4s+4}) = 2q^{\frac{1}{4}}\sin\phi - 2q^{\frac{9}{4}}\sin 3\phi + 2q^{\frac{25}{4}}\sin 5\phi - \ldots$$

Now observing that $q = e^{a}$, and that the factor

$$1 + 2q^{2s+1}\cos 2\phi + q^{4s+2} = (1 + q^{2s+1}e^{2i\phi})\,(1 + q^{2s+1}e^{-2i\phi}),$$

which becomes

$$(1 + e^{(2s+1)a + 2i\phi})\,(1 + e^{(2s+1)a - 2i\phi})$$
$$= \frac{(e^{-i\phi - (s+\frac{1}{2})a} + e^{i\phi + \overline{s+\frac{1}{2}}.a})\,(e^{i\phi - (s+\frac{1}{2})a} + e^{-i\phi + \overline{s+\frac{1}{2}}.a})}{e^{-(2s+1)a}}$$
$$= 4e^{(2s+1)a}\cos(\phi + \overline{s+\tfrac{1}{2}}ia)\cos(\phi - \overline{s+\tfrac{1}{2}} . ia).$$

Put $\phi=0$ in this; we get

$$1+2q^{2s+1}+q^{4s+2}=4e^{(2s+1)a}\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia,$$

and then by division

$$\frac{1+2q^{2s+1}\cos 2\phi+q^{4s+2}}{1+2q^{2s+1}+q^{4s+2}}=\frac{\cos(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\cos(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia}.$$

Hence $$\theta(i\phi)=\theta\,.\,\Pi_0^\infty\frac{\cos(\phi+\overline{s+\tfrac{1}{2}}ia)\cos(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia},$$

so $$\theta'(i\phi)=\theta\,.\,\Pi_0^\infty\frac{\sin(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\sin(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia},$$

whence $$\theta'=\theta\,.\,\Pi_0^\infty\tan^2\overline{s+\tfrac{1}{2}}\,.\,ia,$$

and therefore

$$\theta'(i\phi)=\theta'.\,\Pi_0^\infty\frac{\sin(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\,.\,\sin(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\sin^2\overline{s+\tfrac{1}{2}}\,.\,ia}.$$

Similarly

$$\begin{aligned}1+2e^{(2s+2)a}\cos 2\phi+e^{(4s+4)a}&=(1+e^{\overline{2s+2}a+2i\phi})(1+e^{\overline{2s+2}a-2i\phi})\\&=4e^{\overline{2s+2}a}\cos(\overline{s+1}\,.\,ai+\phi)\cos(s+1\,.\,ai-\phi),\end{aligned}$$

and therefore

$$\theta_1(i\phi)=\theta_1\,.\,\Pi_1^\infty\frac{\cos(\phi+s\,.\,ia)\cos(\phi-s\,.\,ia)}{\cos^2 s\,.\,ia};$$

so also

$$\theta_1'(i\phi)=i\theta_1\,.\,\Pi_1^\infty\frac{\sin(\phi+s\,.\,ia)\sin(\phi-s\,.\,ia)}{\cos^2 s\,.\,ia}.$$

Returning to the factorial expression for θx, which may be written in the form

$$\theta x=\theta\,.\,\Pi_0^\infty\frac{\cos i\,.\,(x+\overline{s+\tfrac{1}{2}}\,.\,a)\,.\cos i(x-\overline{s+\tfrac{1}{2}}\,.\,a)}{\cos^2 i\,.\overline{s+\tfrac{1}{2}}\,.\,a},$$

we observe that

$$\cos x=\Pi_0^\infty\left(1-\frac{x^2}{(t+\tfrac{1}{2})^2\pi^2}\right),\ (t\text{ an integer}),\ =\Pi_0^\infty\left(1+\frac{x}{(t+\tfrac{1}{2})\pi}\right)\left(1-\frac{x}{(t+\tfrac{1}{2})\pi}\right).$$

Hence

$$\frac{\cos i.(x+\overline{s+\tfrac{1}{2}}.a)}{\cos i.(\overline{s+\tfrac{1}{2}}\cdot a)}=\Pi_0^\infty\frac{1+\dfrac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai}{(t+\tfrac{1}{2})\pi}}{1+\dfrac{(s+\tfrac{1}{2})\,.\,ai}{t+\tfrac{1}{2}\pi}}\cdot\Pi_0^\infty\frac{1-\dfrac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai}{(t+\tfrac{1}{2})\pi}}{1-\dfrac{(s+\tfrac{1}{2})\,.\,ai}{(t+\tfrac{1}{2})\pi}}$$

$$=\Pi_0^\infty\frac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai+\overline{t+\tfrac{1}{2}}\pi}{\overline{s+\tfrac{1}{2}}\,.\,ai+\overline{t+\tfrac{1}{2}}\pi}\cdot\Pi_0^\infty\frac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai-(t+\tfrac{1}{2})\pi}{(s+\tfrac{1}{2})ai-(t+\tfrac{1}{2})\pi}$$

$$=\Pi_0^\infty\left(1+\frac{x}{(s+\tfrac{1}{2})a+(t+\tfrac{1}{2})\pi i}\right)\Pi_0^\infty\left(1+\frac{x}{(s+\tfrac{1}{2})a-(t+\tfrac{1}{2})\pi i}\right).$$

Let now $(\bar{s}, \bar{t})$ denote $\pm(s+\frac{1}{2})a \pm (t+\frac{1}{2})\pi i$, and let

$$\Pi\left(1+\frac{x}{(\bar{s}, \bar{t})}\right) = \Pi_0^\infty \Pi_0^\infty \left(1-\frac{x^2}{\{(s+\frac{1}{2})a+(t+\frac{1}{2})\pi i\}^2}\right) \cdot \left(1-\frac{x^2}{\{(s+\frac{1}{2})a-(t+\frac{1}{2})\pi i\}^2}\right),$$

it being understood that the infinite values of t are infinitely greater than the infinite values of s; then we shall have

$$\theta x = \theta \,.\, \Pi\left(1+\frac{x}{(\bar{s}, \bar{t})}\right).$$

VIII.

Cayley's Theory of Doubly-infinite factorials.

A product like $\Pi\left(1+\frac{x}{(\bar{s}, \bar{t})}\right)$ containing a doubly infinite number of factors depending on two variable integers s, t, is not fully defined until we have fixed upon the relations of the infinite values of s and t. Let these be regarded as coordinates of a point in a plane; then we may suppose that the product is formed first with those values of s t which lie in a certain closed curve surrounding the origin, and that then this curve is allowed to expand without limit, remaining always similar to itself and similarly situated in regard to the origin. The doubly-infinite product so obtained will depend in general upon the shape of this curve.

If we suppose the curve to have the origin for a centre, i.e. that every line through the origin is bisected by it, then *the value of the product will be determined with the exception of a factor* e^{Ax^2}. For suppose two curves to be drawn having the origin for centre, and let Π, Π' be the products belonging to them. Then

$$\log \Pi - \log \Pi' = \Sigma \log\left(1+\frac{x}{(\bar{s}, \bar{t})}\right) = -x\Sigma\frac{1}{(\bar{s}, \bar{t})} + \frac{x^2}{2}\Sigma\frac{1}{(\bar{s}, \bar{t})^2} - \ldots,$$

the summation extending over those values of s, t which correspond to points lying between the two curves. Now $\Sigma\frac{1}{(\bar{s}, \bar{t})} = 0$ because of the symmetry; and let A be the limit of $\Sigma\frac{1}{(\bar{s}, \bar{t})^2}$, in comparison with which all subsequent terms must vanish, because s, t become infinite when the curves are increased without limit. Therefore

$$\log \Pi - \log \Pi' = Ax^2,$$

or

$$\Pi = e^{Ax^2} \,.\, \Pi'.$$

Now we have shewn that when $\frac{s_\infty}{t_\infty} = 0$,

$$\theta x = \theta \,.\, \Pi\left(1+\frac{x}{(\bar{s}, \bar{t})}\right) = \theta \,.\, \Pi\left(1+\frac{x}{(s+\frac{1}{2})a+(t+\frac{1}{2})\pi i}\right).$$

But

$$\frac{x}{(s+\frac{1}{2})a+(t+\frac{1}{2})\pi i}=\frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i-(t+\frac{1}{2})\frac{\pi^2}{a}},$$

$$\therefore\ \theta x=\theta\,.\,\Pi\left(1+\frac{x}{\overline{(s,\,t)}}\right)=\theta\,.\,\Pi\frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i+(t+\frac{1}{2})\frac{\pi^2}{a}}$$

$$=\frac{\theta\,.\,e^{Ax^2}\,.\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)}{\theta\left(0,\ \frac{\pi^2}{a}\right)},$$

since, in order that the Π may represent $\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)$, we must have $\frac{t_\infty}{s_\infty}=0$; the former arrangement regarded the plane as an infinite rectangle, infinitely longer in the direction of t than in that of s; the present one makes it infinitely longer in the direction of s than in that of t.

We must now determine A so that $e^{Ax^2}\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)$ may be unaltered when x is increased by πi. We have

$$e^{A(x+\pi i)^2}\,\theta\left(\frac{x\pi i}{a}-\frac{\pi^2}{a},\ \frac{\pi^2}{a}\right)=e^{2Ax\pi i-A\pi^2-\frac{\pi^2}{a}+\frac{2x\pi i}{a}}\,.\,e^{Ax^2}\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right),$$

whence $A=-\frac{1}{a}$, and the formula may be written

$$\theta\left(0,\ \frac{\pi^2}{a}\right).\,\theta x=\theta\,.\,e^{-\frac{x^2}{a}}\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=\theta\,.\,\Sigma e^{-\frac{1}{a}(x+n\pi i)^2}.$$

Observe that the formula may also be written

$$\theta\,.\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=e^{\frac{x^2}{a}}\,\theta\left(0,\ \frac{\pi^2}{a}\right).\,\theta x=\theta\left(0,\ \frac{\pi^2}{a}\right)\Sigma e^{\frac{1}{a}(x+na)^2}.$$

Similarly for the other θ, we have

$$\theta' x=\theta'\,.\,\Pi\frac{\sin i\,.\,(x+\overline{s+\frac{1}{2}}\,.\,a)\sin i\,.\,(x-\overline{s+\frac{1}{2}}\,.\,a)}{\sin^2 i\cdot\overline{s+\frac{1}{2}}\,.\,a},\ \text{and}\ \sin x=x\Pi\left(1-\frac{x^2}{t^2\pi^2}\right);$$

$$\therefore\ \frac{\sin i\,.\,(x+\overline{s+\frac{1}{2}}\,.\,a)}{\sin i\,.\,\overline{s+\frac{1}{2}}a}=\frac{x+\overline{s+\frac{1}{2}}\,.\,a}{(s+\frac{1}{2})a}\,\Pi\left(\frac{1+i\dfrac{x+\overline{s+\frac{1}{2}}\,.\,a}{t\pi}}{1+i\dfrac{\overline{s+\frac{1}{2}}\,.\,a}{t\pi}}\right)$$

$$=\left(1+\frac{x}{\overline{s+\frac{1}{2}}\,.\,a}\right)\Pi\left(1+\frac{x}{\overline{s+\frac{1}{2}}\,.\,a+t\pi i}\right),$$

$$\therefore\ \theta' x=\theta'\,.\,\Pi\left\{1+\frac{x}{\overline{(s,\,t)}}\right\}\qquad\left(\frac{s_\infty}{t_\infty}=0\right).$$

Hence also
$$\theta_1 x = \theta_1 . \Pi \left\{1 + \frac{x}{(s,\, t)}\right\}, \qquad \left(\frac{s_\infty}{t_\infty} = 0\right),$$
$$\theta'_1 x = \theta'_1 . \Pi \left\{1 + \frac{x}{(s,\, t)}\right\} . x. \qquad ,,$$

From these formulæ we may derive the transformations

$$* \, \theta_1{}^0 \left(0,\, \frac{\pi^2}{a}\right) . \theta' x = \theta' . e^{-\frac{x^2}{a}} \theta_1 \left(\frac{x\pi i}{a},\, \frac{\pi^2}{a}\right) = \theta' . \Sigma e^{-\frac{1}{a}(x + \overline{n+\frac{1}{2}} . \pi i)^2},$$
$$\theta' \left(0,\, \frac{\pi^2}{a}\right) . \theta_1 x = \theta_1 . e^{-\frac{x^2}{a}} \theta' \left(\frac{x\pi i}{a},\, \frac{\pi^2}{a}\right) = \theta_1 . \Sigma (-)^n e^{-\frac{1}{a}(x + n\pi i)^2},$$
$$i\theta'_1 \left(0,\, \frac{\pi^2}{a}\right) . \theta'_1 x = \theta'_1 . e^{-\frac{x^2}{a}} \theta'_1 \left(\frac{x\pi i}{a},\, \frac{\pi^2}{a}\right) = \theta'_1 . \Sigma (-)^n e^{-\frac{1}{a}(x + \overline{n+\frac{1}{2}} . \pi i)^2},$$

and, expressing these in terms of $f\ g\ h$, we find $\left\{f'u \text{ for } f\left(u,\, \frac{\pi^2}{a}\right)\right\}$

$$f'\left(\frac{x\pi i}{a}\right) = \frac{ifx}{gx}, \quad g'\left(\frac{x\pi i}{a}\right) = \frac{1}{gx}, \quad h'\left(\frac{x\pi i}{a}\right) = \frac{hx}{gx}.$$

Whence

$$h'^2 + k^2 f'^2 = g'^2 = 1 + f'^2, \text{ or } h'^2 = 1 + (1 - k^2) f'^2 = 1 + k'^2 f'^2,$$

from which it follows that k' is the k of the transformed functions.

To express the same result in terms of the elliptic functions we have only to observe that the transformation changes K into K' and vice versa; so that we have, if $x = \frac{\pi u i}{2K}$, $\frac{x\pi i}{a} = -\frac{\pi^2 u}{2Ka} = \frac{\pi u}{2K'}$ and consequently

$$\text{sn}\,(ix) = i\frac{\text{sn}\,(x,\, k')}{\text{cn}\,(x,\, k')} = i\,\text{tn}\,(x,\, k'), \qquad \text{cn}\,(ix) = \frac{1}{\text{cn}\,(x,\, k')}, \qquad \text{dn}\,(ix) = \frac{\text{dn}\,(x,\, k')}{\text{cn}\,(x,\, k')}.$$

The transformation may also be represented as follows. In virtue of the equation $\text{sn}^2 x + \text{cn}^2 x = 1$, we are at liberty to write

$$\text{sn}\, x = i \tan\phi, \quad \text{cn}\, x = \sec\phi, \quad \text{dn}\, x = \sqrt{(1 + k^2 \tan^2\phi)},$$

and then
$$\text{cn}\, x\, \text{dn}\, x\, dx = i \sec^2\phi\, d\phi,$$

$$dx = \frac{i\, d\phi \sec\phi}{\sqrt{(1 + k^2\tan^2\phi)}} = \frac{i\, d\phi}{\sqrt{(\cos^2\phi + k^2\sin^2\phi)}}$$
$$= \frac{i\, d\phi}{\sqrt{(1 - k'^2\sin^2\phi)}},$$

whence
$$\phi = -\,am\,(ix),$$

$$\text{sn}\,(ix,\, k') = -\sin\phi = -\frac{\tan\phi}{\sec\phi} = i\frac{\text{sn}\, x}{\text{cn}\, x}, \quad \text{cn}\,(ix,\, k') = \cos\phi = \frac{1}{\text{cn}\, x},$$

$$\text{dn}\,(ix,\, k') = \sqrt{(1 - k'^2\sin^2\phi)} = \frac{\text{dn}\, x}{\text{cn}\, x},$$

which formulæ are equivalent to the former. Observe that the transformation gives

$$\int_0^1 \frac{du}{\sqrt{(1 - u^2)(1 - k'^2 u^2)}} = K'.$$

* [? θ_1].

IX.

Problem of linear transformations.

The transformation just considered amounts to an interchange of the period and quasi-period; for the function $\theta\left(\frac{x\pi i}{a}, \frac{\pi^2}{a}\right)$ has the period a and the quasi-period πi. If $\alpha\ \beta\ \gamma\ \delta$ are whole numbers, the problem *to express the θ-function which has the period $\alpha\pi i+\beta a$ and the quasi-period $\gamma\pi i+\delta a$ in terms of the θ-function with period πi and quasi-period a is called the* Transformation-Problem. If we write $\alpha\pi i+\beta a=\omega$, $\gamma\pi i+\delta a=\omega'$, the θ-function is $\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)$. Now if $\alpha\delta-\beta\gamma=-1$, [in which case the transformation is said to be linear] we shall find $-\pi i=\delta\omega-\beta\omega'$, $-a=\alpha\omega'-\gamma\omega$; and on the plane of complex numbers the area of the parallelogram included by $\omega\ \omega'$ is the same as that included by πi and a. The quotient $\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right):\theta\left(0, \frac{\omega'\pi i}{\omega}\right)$ is then equal to the doubly infinite product $\Pi\left(1+\frac{x}{(s+\frac{1}{2})\,\omega+(t+\frac{1}{2})\,\omega'}\right)$, $\frac{t_\infty}{s_\infty}=0$, that is when the plane is regarded as an infinite parallelogram whose sides are parallel to $\omega\ \omega'$, but the former side infinitely greater than the latter. Hence we must have

$$\frac{\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)}{\theta\left(0, \frac{\omega'\pi i}{\omega}\right)}=e^{Ax^2}\frac{\theta_q^p(x)}{\theta_q^p} \text{ if } pq=0, \text{ or } =e^{Ax^2}\frac{\theta_1'(x)}{\theta_1'} \text{ if } p=1,\ q=1.$$

To determine A, observe that the left-hand side is unaltered when x is increased by ω, which is $=\alpha\pi i+\beta a$. Consequently

$$e^{A\,(x+\alpha\pi i+\beta a)^2}\,e^{q\alpha\pi i-\beta^2 a-2\beta x}\,\theta_q^p x=e^{Ax^2}\,\theta_q^p x,$$

therefore $$A\,(\alpha\pi i+\beta a)=\beta, \quad \therefore\ A=\frac{\beta}{\omega},$$

$$A\,(\alpha\pi i+\beta a)^2=-q\alpha\pi i+\beta^2 a+2m\pi i,$$

or $$\beta\,(\alpha\pi i+\beta a)=-q\alpha\pi i+\beta^2 a+2m\pi i.$$

Hence $\quad \alpha q=\alpha\beta$ mod. 2 but $q\equiv\beta+\delta+1$, $\therefore\ \alpha\delta=\alpha$ mod. 2.

Now $$(s+\tfrac{1}{2})\,\omega+(t+\tfrac{1}{2})\,\omega'=(s+\tfrac{1}{2})\,(\alpha\pi i+\beta a)+(t+\tfrac{1}{2})\,(\gamma\pi i+\delta a)$$
$$=(\alpha s+\gamma t+\tfrac{1}{2}\,\overline{\alpha+\gamma})\,\pi i+(\beta s+\delta t+\tfrac{1}{2}\,.\,\overline{\beta+\delta})\,a.$$

Hence the formulæ must be

$$\frac{\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)}{\theta\left(0, \frac{\omega'\pi i}{\omega}\right)}=e^{\frac{\beta x^2}{\omega}}\,\frac{\theta_1'(x+\frac{1}{2}\omega+\frac{1}{2}\omega')}{\theta_1'(\frac{1}{2}\omega+\frac{1}{2}\omega')},$$

or generally since if p, $q=0$ or 1,

$$\left(s+\frac{p}{2}\right)\omega+\left(t+\frac{q}{2}\right)\omega'=\left(s+\frac{p}{2}\right)(\alpha\pi i+\beta a)+\left(t+\frac{q}{2}\right)(\gamma\pi i+\delta a)$$

$$=(\alpha s+\gamma t+\overline{\tfrac{1}{2}p\alpha+q\gamma})\,\pi i+(\beta s+\delta t+\overline{\tfrac{1}{2}p\beta+q\delta})\,a,$$

$$\frac{\theta^{p}_{q}\left(\dfrac{x\pi i}{\omega},\dfrac{\omega'\pi i}{\omega}\right)}{\theta^{p}_{q}\left(0,\dfrac{\omega'\pi i}{\omega}\right)}=e^{\frac{\beta x^2}{\omega}}\frac{\theta^{p+1}_{q+1}(x+\frac{1}{2}\omega+\frac{1}{2}\omega')}{\theta^{p+1}_{q+1}(\frac{1}{2}\omega+\frac{1}{2}\omega')}.$$

There are six cases of this. Since $\beta\gamma-\alpha\delta=1$, the four numbers cannot be all even or all odd. If $\beta\gamma$ is odd, α or δ or both may be even; if $\alpha\delta$ is odd, β or γ or both may be even. Besides the case already considered, the only one of interest is $\alpha=\gamma=\delta=1$, $\beta=0$ or $\omega=\pi i$, $\omega'=\pi i+a$; in this we have

$$\theta(x,a+\pi i)=\Sigma e^{n^2(a+\pi i)+2nx}=\theta'(x,a),$$

$$\theta'(x,a+\pi i)=\Sigma e^{n^2\pi i+n^2(a+\pi i)+2nx}=\theta(x,a),$$

$$\theta_1(x,a+\pi i)=\Sigma e^{(n+\frac{1}{2})^2(a+\pi i)+2(n+\frac{1}{2})x}=i\theta_1(x,a),$$

and $$\theta_1'(x,a+\pi i)=\Sigma e^{n^2\pi i+(n+\frac{1}{2})^2\overline{a+\pi i}+2\overline{n+\frac{1}{2}}\,.\,x}=i\theta_1'(x,a).$$

Hence $$\mathrm{sn}'\,x=k'\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x},\quad \mathrm{cn}'\,x=\frac{\mathrm{cn}\,x}{\mathrm{dn}\,x},\quad \mathrm{dn}'\,x=\frac{1}{\mathrm{dn}\,x}.$$

So if λ be the new modulus,

$$\mathrm{dn}'^2\,x+\lambda^2\,\mathrm{sn}'^2\,x=\frac{1+\lambda^2k'^2\,\mathrm{sn}^2\,x}{\mathrm{dn}^2\,x}=1,\ \text{ or }\ \lambda^2=-\frac{k^2}{k'^2},\ \lambda=\frac{ik}{k'}.$$

Also $$\mathrm{sn}'\,x=k'\frac{\mathrm{cn}\,x\,\mathrm{dn}^2\,x+k^2\,\mathrm{cn}\,x\,\mathrm{sn}^2\,x}{\mathrm{dn}^2\,x}=k'\,\mathrm{cn}'\,x\,\mathrm{dn}'\,x,$$

whence

$$\mathrm{sn}'\,x=\mathrm{sn}\left(k'x,\frac{ik}{k'}\right)=k'\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x},\quad \mathrm{cn}\left(k'x,\frac{ik}{k'}\right)=\frac{\mathrm{cn}\,x}{\mathrm{dn}\,x},\quad \mathrm{dn}\left(k'x,\frac{ik}{k'}\right)=\frac{1}{\mathrm{dn}\,x}.$$

X.

General Problem of Transformation.

Jacobi's Theorem for product of n θ-functions.

Consider the product of the n functions

$$\theta(x+u_s,a)=\Sigma e^{m_s^2a+2m_s(x+u_s)},\qquad (s=1,2,\ldots n)$$

we shall have

$$\Pi\theta(x+u_s)=\Sigma^n e^{a\Sigma m^2+2x\Sigma m+2\Sigma mu},$$

the summation being so taken that the numbers $m_1\,m_2\ .\ m_n$ take all values from $-\infty$ to $+\infty$. For any particular set of values of the m, let their sum Σm be

divided by n; let β be the quotient and a the remainder, so that a is less than n and $\Sigma m = n\beta + a$. Then let $m_s - \beta$ be denoted by μ_s; we shall have $\Sigma\mu = \Sigma m - n\beta = a$, or $\Sigma\mu$ is positive and less than n. This being so, the numbers $\mu_s + \beta$ will take all values from $-\infty$ to $+\infty$, provided that (1) the μ take all values consistent with $0 < \Sigma\mu < n$, and (2) β take all values from $-\infty$ to $+\infty$. Now we have for any given set of values of the m,

$$\begin{aligned}\Sigma m &= a + n\beta,\\ \Sigma m^2 &= \Sigma(\mu+\beta)^2 = \Sigma\mu^2 + 2\beta(a) + n\beta^2,\\ \Sigma mu &= \Sigma\mu u + \beta\Sigma u.\end{aligned}$$

The exponent of the general term becomes therefore

$$a(\Sigma\mu^2 + 2a\beta + n\beta^2) + 2x(a + n\beta) + 2\Sigma\mu u + 2\beta\Sigma u$$

and we have

$$\begin{aligned}\Pi\theta(x+u_s) &= \sum_{a=0}^{a=n-1} P_a e^{2ax} \sum_{\beta=-\infty}^{\beta=+\infty} e^{\beta^2 . na + 2\beta(nx + aa + \Sigma u)}\\ &= \sum_{a=0}^{a=n-1} P_a e^{2ax}\,\theta(nx + aa + \Sigma u, na),\end{aligned}$$

where $P_a = \Sigma e^{a\Sigma\mu^2 + 2\Sigma\mu u}$, the summation extended over all those values of μ which make $\Sigma\mu = a$. The values of the P_a may be determined as follows. Write in the formula $x + \frac{h\pi i}{n}$ for x; then we find

$$\Pi\theta\left(x + u_s + \frac{h\pi i}{n}\right) = \sum_{a=0}^{a=n-1} P_a e^{2ax} e^{2a\frac{h\pi i}{n}}\,\theta(nx + aa + \Sigma u, na).$$

By giving to h the values $0, 1, 2 \,..n-1$ we obtain n equations between the n quantities $\theta(nx + aa + \Sigma u, na)$ for values of a from 0 to $n-1$. Solving these by means of the known properties of the n^{th} roots of unity, we find

$$nP_a e^{2ax}\,\theta(nx + aa + \Sigma u, na) = \sum_{h=0}^{h=n-1} e^{-2a\frac{h\pi i}{n}} \prod_{s=1}^{s=n} \theta\left(x + u_s + \frac{h\pi i}{n}\right),$$

this determines the P_a when we put $x = 0$.

Suppose now that n is a prime number, and that $u_s = \frac{s\pi i}{n}$. Then $\Pi\theta(x+u_s)$ is unaltered when we increase x by $\frac{h\pi i}{n}$; moreover we have $\Sigma u = \frac{1}{2}(n-1)\pi i$. Therefore, since $\Sigma e^{-2a\frac{h\pi i}{n}} = 0$ unless $a = 0$, and then $= n$, it follows that

$$P_0\theta(nx, na) = \Pi\theta\left(x + \frac{s\pi i}{n}\right).$$

Putting $x = 0$, we convert the equation into

$$\frac{\theta(nx, na)}{\theta(0, na)} = \frac{\Pi\theta\left(x + \frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)}.$$

Write $x+\frac{1}{2}\pi i$ for x, and remember that n is odd; thus

$$\frac{\theta'(nx, na)}{\theta(0, na)}=\frac{\Pi\theta'\left(x+\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)} \text{ whence } \frac{\theta'(0, na)}{\theta(0, na)}=\frac{\Pi\theta'\left(\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)}.$$

Again, writing successively $x+\frac{1}{2}a$, $x+\frac{1}{2}\pi i+\frac{1}{2}a$ for x, we find

$$\frac{\theta_1(nx, na)}{\theta_1(0, na)}=\frac{\Pi\theta_1\left(x+\frac{s\pi i}{n}\right)}{\Pi\theta_1\left(\frac{s\pi i}{n}\right)},$$

$$\frac{\theta'_1(nx, na)}{\theta(0, na)}=\frac{\Pi\theta'_1\left(x+\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)},$$

whence

$$\frac{\theta'_1(0, na)}{\theta(0, na)}=\frac{\theta'_1\prod_{s=1}^{s=n-1}\theta'_1\left(\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)},$$

$$\therefore \operatorname{sn}\left(\frac{x}{M}, \lambda\right)=\frac{\prod_{s=0}^{s=n-1}\operatorname{sn}\left(x+\frac{2sK}{n}\right)}{\prod_{s=0}^{s=n-1}\operatorname{sn}\left(K+\frac{2sK}{n}\right)},$$

where

$$\frac{1}{M}=\frac{\prod_{s=1}^{s=n-1}\operatorname{sn}\left(\frac{2sK}{n}\right)}{\Pi\operatorname{sn}\left(K+\frac{2sK}{n}\right)},$$

$$\operatorname{cn}\left(\frac{x}{M}, \lambda\right)=\frac{\Pi\operatorname{cn}\left(x+\frac{2sK}{n}\right)}{\Pi\operatorname{cn}\left(\frac{2sK}{n}\right)},$$

$$\operatorname{dn}\left(\frac{x}{M}, \lambda\right)=\frac{\Pi\operatorname{dn}\left(x+\frac{2sK}{n}\right)}{\Pi\operatorname{dn}\left(\frac{2sK}{n}\right)}.$$

XI.

Schröter's Theorem for product of two θ-functions.

We propose to find the value of the product

$$\theta(x, pa)\,.\,\theta(y, qa)=\Sigma e\,|\,m^2pa+2mx\,|\,.\,\Sigma e\,|\,n^2qa+2ny\,|$$
$$=\Sigma\Sigma e\,|\,\overline{m^2p+n^2q}\,.\,a+2mx+2ny\,|.$$

Let $n-m$ when divided by $p+q$ give s for quotient and μ for remainder, so that $n-m=s(p+q)+\mu$, $(\mu<p+q)$.

If then we make

$$m=t-qs,$$

we must have

$$n=t+ps+\mu,$$

and the numbers m, n will take all integer values from $-\infty$ to $+\infty$ if μ takes all positive integer values $<p+q$, and s, t take all integer values from $+\infty$ to $-\infty$. Making this substitution, the exponent of the general term becomes

$$(t-qs)^2pa+(t+ps+\mu)^2qa+2(t-qs)x+2(t+ps+\mu)y$$
$$=\mu^2qa+2\mu y+(t^2\,.\,\overline{p+q}\,.\,a+2t\,.\,\overline{x+y+\mu qa})+(s^2\,.\,pq\,.\,\overline{p+q}\,.\,a+2s\,.\,\overline{py-qx+\mu pqa}).$$

Consequently we have

$$\theta(x, pa)\,\theta(y, qa)$$
$$=\sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2qa+2\mu y}\,\theta(x+y+\mu qa,\ \overline{p+q}\,.\,a)\,.\,\theta(py-qx+\mu pqa,\ pq\,.\,\overline{p+q}\,.\,a)$$
$$=\sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2pa+2\mu x}\,\theta(x+y+\mu pa,\ \overline{p+q}\,.\,a)\,\theta(qx-py+\mu pqa,\ pq\,.\,\overline{p+q}\,.\,a).$$

For x, y write $y+x$, $ny-x$, and for p, q, 1 and n respectively, in the second formula; then

$$\theta(x+y, a)\,\theta(ny-x, na)$$
$$=\sum_{\mu=0}^{\mu=n} e^{\mu^2a+2\mu\,.\,\overline{x+y}}\,\theta(\overline{n+1}\,.\,y+\mu a,\ \overline{n+1}a)\,\theta(\overline{n+1}\,.\,x+\mu na,\ n\,.\,\overline{n+1}\,a).$$

Assuming now that n is odd, write $x+\tfrac{1}{2}\pi i$ for x; then

$$\theta'(x+y, a)\,\theta'(ny-x, na)$$
$$=\Sigma e^{\mu^2a+2\mu\,.\,\overline{x+y}}(-)^{\mu}\,\theta(\overline{n+1}\,.\,y+\mu a,\ \overline{n+1}a)\,\theta(\overline{n+1}x+\mu na,\ n\,.\,\overline{n+1}\,a).$$

When we add together these formulæ, only those terms remain on the right for which μ is even; we may therefore write 2μ instead of μ, and then

$$\theta\overline{x+y,\ a}\,.\,\theta\overline{ny-x,\ na}+\theta'\overline{x+y,\ a}\,.\,\theta'\overline{ny-x,\ na}$$
$$=2\sum_{\mu=0}^{\mu=\frac{1}{2}(n-1)} e^{4\mu^2a+4\mu\overline{x+y}}\,\theta(\overline{n+1}\,.\,y+2\mu a,\ \overline{n+1}\,a)\,\theta(n+1\,.\,x+2\mu na,\ n\,.\,\overline{n+1}\,a).$$

Now $\theta x+\theta' x=2\theta(2x, 4a)$, $\theta x-\theta' x=2\theta_1(2x, 4a)$, by direct addition of series, and hence

$$\theta(x, a)\,\theta(y, b)+\theta'(x, a)\,\theta'(y, b)=2\theta(2x, 4a)\,\theta(2y, 4b)+2\theta_1(2x, 4a)\,\theta_1(2y, 4b).$$

Transforming by this formula the left-hand of the equation last arrived at, and then writing $\frac{1}{2}x$, $\frac{1}{2}y$, $\frac{1}{4}a$ instead of x, y, a, we find

$$\theta\overline{x+y,\, a}\;\theta\overline{ny-x,\, na}+\theta_1\overline{x+y,\, a}\,.\,\theta_1\overline{ny-x,\, na}$$
$$=\Sigma e^{\mu^2a+2\mu\overline{x+y}}\,\theta\left(\frac{n+1}{2}\,y+\frac{\mu}{2}a,\ \frac{n+1}{4}\,a\right)\theta\left(\frac{n+1}{2}\,x+\tfrac{1}{2}\mu na,\ \frac{n\,.\,\overline{n+1}}{4}\,a\right).$$

Assuming further that $n+1$ is divisible by 4, let $n=4m-1$, so that

$$\theta(x+y, a)\,\theta(ny-x, na)+\theta_1(x+y, a)\,\theta_1(ny-x, na)$$
$$=\Sigma e^{\mu^2a+2\mu\overline{x+y}}\,\theta(2my+\tfrac{1}{2}\mu a,\ ma)\,\theta(2mx+\tfrac{1}{2}\mu na,\ mna).$$

Now writing $x+\frac{1}{2}\pi i$ for x, adding the two formulæ, and putting 2μ for μ, since only those terms remain in which μ is even, we get

$$\left.\begin{aligned}&\theta\ (x+y,\ a)\ \theta\ (\overline{4m-1}\,.\,y-x,\ \overline{4m-1}\ a)\\+\ &\theta'\ (x+y,\ a)\ \theta'\ (\overline{4m-1}\,.\,y-x,\ \overline{4m-1}\ a)\\+\ &\theta_1\,(x+y,\ a)\ \theta_1\,(\overline{4m-1}\,.\,y-x,\ \overline{4m-1}\ a)\\-\ &\theta'_1\,(x+y,\ a)\ \theta'_1\,(\overline{4m-1}\,.\,y-x,\ \overline{4m-1}\ a)\end{aligned}\right\}$$
$$=2\sum_0^{m-1} e^{4\mu^2a+2\mu\overline{x+y}}\,\theta(2my+\mu a,\ ma)\,\theta(2nx+\mu na,\ mna).$$

In the case $m=1$, $n=3$, $x=y=0$, we find, writing Θ for $\theta(0, 3a)$

$$\theta_1\Theta_1+\theta'\Theta'=\theta\Theta, \text{ or } \sqrt{kl}+\sqrt{k'l'}=1,$$

which is the modular equation in a form given by Jacobi. It is worth observing that we have more generally

$$\theta_1 x\Theta_1 x+\theta' x\Theta' x-\theta_1' x\Theta_1' x=\theta x\Theta x.$$

XII.

Rosenhain's functions, and integrals of the third kind.
[*Unfinished.*]

If we write $R=\sqrt{x(1-x)(1-k^2x)(1-l^2x)(1-m^2x)}$, and if

$$u=\int_0^{x_1}\frac{\alpha+\beta x}{R}\,dx\pm\int_0^{x_2}\frac{\alpha+\beta x}{R}\,dx,$$

$$v=\int_0^{x_1}\frac{\alpha'+\beta' x}{R}\,dx\pm\int_0^{x_2}\frac{\alpha'+\beta' x}{R}\,dx,$$

then x_1 x_2 are the roots of a quadratic equation $L+2Mx+Nx^2=0$, whose coeffi-

cients LMN can be rationally expressed in terms of [double] θ-functions of u, v. Such a [double θ-] function is defined by the equation

$$\theta(u, v) = \sum_m \sum_n e \mid m^2a + 2mnb + n^2c + 2mu + 2nv \mid$$

and the most general form is

$$\theta_{rs}^{pq}(u, v) = \sum_{mn} \sum e \mid \left(m+\frac{r}{2}\right)^2 a + 2\left(m+\frac{r}{2}\right)\left(n+\frac{s}{2}\right) b + \left(n+\frac{s}{2}\right)^2 c$$
$$+ 2\left(m+\frac{r}{2}\right)(u+\tfrac{1}{2}p\pi i) + 2\left(n+\frac{s}{2}\right)(v+\tfrac{1}{2}q\pi i)\mid,$$

where $p\ q\ r\ s$ are either 1 or 0. This is Jacobi's form of the inversion problem for the integrals $\int \frac{\alpha+\beta x}{R}\,dx$.

We now consider the particular case $l=m$, in which these integrals are reduced to elliptic integrals. We then have

$$R = (1-l^2x)\sqrt{x(1-x)(1-k^2x)},$$

and as the quantities $\alpha\beta\ \alpha'\beta'$ are arbitrary we will make $2\alpha=1$, $2\beta=-l^2$, so that $2(\alpha+\beta x)=1-l^2x$. The first of our two equations then becomes

$$u = \int_0^{x_1} \frac{\frac{1}{2}dx}{\sqrt{x(1-x)(1-k^2x)}} \pm \int_0^{x_2} \frac{\frac{1}{2}dx}{\sqrt{x(1-x)(1-k^2x)}} = u_1 + u_2,$$

where $$x_1 = \text{sn}^2 u_1, \quad x_2 = \text{sn}^2 u_2.$$

To determine conveniently $\alpha'\ \beta'$, observe that we have

$$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}} + \frac{\theta'y}{\theta'y}x = \lambda fy \cdot gy \cdot hy \int_0^x \frac{k^2fx^2dx}{1-k^2fy^2 \cdot fx^2},$$

or $$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}} = \int_0^x \frac{-\frac{\dot\theta' y}{\theta' y} + k^2\left(fy^2\frac{\dot\theta' y}{\theta' y} + \lambda fy \cdot gy \cdot hy\right)fx^2}{1-k^2fy^2 \cdot fx^2}\,dx,$$

which becomes, if we make $x=\dfrac{i\pi u_1}{2K}=\lambda u$, $y=\dfrac{i\pi a}{2K}=\lambda a$, $Za=\dfrac{\dot\theta'\left(\frac{i\pi a}{2K}\right)}{\theta'\left(\frac{i\pi a}{2K}\right)}$,

$$\tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_1-a)}{\theta'\frac{i\pi}{2K}(u_1+a)} = \int_0^{u_1}\frac{-Za+k^2(\text{sn}^2 aZa + \text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a)\,\text{sn}^2u_1}{1-k^2\text{sn}^2a\,\text{sn}^2u_1}\,du_1.$$

Hence if we write $k^2\,\text{sn}^2 a = l^2$, $2\alpha' = -Za$, $2\beta' = k^2(\text{sn}^2 aZa + \text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a)$ we shall have from the second equation

$$v = \tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_1-a)}{\theta'\frac{i\pi}{2K}(u_1+a)} + \tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_2-a)}{\theta'\frac{i\pi}{2K}(u_2+a)},$$

whence $$e^{2v}=\frac{\theta'\frac{i\pi}{2K}(u_1-a)\,\theta'\frac{i\pi}{2K}(u_2-a)}{\theta'\frac{i\pi}{2K}(u_1+a)\,\theta'\frac{i\pi}{2K}(u_2+a)}.$$

Now since $u=u_1+u_2$, [we have, transforming the expressions for

$$\vartheta'\,.\,\vartheta'u\,.\,\vartheta'\overline{u_1+a}\,.\,\vartheta'\overline{u_2+a},\quad \vartheta_1'\,.\,\vartheta_1'u\,.\,\vartheta_1'\overline{u_1+a}\,.\,\vartheta_1'\overline{u_2+a},$$

of which the second vanishes, by the formula for the multiplication of four theta-functions, and adding the results]

$$\tfrac{1}{2}\vartheta'\,.\,\vartheta'u\,.\,\vartheta'\overline{u_1+a}\,.\,\vartheta'\overline{u_2+a}=\vartheta'\overline{u+a}\,.\,\vartheta'a\,.\,\vartheta'u_1\,.\,\vartheta'u_2+\vartheta_1'\overline{u+a}\,.\,\vartheta_1'a\,.\,\vartheta_1'u_1\,.\,\vartheta_1'u_2,$$

whence $$e^{2v}=\frac{\vartheta'\overline{u-a}\,.\,\vartheta'a\,.\,\vartheta'u_1\,.\,\vartheta'u_2-\vartheta_1'\overline{u-a}\,.\,\vartheta_1'a\,.\,\vartheta_1'u_1\,.\,\vartheta_1'u_2}{\vartheta'\overline{u+a}\,.\,\vartheta'a\,.\,\vartheta'u_1\,.\,\vartheta'u_2+\vartheta_1'\overline{u+a}\,.\,\vartheta_1'a\,.\,\vartheta_1'u_1\,.\,\vartheta_1'u_2},$$

and $$\frac{\vartheta_1'u_1\,.\,\vartheta_1'u_2\,.\,\vartheta_1'a}{\vartheta'u_1\,.\,\vartheta'u_2\,.\,\vartheta'a}=\frac{e^{-v}\vartheta'(u-a)-e^{v}\vartheta'(u+a)}{e^{-v}\vartheta_1'(u-a)+e^{v}\vartheta_1'(u+a)}=k^{\frac{3}{2}}\operatorname{sn}a\sqrt{x_1x_2}.\text{*}$$

* [Here the MS. ends.]

ON ELLIPTIC FUNCTIONS.

On the multiplication of infinite series *.

The properties of the θ-functions are most easily investigated by multiplying together the series which expand them and rearranging the terms. We shall now therefore examine some of the conditions under which such rearrangement is justified.

A singly infinite series consisting entirely of positive terms, if it converges, must converge independently of the order of the terms. For let P be the sum of the series, if the terms are arranged in a certain order; and let Q_n be the sum of n terms, when they are arranged in any other order. Then Q_n cannot exceed P, and therefore must have some limit when n is increased indefinitely. Let Q be this limit, and let P_n be the sum of n terms of the former arrangement. Then P_n cannot exceed Q, because the terms are all positive. Hence $P = Q$, because neither of them can exceed the other.

If the series consist of positive and negative terms, its sum will be independent of the order if the positive and negative parts converge separately. Let P and $-Q$ be the sums, P_m and $-Q_n$ the sums of m and n terms, of the positive and negative parts respectively. Then $P - P_m$ and $Q - Q_n$ can be made as small as we like by taking m and n large enough. Hence $P - Q - P_m + Q_n$ can be made as small as we like, and therefore $P - Q$ is the sum of the compound series, whatever the order of the terms.

(The proof applies to *any* two convergent series P and Q, provided that the order of the terms in each is preserved in mixing them up.)

Similarly, in a series of complex terms, if the positive and negative real and imaginary parts converge separately, no change can be made in the sum by altering the order of the terms.

This will be the case if the series converges when we substitute for each term $p + iq$ its modulus $\sqrt{(p^2 + q^2)}$; for this is at least as great as p or q, and hence the series of the real and imaginary parts of the positive and negative terms must converge to a sum less than that of the moduli.

* [The *Tract on Elliptic Functions*, cf. p. 442, consists of 16 pages of MS. with the following headings:—Definition of the Theta-series (3 pp.); the four Theta-functions (3 pp.); on the Multiplication of Infinite Series (6 pp.); Reciprocal Sets of Numbers (2 pp.); the Multiplication of Four Theta-functions (2 pp.)]

If we multiply together two series—for example the series for e^x and e^y, we get a result of this kind:

$$\begin{array}{cccccc}
1 + & x + & \dfrac{x^2}{\Pi 2} + & \dfrac{x^3}{\Pi 3} + & \dfrac{x^4}{\Pi 4} + & . \\
+ & + & + & + & + & \\
y + & xy + & \dfrac{x^2 y}{\Pi 2} + & \dfrac{x^3 y}{\Pi 3} + & \dfrac{x^4 y}{\Pi 4} + & \ldots \\
+ & + & + & + & + & \\
\dfrac{y^2}{\Pi 2} + & \dfrac{xy^2}{\Pi 2} + & \dfrac{x^2y^2}{\Pi 2 . \Pi 2} + & \dfrac{x^3y^2}{\Pi 3 . \Pi 2} + & \dfrac{x^4y^2}{\Pi 4 . \Pi 2} + & \ldots \\
+ & + & + & + & + & \\
\dfrac{y^3}{\Pi 3} + & \dfrac{xy^3}{\Pi 3} + & \dfrac{x^2y^3}{\Pi 2 . \Pi 3} + & \dfrac{x^3y^3}{\Pi 3 . \Pi 3} + & \dfrac{x^4y^3}{\Pi 4 . \Pi 3} + & \ldots \\
+ & + & + & + & + & \\
\vdots & \vdots & \vdots & \vdots & . &
\end{array}$$

This may be called a *doubly infinite* series; the general term is $\dfrac{x^m y^n}{\Pi m . \Pi n}$, containing *two* variable integers m, n, and we obtain the series by giving to each of these all positive integral values, zero included. In summing this series, the terms may be taken in various orders. We may first take all the terms in the first horizontal line, then all those in the second, and so on; this presents the series in the singly infinite form

$$e^x + ye^x + \tfrac{1}{2}y^2e^x + \ldots$$

the sum of which we know to be $e^x e^y$. Or we may take the left-hand column first, then the second column, and so on; this presents the series in the form

$$e^y + xe^y + \tfrac{1}{2}x^2e^y + \ldots$$

whose sum is again $e^y e^x$. The former process approaches the quarter of an infinite plane on which the series is spread out as an infinitely long horizontal rectangle whose breadth is increased without limit; the latter as an infinitely long vertical rectangle whose breadth is increased without limit.

We may also reduce the series to a singly infinite one in the following way. Namely, it is equal to

$$1 + (x+y) + \left(\frac{x^2}{\Pi 2} + xy + \frac{y^2}{\Pi 2}\right) + \left(\frac{x^3}{\Pi 3} + \frac{x^2 y}{\Pi 2} + \frac{xy^2}{\Pi 2} + \frac{x^3}{\Pi 3}\right) + \ldots$$

$$= 1 + x + y + \frac{(x+y)^2}{\Pi 2} + \frac{(x+y)^3}{\Pi 3} + \ldots = e^{x+y}$$

since, by the binomial theorem for a positive integer exponent,

$$\frac{(x+y)^n}{\Pi n} = \frac{x^n}{\Pi n} + \frac{x^{n-1}y}{\Pi(n-1)} + \frac{x^{n-2}y^2}{\Pi(n-2) . \Pi 2} + \ldots + \frac{y^n}{\Pi n} = \Sigma \frac{x^a y^b}{\Pi a . \Pi b}. \qquad (a+b=n)$$

Now this mode of summing the doubly infinite series takes together the terms on

an oblique line joining corresponding terms of the top row and left-hand column. It approaches the quarter of an infinite plane as a triangle in the shape of half a square whose size is indefinitely increased.

We may sum this doubly infinite series in yet another way which leads to a useful result. Let $x=uv$, $y=\frac{u}{v}$; and let moreover

$$fu=1+u+\frac{u^2}{(\Pi 2)^2}+\frac{u^3}{(\Pi 3)^2}+\frac{u^4}{(\Pi 4)^2}+\dots$$

$$f_k u=(\partial_u)^k fu=\frac{1}{\Pi k}+\frac{u}{\Pi(k+1)}+\frac{u^2}{\Pi 2\,.\,\Pi(k+2)}+\dots$$

$$=\Sigma\frac{u^k}{\Pi k\,.\,\Pi(n+k)}.$$

Then we shall find for the product $e^{uv}\,.\,e^{\frac{u}{v}}$ the doubly infinite series

$$\begin{array}{ccccccccc}
1 & + & uv & + & \frac{u^2v^2}{\Pi 2} & + & \frac{u^3v^3}{\Pi 3} & + & \dots \\
+ & & + & & + & & + & & \\
uv^{-1} & + & u^2 & + & \frac{u^3v}{\Pi 2} & + & \frac{u^4v^2}{\Pi 3} & + & \dots \\
+ & & + & & + & & + & & \\
\frac{u^2v^{-2}}{\Pi 2} & + & \frac{u^3v^{-1}}{\Pi 2} & + & \frac{u^4}{\Pi 2\,.\,\Pi 2} & + & \frac{u^5v}{\Pi 2\,.\,\Pi 3} & + & \\
+ & & + & & + & & + & & \\
\frac{u^3v^{-3}}{\Pi 3} & + & \frac{u^4v^{-2}}{\Pi 3} & + & \frac{u^5v^{-1}}{\Pi 3\,.\,\Pi 2} & + & \frac{u^6}{\Pi 3\,.\,\Pi 3} & + & \dots \\
+ & & + & & + & & + & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

And if we now sum first all the terms on the middle diagonal going downwards to the right from the term 1, then those in the two lines parallel to this on either side, and so on, we shall obtain

$$e\,|\,u\,(v+v^{-1})\,|=f\,(u^2)+(v+v^{-1})\,uf_1\,(u^2)+(v^2+v^{-2})\,u^2f_2\,(u^2)+(v^3+v^{-3})\,u^3f_3\,(u^2)+\dots$$

In this formula write ix for u, and $ie^{i\theta}$ for v, so that

$$u\,(v+v^{-1})=-2x\sin\theta,\quad u^k(v^k+v^{-k})=(+i)^k\,2x^k\cos k\left(\theta+\frac{\pi}{2}\right),$$

then $\quad e\,|-2ix\sin\theta\,|=f\,(-x^2)-2ixf_1\,(-x^2)\,.\,\sin\theta-2x^2f_2\,(-x^2)\cos 2\theta-$

$$=f\,(-x^2)+2\Sigma_1^\infty\,(i)^n\,.\,x^nf_n\,(-x^2)\,.\cos n\,(\theta+\tfrac{1}{2}\pi),$$

so that

$$\cos\,(2x\sin\theta)=f\,(-x^2)-2x^2f_2\,(-x^2)\cos 2\theta+2x^4f_4\,(-x^2)\cos 4\theta-\dots$$

$$\sin\,(2x\sin\theta)=2xf_1\,(-x^2)\sin\theta-2x^3f_3\,(-x^2)\sin 3\theta+2x^5f_5\,(-x^2)\sin 5\theta-\dots$$

The function $x^nf_n\,(-x^2)$ is called Bessel's function of the n^{th} order, and is generally denoted by $J_n\,(2x)$.

This process of summing the doubly infinite series approaches the infinite area as an infinitely long figure parallel to the middle diagonal, whose breadth is indefinitely increased.

Thus we have considered four ways of approaching the infinite plane, and each of them consists in taking an area of a certain shape and then allowing it to expand indefinitely.

If the two numbers m, n which determine the place of any term in the series are allowed to take negative as well as positive values, the doubly infinite series will cover the whole plane instead of only a quarter of it. The process of summing the series will still consist in taking an area of a certain shape, and allowing it to expand indefinitely while it remains similar to itself and similarly situated in regard to the origin.

The question is, does the sum of the series depend upon the shape of this area? We may shew very easily that it does *not* so depend when the terms are all real and positive. For let P be the sum of the series when it is summed in one way, then the sum of no portion of the series, however selected, can exceed P. If then P' be an approximation to P made by taking a large number of steps towards the summation that way, Q' a similar approximation to Q, the sum obtained by another arrangement; then P' cannot exceed Q, nor Q', P. And since $P-P'$, $Q-Q'$ can be made as small as we like by proceeding sufficiently with the summation, it follows that P cannot exceed Q, nor Q, P, and consequently $P=Q$.

Hence as before the sum is independent of the mode of summation if the series converges when we substitute for each term its modulus.

Similar reasoning applies to multiply infinite series, or, as a particular case, to products of any number of singly infinite series.

Now in the case of the product just considered, and in that of the products of θ-functions to be presently treated, it is plain that this condition is satisfied. The transformations of this section, therefore, are all of them valid.

NOTES OF LECTURES ON QUATERNIONS *.

We will define the signs + and − as indicating steps, whereby any magnitude may be increased or diminished, or by means of which we may move from one point of a progression to another.

Let the quantity which has the progressive values be measured along a line; but we shall suppose at first that the numbers stand for amounts of any kind of quantity, not necessarily length.

+			0	1	2	3	4	5
−3	−2	−1	0	+1	+2	+3		

Then we should have a zero value and successive values marked off, *above* the line, by the numbers 1 2 3 etc. Numbers standing alone mean amounts of the given quantity, while the sign + or − before them means a step is to be taken through successive values of that quantity so as to lead to a different value from that from which we set out.

Thus the equation

$$2+3=5$$

means that if we start with 2 things represented by 2 places on the line and step through 3 following places we get to the place indicated by 5.

But there is another way of considering this equation. Instead of marking off on our line a scale of numbers merely it may be a scale of steps as by the positive and negative numbers below the line.

And then our equation would be

$$+2+3=+5.$$

Or whatever number we set out from and take two steps to the right and then three again to the right it is the same as taking 5 at once to the right. Here are two interpretations of the equation.

Again consider this form of the same truth

$$5-3=2.$$

This may mean first—

Starting with a number 5 denoted by its place on the line, and taking three steps backward, to the left, we get to the number 2. Or it may mean, if we consider the scale as one of steps,

* [I am indebted to Miss E. Watson for these *Notes* of lectures delivered at University College, towards the close of 1877.]

A step 5 to the right and then a step 3 to the left leads to the same number as if starting with the same we took a step 2 to the right.

Now let us combine two forward steps,

$$2+3+4=9=2+7.$$

This means that if we start at the place 2 and take 3 steps to the right and then 4 to the right we get to the place 9; or this is the same as starting with the place 2 and taking a step 7 forward.

But if this is written

$$+2+3+4=+9,$$

we should interpret it thus:

Starting with any number and making the steps to the right represented by 2, 3, 4, is the same as making the step 9 to the right.

Further

$$2=9-4-3$$

may mean either that starting at the place 9 and taking 4 steps to the left and then 3 again to the left we get to the place: or it may mean that starting anywhere, making a step 9 to the right and then steps 4 and 3 to the left is the same as making the step 2 to the right.

On the whole then we have two modes of interpreting these operations of addition and subtraction.

The corresponding forms of the last written equation are

$$2+3+4=9,$$

$$+2+3+4=+9.$$

The common interpretation would be 2 things and 3 things and 4 things all treated in the same way make 9 things.

I. The first equation consists of a number with two steps performed on it, leading to a number.

II. The second means altogether steps, which lead to a step performed on *any* number.

Whenever as a result we get a negative expression, it means a step of *decrease* of a quantity or of direction.

We now go on to the symbols of Multiplication.

Again every equation has two distinct interpretations, as well as the common arithmetical one:

$$2\times 3=6,$$

may be read '6 is the product of 2 and 3', in which case numbers are treated in just the same way.

But we shall leave this meaning out of account and read the equation either as

I. Twice three are 6. I perform an operation on three things by the operator 2; or,

II. I take any number, double and triple it. This is equivalent to the operation of sextupling it.

Now let us consider the multiplication of steps instead of things.

In the equation

$$2 \times (+3) = +6$$

the last term on each side is a step, the first is an operator and the equation means by means of doubling I can turn a step 3 to the right into a step 6 to the right.

We notice in passing that in Multiplication the operation to be performed comes first; in addition it comes after the thing or step operated on.

Now what operator is required to turn the step -3 to the left into the step $+6$ to the right?

First we reverse the step by an operator which we will call r, $\{r(-3) = +3\}$: thus it becomes $+3$. Now double it, and the whole operation is written,

$$2r(-3) = +6,$$

so the required operator is $2r$, which means reverse and *then* multiply by 2.

But we may change the order of the process, viz. double and then reverse and we get the same result

$$r2(-3) = +6.$$

Now we will construct an equation analogous to this but which shall consist entirely of operators. Here we have two steps.

Let $k3$ mean triple without reversing. And let us suppose any step taken, tripled, reversed, doubled and reversed again. The two reversals will clearly destroy each other and give "no reversal" or k, and we shall have our step sextupled without reversal. This may be written as an equation like the last. And in the same way we have two others in which the direction is reversed:

I.	$r2(-3) = +6,$	$r2(+3) = -6,$
II.	$r2(r3) = k6,$	$r2(k3) = r6:$

and we are led to assign a new meaning to the symbols $+$, $-$; we may use them instead of k and r respectively.

Thus their meaning is extended from that of indicators of steps to operators on steps. Unless this extended meaning is borne in mind, many equations would be unmeaning, e.g. the familiar one,

$$-2(-3) = +6.$$

For there is no meaning in Multiplying one step by another.

We may assign two reasonable interpretations. Either both $-$ signs mean reverse, or the second is a step and the first means reverse.

In every equation of multiplication the last factor on each side may mean a step and then all the rest must be operations, or they may all be operations. It will be necessary to examine in the particular cases whether both meanings are allowable or whether only one may be given.

Let us now take lengths to stand for quantities either commensurable or incommensurable. If they are incommensurable (that is if they are among the values of a continuous quantity) the only way to represent them is on a certain scale.

Quantities of length are generally added by placing them end to end, treating them simultaneously in exactly the same way.

But we shall always suppose one length given and the other numbers with their signs to mean operations to be performed on it. Addition shall be represented by a step to the right along the line of the given length and subtraction a step to the left along the same line.

With the meaning of a quantity so far developed the product of two quantities represented by lines will be given by the rectangle on these lines. Then any product of higher degree than 3 will be unmeaning because it will be of higher dimensions in space than 3.

We learn however that before Descartes, this linear way of representing quantities was the only one used for the solution of equations Vieta, in his treatise on Algebra imagines space of 9 dimensions in order to explain his equations.

The different orders began with Linear, Planum, Plano-planum, and went down to, solido-solido-solidum.

The equation

$$x^3+ax^2+bx+c$$

was interpreted as the sum of a number of solids, viz.

cube of $x+a$ linear x^2+b planum $x+c$ solidum,

and the equation was actually solved by cutting up a cube.

Just as in the first treatise on Algebra introduced into Europe from Arabia, by Cardan*, the equation $x^2+2x=15$

was solved by a construction in a plane [see Fig 60] which gives the value

$$x=3.$$

Descartes first gave another meaning to the product of two quantities. He arrived at this by letting numbers stand for the *ratios of quantities*. The length of any line would then be the operation which is necessary to convert the unit of length into that line.

And the product of two quantities becomes the *ratio* compounded of the ratios which they bear to the unit.

With such a change of meaning we can write x^n without supposing a figure of n dimensions. It will mean simply the nth power of the ratio of the line x to the unit. This is clearly equivalent to our second way of looking at multiplication. But we may also use the first way as well. ab may now mean, not the rectangle on ab, but either

I. The line which bears the same ratio to b which a does to 1.

II. Where a and b are both operations; *the ratio* compounded of the ratios of a and b to the unit.

Just as before, the only choice we have is with the last number on either side, which may be either a *quantity* or an *operation*.

* [*de Arithmetica*, lib. x. cap. v.; cf. also Chasles, *Aperçu historique*, pp. 489, 541.]

Or if instead of a scale of quantities we have a scale of steps,

$$(a)(-b)=-c$$

may mean

I. Take a step b to the left and increase it in the ratio a to 1 and you get a step c to the left, or

II. Take any step, multiply it by the ratio b, reverse it, multiply it by the ratio a,—then the result is the step multiplied by c and reversed.

As yet we have considered only scalar quantities; which are either steps of addition or subtraction, i e. steps of position on a straight line, or operations performed on those steps. These last may be equally considered as ratios of steps. We extended the meaning of quantity from simple number to these two: steps and operations. We were led to 'steps' by the appearance of negative quantity. But there is still another unexplained symbol, namely the square root of a negative quantity. Descartes' method takes no account of such a quantity.

We again define the symbol + by the equation

$$OP=OM+MP,$$

so that + now means a step in a plane [Fig. 61].

This equation holds as a definition whenever a step can be made in two instalments.

Whatever may be the angles made between the component steps we shall write [Fig. 62]

$$ac=ab+bc.$$

If OM and MP are to stand for the ratios of these lines to a unit line, we must have a unit in the direction of OM and another in that of MP. Then if x and y are ratios,

$$OP=x \text{ times } OI+y \text{ times } OI'.$$

x and y are scalars, ratios of steps to the unit steps. x is positive whenever P is on the right of O and y is positive whenever P is above OX.

Thus two scalar numbers x and y must be assigned in order to determine OP.

We will lay down a rule for adding two steps. Place the beginning of the second at the end of the first, then the line joining the beginning of the first to the end of the second will be their sum.

In giving this rule we make an assumption, viz. that the step PR is the same as OQ. Assuming this,

$$OR=OP+OQ.$$

But we have also

$$OR=ON+NR.$$

And ON is the sum of the horizontal parts of OP and OQ and NR of their vertical parts. We may express this by the equations—

if $\quad OP=x$ times $OI+y$ times OI'; and $OQ=x'$ times $OI+y'$ times OI';

$$OP+OQ=\overline{x+x'} \text{ times } OI+\overline{y+y'} \text{ times } OI'.$$

So that if (xy) are the pair of numbers required to describe OP and $(x'y')$ the pair required to describe OQ we must add corresponding numbers from these two pairs, if we want to get the pair required to determine $OP+OQ$, or

$$(x,\ y)+(x',\ y')=(x+x',\ y+y'),$$

the reason of this from the figure is that OQL is the same triangle as PRS only in a different position.

We have considered

1. Quantities. 2. Steps of those quantities. 3. Ratios of steps of quantities.

numbers, lengths on a line, lengths on a plane, lengths in space to be considered hereafter.

We have seen that with the extended meanings given to $+$ and $=$,

$$ab+bc=ac,$$

and extending this result to vectors in space

$$ab+bc+cd+de+ef=af.$$

Or adopting a notation used by Hamilton,

$$a+ab=b, \qquad b-a=ab,$$

(point + line = point, point − point = line;

or in Hamilton's terms,

vehend + vector = vectum, vectum − vehend = vector.

He calls the operation symbolized by the first equation "ordinal synthesis." It is a putting together of a line and a point—and the result is a point. The second operation he calls "ordinal analysis." And he enunciates this theorem: If we start with the result of a synthesis and perform on it the corresponding analysis we shall get the instrument of synthesis.)

Using this notation, the equation for addition of vectors in space becomes

$$b-a+c-b+\ldots+f-e=f-a,$$

which is an identity in this form.

It is convenient to represent steps by points in this way when we have to consider steps beginning at the same point.

We have here [Fig. 63],

$$oa+ob=oa+ac=oc=2of,$$

that is in the symbolical point form

$$a-o+b-o=2\,(f-o),$$

and hence

$$a+b=2f;$$

f may then be called the 'mean' of the two points a, b.

Generally, if we have to express of drawn to any point of ab in terms of oa, ob, we have, [Fig. 64],

$$of=oa+af=oa+\frac{m}{l+m}\cdot ab,$$

where $l : m$ is the ratio in which f divides ab. That is

$$of = oa + \frac{m}{l+m}(ob - oa),$$

and

$$(l+m)\,of = (l+m)\,oa + m\,.\,ob - m\,.\,oa = l\,.\,oa + m\,.\,ob,$$

and hence

$$of = \frac{l\,.\,oa + m\,.\,ob}{l+m}.$$

Expressing this in terms of the points

$$(l+m)(f-o) = l(a-o) + m(b-o),$$

$$\therefore\ (l+m)f = la + mb \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

From this we are led to consider points in a plane as having masses. The expression for f shews that it is the centre of gravity (or centre of mass) of the masses l and m placed at b and a respectively. The point f divides ab in the inverse ratio of the masses.

It may also be expressed in terms of steps, transposing in the last equation

$$l(f-a) + m(f-b) = 0,$$

that is

$$l\,.\,af + m\,.\,bf = 0.$$

We may make use of this result to prove the vector form of equation (A).

We have of course

$$of = oa + af \text{ and } of = ob + bf.$$

Hence

$$(l+m)\,of = l\,.\,oa + m\,.\,ob + (l\,.\,af + m\,.\,bf) = 0, \text{ as just found,}$$

or as before

$$(l+m)\,of = l\,.\,oa + m\,.\,ob.$$

If we want to extend the rule for finding the mean of two points to a greater number of points, the sought point is called their mid-centre and is defined by this equation.

If m is mid-centre and $abc..f$ etc. the points,

$$ma + mb + mc + \ldots + mf = 0,$$

that is

$$a - m + b - m + \ldots + f - m = 0,$$

or

$$a + b + c + \ldots + f = nm,$$

where n = number of points.

Then

$$m = \frac{a+b+c..+f}{n}.$$

For example, the mid-centre of four points a, b, c, d is by this rule

$$\frac{a+b+c+d}{4}.$$

Clearly it lies at the point B in the figure. [Fig. 65].

In the case of an odd number of points, $2n+1$, we should first find the mid-centre B of $2n$ of them and then joining it to the last divide this line in the ratio $1 : 2n$.

We can now find the resultant of n steps. It is n times the step from the mid-centre of the beginnings to the mid-centre of the ends.

This is obvious in the case of two steps.

We have then, [Fig. 66],

$$ab+cd=2fg,$$

that is

$$(b+d)-(c+a)=2g-2f,$$

which follows from the definition of the mean of two points.

If there are four steps and p is the mid-centre of the a's and q of the b's, we have [Fig. 67]

$$\begin{aligned} &a_1b_1+a_2b_2+a_3b_3+a_4b_4 \\ &=b_1+b_2+b_3+b_4-(a_1+a_2+a_3+a_4) \\ &= 4q - 4p \\ &= 4(q-p), \end{aligned}$$

and this is the resultant.

This construction is sometimes more convenient than the tandem arrangement. For the mid-centres lie always within the area covered by the steps. The construction can always be carried out provided the steps are given.

If we want to add together certain multiples l, m, n, r of steps we attribute masses l, m, n, r to the points a and also to the points b. Then we should have,

$$lb_1+mb_2+nb_3+rb_4-(la_1+ma_2+na_3+ra_4)=(l+m+n+r)(q-p),$$

as may be easily seen, by joining each of the points b_1 etc., a_1 etc. to the mid-centre of the set.

We will now apply the new interpretation of the signs $+$, $-$, $=$ to the description of motions.

Let p be a point moving uniformly along a line. Then if r is the space traversed in $1''$, $rt=$ space gone over in t''.

We must find an expression for the step necessary to get from o to the position of p at any time t. Suppose p goes over ab [Fig. 68] in $1''$.

Then we have,

$$op=oa+t\,.\,ab,$$

$$\rho=\alpha+t\beta.$$

This is the equation of uniform motion of a point on a line.

We may also find the equation of uniform circular motion.

Let the arc gone over in $1''$ be an [Fig. 69], then the arc described in $t''=t\,.\,an$. If this is represented by the arc ap, the circular measure of the angle aop is nt. We have

$$\overline{om}=\overline{oa}\,.\cos nt\,;\ \overline{mp}=\overline{ob}\,.\sin nt.$$

Hence

$$op = \overline{om} + \overline{mp} = \overline{oa} \, . \cos nt + \overline{ob} \, . \sin nt,$$

or

$$\rho = \alpha \cos nt + \beta \sin nt.$$

This is the step from the centre to the position of p at any time t.

Suppose now this uniform plane circular motion projected on an inclined plane. The curve becomes an ellipse, the parts of any radius vector to which from the centre are always proportional to the corrresponding ones of the circle [Fig. 70],

$$\frac{o'm'}{o'a'} = \frac{om}{oa} = \cos nt,$$

$$\frac{m'p'}{o'b'} = \frac{mp}{ob} = \sin nt.$$

Hence we have the same form of equation for harmonic motion on an ellipse as for uniform motion on a circle, viz.

$$\rho = \alpha \cos nt + \beta \sin nt.$$

But here α and β are not at right angles to each other.

The rate of change of op is the same as the velocity of the point p, as we see at once.

If $op = r$,

$$\dot{r} = \dot{p} - \dot{o},$$

$$= \dot{p}, \text{ since we suppose } o \text{ fixed.}$$

If we extend this to a step $\overline{op}$ in space, the rate of change of $\overline{op}$ is in this case a step,

i.e. if $\rho = \alpha + \beta t, \ \dot{\rho} = \beta, \ (ab = \beta = pq)$ [Fig. 71].

We have found, putting $oa = \alpha$ (of length a),

$$\rho = \alpha \cos nt + \beta \sin nt,$$

and we can at once obtain from this an expression for the velocity of p at any instant.

Since the angle described in $1'' = n$,

the arc ,, $= na$,

this then is the numerical value of the velocity. The direction is in the tangent line at p [Fig. 72].

Hence if we draw a radius oq perpendicular to op, we have, since $oq = a$,

$$\dot{\rho} = n \, . \, \overline{oq}.$$

But $\overline{oq}$ is what op becomes when nt is increased by $\frac{\pi}{2}$,

$$\therefore \ \dot{\rho} = n\alpha \cos\left(nt + \frac{\pi}{2}\right) + n\beta \sin\left(nt + \frac{\pi}{2}\right)$$

$$= -n\alpha \sin nt + n\beta \cos nt.$$

The rule then for finding the vector-velocity from the position or vector-radius is—multiply this last vector by n and increase its angle by $\frac{\pi}{2}$.

We have gone on the supposition that α and β are at right angles, and equal to each other; but the result is true independently of this, since

$$\partial_t \cos nt, \ \partial_t \sin nt$$

always equal $-n \sin nt$, $n \cos nt$ respectively.

If then we take the elliptic projection of the circular motion, we may still conclude that velocity of $p = n . \overline{oq}$ if $\overline{oq}$ is parallel to the tangent at p [Fig. 73]. For the velocity at p is always parallel to the tangent at p.

Now consider the velocity of the point when it arrives at q. For this point

$$\rho = \alpha \cos\left(nt + \frac{\pi}{2}\right) + \beta \sin\left(nt + \frac{\pi}{2}\right).$$

Hence, multiplying by n and increasing angle by $\frac{\pi}{2}$,

(for q)
$$\begin{aligned} \dot\rho &= n\alpha \cos(nt + \pi) + n\beta \sin(nt + \pi) \\ &= -n\alpha \cos nt - n\beta \sin nt \\ &= -n . \overline{op} = n . \overline{po}. \end{aligned}$$

But this velocity must be in the direction qs of the tangent at q. Hence we may conclude, if we draw oq parallel to the tangent at p, then op will be parallel to the tangent at q.

The characteristic property of conjugate diameters is proved of the projections of diameters of a circle at right angles to each other.

We have seen that

$$op = \overline{oa} . \cos nt + \overline{ob} . \sin nt, \qquad \text{and also } op = om + mp,$$

$$oq = -\overline{oa} . \sin nt + \overline{ob} . \cos nt, \text{ and} \qquad oq = on + nq.$$

Hence, we have

$$\frac{om}{oa} = \cos nt, \qquad \frac{mp}{ob} = \sin nt,$$

$$\frac{nq}{ob} = \cos nt, \qquad \frac{on}{oa} = -\sin nt,$$

and from these equations

$$\frac{om}{oa} = \frac{nq}{ob}, \qquad \frac{on}{oa} = -\frac{mp}{ob},$$

a result which is thus interpreted:

We have two pairs of semi-conjugate diameters, oa and ob; and op and oq. The projection of op on oa is to oa as the projection of oq on ob to ob. And the projection of oq on oa is to oa as the reversed projection of op on ob to ob.

The first of these equalities is the same as

$$\frac{om}{oa} = \frac{or}{ob}.$$

Since ρ is a vector the expression for it may be treated in the same way as that for ρ. If we take n an improper fraction so that $oq' = n . oq$, the extremity of the vector $\dot{\rho}$ will trace out an ellipse similar and similarly situated to the first. Generally, the curve which describes the way in which the path of a moving point is gone over is called the hodograph (of that path). In the case of uniform rectilinear motion $\rho = a + \beta t$, $\dot{\rho} = \beta$, and the hodograph reduces to a point.

For uniform circular motion it is a circle with dimensions n times those of the first circle; and for harmonic motion in an ellipse, n times that ellipse.

We will now consider the velocity of q as it moves round its ellipse.

We have

$$\overline{oq} = \dot{\rho} = n\alpha \cos(nt + \tfrac{1}{2}\pi) + n\beta \sin(nt + \tfrac{1}{2}\pi).$$

The process for finding its velocity is the same as in the case of op. Hence

$$\begin{aligned}\dot{\rho} &= n^2 \alpha \cos(nt + \pi) + n^2 \beta \sin(nt + \pi) \\ &= -n^2 . \overline{op} \\ &= n^2 . \overline{po}.\end{aligned}$$

Thus we learn that in harmonic motion in an ellipse the acceleration of the moving point is directed towards the centre of the ellipse and is proportional to the distance from the centre.

We will now consider a new kind of motion of which the equation is

$$\rho = \alpha e^{nt} + \beta e^{-nt}.$$

These exponential functions are quantities which are equally multiplied in equal times.

Consider quantities, s, defined by this property.

We may see at once that their rate of change at any instant is proportional to the value of the quantity at that instant.

We have by def. $s_2 = ks$, and moving the interval $t_2 - t$, along the axis [Fig. 74], we still have $s_4 = ks_3$. Hence we infer the rates of change of the two quantities at the beginning and end of the interval are to each other as $1 : k$, or

$$\dot{s}_2 = k\dot{s}_1.$$

Hence we have the proportion

$$\dot{s}_2 : \dot{s}_1 :: s_2 : s_1,$$

and therefore $$s = ps.$$

Calling p the logarithmic rate, the last equation declares that s is a quantity which increases at uniform logarithmic rate.

If we change our unit of time from $1''$ to n'', $\dot{s}$ becomes $n\dot{s}$, and therefore p becomes np.

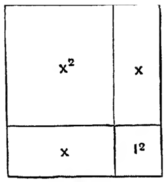


Fig 60.

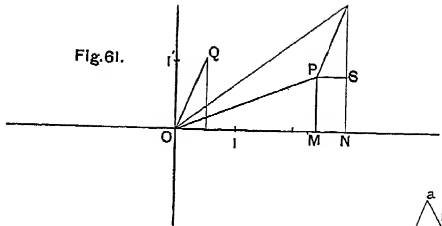


Fig.61.

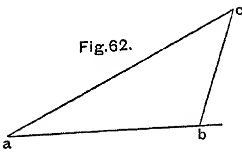


Fig.62.

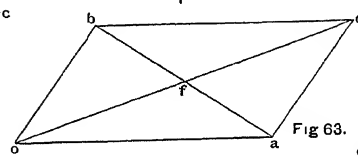


Fig 63.

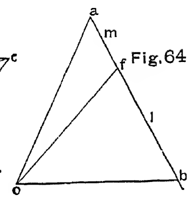


Fig.64

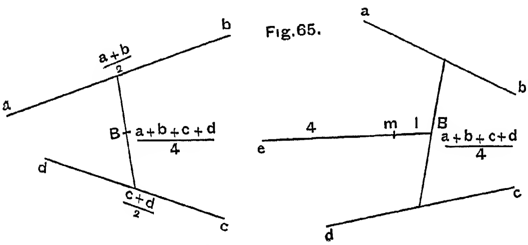


Fig.65.

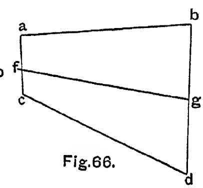


Fig.66.

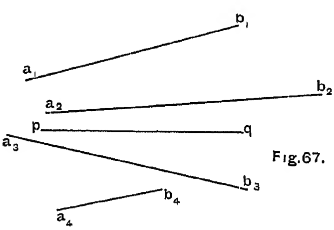


Fig.67.

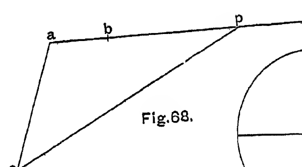


Fig.68.

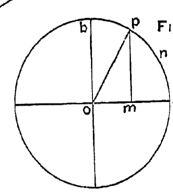


Fig 69.

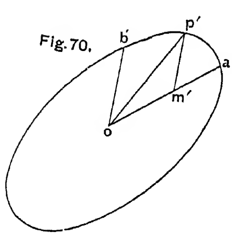


Fig.70.

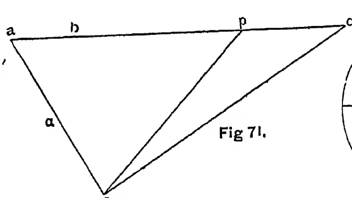


Fig 71.

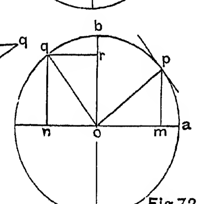


Fig.72.

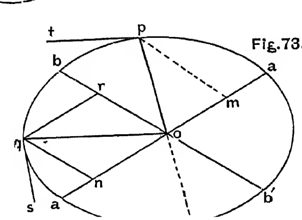


Fig.73.

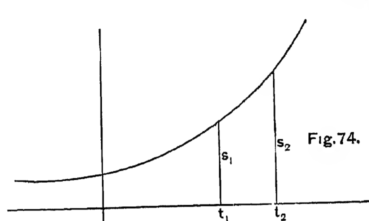


Fig.74.

Having given that the quantity varies at logarithmic rate p, we are required to find P the number by which it is multiplied in $1''$. In n seconds s is multiplied by P^n; and n must be a commensurable number, whole or fractional.

If it is a fraction, e.g. $\frac{1}{2}$, we must give one value only to $\sqrt{P}$, because at half a second after the time at which we begin, s will have a perfectly definite value.

If n is incommensurable, it may be defined by this physical property.—Let P^n mean the multiplier which s gets in this incommensurable time. Let us take a quantity $=1$, which increases at logarithmic rate 1. Then at the end of $1''$ it will have reached a certain definite value. Call it e. Then e is the result of making unity grow at logarithmic rate 1 for $1''$.

Then, I say, the result of making unity grow at logarithmic rate 1 for $t'' = e^t$.

For since in $1''$ it is multiplied by e, in t seconds it will be multiplied by e^t.

Now change the unit of time to n seconds. We see then that the result of making unity grow at logarithmic rate 1 for nt'' of old units $= e^{nt}$; therefore in the new unit e^{nt} results from making 1 grow at logarithmic rate n for t''.

The rate of change of e^{nt} is then ne^{nt} $(\dot{s} = ps)$.

If the logarithmic rate is negative the quantity decreases. Rate of change of $e^{-nt} = -ne^{-nt}$.

We wrote the equation of the curve $\rho = \alpha e^{nt} + \beta e^{-nt}$, where $\alpha = oa$, $\beta = ob$.

But $\rho = op = om + mp$. [Fig. 75].

Hence $\frac{om}{oa} = e^{nt}$, $\frac{mp}{ob} = e^{-nt}$.

Hence $om \,.\, mp = oa \,.\, ob = \text{const.}$

This is the relation between the vector coordinates of p, and hence its locus is a hyperbola. We see that the axes of α and β are asymptotes to the curve. For by taking nt large enough, that is the length of $om = e^{nt}$ very large, we can make e^{-nt} or the length of mp as small as we like. And by taking nt small enough we can make e^{-nt} or mp as large and e^{nt} or om as small as we like.

For the flux of ρ we have

$$\dot{\rho} = n\alpha e^{nt} - n\beta e^{-nt},$$

a similar curve to that for ρ; but it is turned round the α axis so that the β coordinates are negative. It is shewn by the dotted line in the figure.

We see that the tangent at p is parallel to oq,

$$oq = om + mq = om - mp.$$

And the velocity which is measured in this direction is n times oq.

The figure is drawn for $n = \frac{1}{2}$.

The locus of q is the conjugate hyperbola, and op and oq are semi-conjugate diameters. In the ellipse both of them meet the curve, but in the hyperbola one meets the conjugate branch.

We have,

$$\rho = \alpha e^{nt} + \beta e^{-nt} = (\alpha+\beta)\tfrac{1}{2}(e^{nt}+e^{-nt}) + (\alpha-\beta)\tfrac{1}{2}(e^{nt}-e^{-nt}).$$

We will call

$\frac{1}{2}(e^{nt}+e^{-nt})$, the hyperbolic cosine of nt, $hc\,.\,nt$,

and $\frac{1}{2}(e^{nt}-e^{-nt})$, " " sine of nt, $hs\,.\,nt$.

If we also write

$$\alpha+\beta=\gamma, \quad \alpha-\beta=\delta,$$

the last formula becomes

$$\rho = \gamma hc\,.\,nt + \delta hs\,.\,nt.$$

We thus have an expression for ρ in terms of two semi-conjugate diameters, analogous to the one we found for the ellipse.

γ and δ are clearly the values of op and oq for $t=0$.

The corresponding expression for the flux is

$$\dot{\rho} = n(\gamma hs\,.\,nt + \delta hc\,.\,nt),$$

since each of the functions $hs\,.\,nt$, $hc\,.\,nt$ has for its flux n times the other. Or we may see that this follows from the first expression found for $\dot{\rho}$.

If we take any two semi-conjugate diameters, OC and OD, and also two others OP and OQ [Fig. 76], we have,

$$\rho = OP = OM + MP$$
$$= \gamma hc\,.\,nt + \delta hs\,.\,nt.$$

Hence,

$$\frac{OM}{OC} = hc\,.\,nt; \quad \frac{MP}{OD} = hs\,.\,nt,$$

and

$$\frac{\dot{\rho}}{n} = OQ = ON + NQ$$
$$= \gamma hs\,.\,nt + \delta hc\,.\,nt.$$

Hence,

$$\frac{ON}{OC} = hs\,.\,nt; \quad \frac{NQ}{OD} = hc\,.\,nt.$$

From these we find

$$\frac{OM}{OC} = \frac{NQ}{OD} \text{ and } \frac{ON}{OC} = \frac{MP}{OD},$$

for any two pairs of conjugate diameters.

These equations declare that,

The projection of OP on OC is to OC as the projection of OQ on OD is to OD; and the projection of OQ on OC is to OC, as the projection of OP on OD is to OD.

From the equation for $\dot{\rho}$, we see that the hodograph is a curve of n times the dimensions of the conjugate hyperbola, and is similarly situated to that hyperbola.

For the acceleration, we find,

$$\ddot{\rho} = n^2 (\alpha e^{nt} + \beta e^{-nt})$$
$$= n^2 \rho,$$

by repeating the operation by which $\dot{\rho}$ was found.

Thus the acceleration of a point P moving on a hyperbola is in the direction of OP, but away from the centre.

In the ellipse the acceleration is directed towards the centre.

Velocities and accelerations resemble steps in having magnitude and direction. All three belong to the class of "vectors," which are characterised by these two properties, and also by a third, not pointed out by Hamilton, namely, that they have no definite position.

Hence we must consider vectors as steps, not of points, but of a rigid body. And similarly, velocities and accelerations must be regarded as velocities and accelerations, of translation, of a rigid body.

Thus the uniform circular motion which we have discussed, is to be considered as the revolution of a rigid body, round a centre, so that all its particles describe circles of the same size.

Any enclosed area may be represented by a line drawn perpendicular to its plane, and of length proportional to its magnitude, so that there are as many linear units in the line, as there are square units in the area.

Now the area is not completely defined until the direction in which we go round it is known. This direction may be either clock-wise or the reverse, and the corresponding vectors would be drawn in opposite directions, upwards in the one case, downwards in the other.

The rule for drawing the representative vector is this :—It must be drawn so that when we look back along it the area appears to be gone round counter clock-wise.

For the area shewn in the figure [Fig. 77] it must therefore be drawn upwards.

If we have a figure of 8 [Fig. 78] bounding two areas which are gone round in opposite directions, looked at from one side, the true value of the whole area is the difference of the two parts. It is represented by the difference of the lengths of the vectors whose lengths correspond to their magnitudes; for these are drawn in opposite directions. (We will call the rotation opposite to that of the clock-hands, positive rotation.) This method for the addition of areas was given by Möbius.

We may take a more complex plane curve [Fig. 79]. It is easy to see that the two areas in the centre are to be counted as 0. The lower one for example is seen to be formed by the overlapping parts of a positive and a negative area, and the superposed parts destroy one another.

The addition of areas in space corresponds to the addition of vectors which are not in the same straight line.

If we want to represent an area on a curved surface, for example, a cone, we break it up into small areas, nearly plane, and draw from each of these its representative vector in the manner before described. These lines will not be parallel, and must be added together according to one of the rules for finding the resultant of vectors in space. Their resultant represents the area. It represents not the actual size of the area, but in a certain physical sense, the area of the contour. Hence every closed surface may be considered as a quantity having a certain magnitude and direction. This point of view was first taken by Hayward*.

If A is any plane area, its projection on a plane making an angle θ with its plane is $A \cos \theta = A'$.

We know that for any plane area there is a set of planes on which the projection is a maximum, viz. all parallel planes, and another set on which the projection is a minimum, viz. planes perpendicular to the first.

All this holds good for areas, not plane, which are represented in the way we have described.

There is a certain set of planes on which the projection is a maximum. Call it A. On all other planes making an angle θ with the maximum planes, the projection is $A \cos \theta$. And there is a plane on which the projection is zero.

Take as an example of a non-plane area, two triangles and a parallelogram in different planes. [Fig. 80.]

Let ab, bc, cd be the vectors representing the areas 1, 2, 3, then ad represents the whole area.

If now we project the areas 1, 2, 3 on a plane D, it is clear that the new representative vectors are the projections of the lines 1, 2, 3 on a line perpendicular to the plane D.

For the angle between two representative vectors = the angle between their planes. This being so, the projection of the compound area (1, 2, 3) on any plane is represented by the projection of ad, on a normal D to that plane.

Hence all planes perpendicular to ad are maximum planes for the compound area, and its projection on them is A, the projection on any plane making an angle θ with them is $A \cos \theta$. And the projection on planes parallel to ad is a minimum. If then the maximum planes are known, and the maximum projection, we find the line representing the compound area by this rule:—Draw a line perpendicular to the maximum planes, of length A.

Consider first two triangles in a plane, having a common side and vertex; we can prove that their sum is the triangle having the same side, with its base equal to the (vector-) sum of their bases.

If the two triangles are [Fig. 81] OAB, OAC, it is to be shewn that

$$OAD = OAB + OAC.$$

* [*Proc. of L. Mathematical Society*, Vol. IV. pp. 289—91, 417.]

If we complete the parallelogram with sides AC, AB, the triangle

$$CAD = DAB,$$

for their areas are equal and the motion round both is positive.

Taking account of the way in which we go round their contours,

$$OAD = OAC + OCD + CAD$$

$$= OAC + OCD + DAB.$$

Now, $$OAB = OCD + DAB,$$

(for, drawing perpendiculars to a line at right angles to AB, the areas are,

$$OAB = \tfrac{1}{2}ln \,.\, AB,$$

$$DAB = \tfrac{1}{2}mn \,.\, AB,$$

$$OCD = \tfrac{1}{2}lm \,.\, AB.$$

Hence $$OAB = OCD + DAB.)$$

substituting, this gives

$$OAD = OAB + OAC.$$

If we agree to call the common side OA the height of the triangles, we may write the last result thus:—The sum of two triangles with the same height, is a triangle having the same height and with its base equal to the sum of their bases.

We may now extend this to triangles in space, which are represented by vectors.

Take two triangles standing up from the plane of the paper, and first suppose their common "height" perpendicular to the plane.

We have to shew that the line representing OAD [Fig. 82] is the sum of the lines representing OAC and OAB.

This is at once seen by looking at the space-figure. We may represent OAC by a vector perpendicular to AC, numerically equal in length to AC. Let this be AC'. Then OAB and OAD will be represented in the same way by AB', AD'. These representative vectors must be all drawn in the plane of AB and AC, that is, the plane of the paper. For each of the triangles is perpendicular to this plane. The new vectors are the sides and diagonals of the same parallelogram turned through a right angle, and we see that $AC' + AB' = AD'$.

This proves the proposition.

We may easily pass to the general case where the height OA is not supposed perpendicular to the plane of AB and AC.

If we draw through O a plane perpendicular to OA it cuts the plane of each triangle in a line perpendicular to OA. Take OAD [Fig. 83] for example; then

$$\text{area } 2OAD = OP \times AD = OA \,.\, \frac{OP}{OA} \,.\, AD$$

$$= OA \,.\, AD \,.\, \cos\theta = OA \,.\, OF.$$

Similarly, if OG, OH are the projections on the plane perpendicular to OA of the bases AC, AB, their areas will be $OA \,.\, OG$, $OA \,.\, OH$, respectively. Now the lines OG, OH, OF are the sides and diagonals of a parallelogram; hence by the addition of vectors, $OF = OG + OH$. And since we may represent the triangles by lines drawn from O in the plane perpendicular to all their planes, at right angles to the plane of each, and equal in length to OG, OH, OF, these representative vectors form the same parallelogram turned through a right angle. If they are OF', OG', OH' we still have

$$OF' = OG' + OH',$$

and hence for the corresponding areas

$$OAD = OAB + OAC.$$

In the ancient geometry the product of two lines was the rectangle contained by them. We will now extend this representation of a product, and say that the product of two lines inclined at any angle is the area of the parallelogram contained by them. Thus the product of two vectors is the vector perpendicular to their plane and proportional in length to the area they enclose. Thus we make the definition $OA \,.\, OB = 2 \,.\,$ triangle OAB.

Remembering the convention about signs of areas, we see that OAB is positive.

By interchange of letters, the formula gives,

$$OB \,.\, OA = 2 \,.\, \text{triangle } OBA.$$

Now OBA is $-OAB$.

Hence we learn that the product of two vectors is altered in sign when the order of multiplication is reversed.

We may now interpret the proposition about the sum of two triangles in space, by this formula.

Their areas will be

$$2AOD = AO \,.\, AD,$$
$$2AOB = AO \,.\, AB,$$
$$2AOC = AO \,.\, AC,$$

and since $$AOD = AOB + AOC,$$

$$AO \,.\, AD \,\{= AO\,(AB + AC)\} = AO \,.\, AB + AO \,.\, AC.$$

Hence we conclude that vector multiplication is distributive.

Using shorter symbols the theorem may be written

(1) $$V\alpha\,(\beta + \gamma) = V\alpha\beta + V\alpha\gamma.$$

We have seen that ($OA \,.\, OB = -OB \,.\, OA$, or)

(2) $$V\alpha\beta = -V\beta\alpha.$$

This shews that vector multiplication is not commutative.

If we change the order in each term of equation (1), we change the sign of each term and get,

(3) $$V\,(\beta + \gamma)\,\alpha = V\beta\alpha + V\gamma\alpha.$$

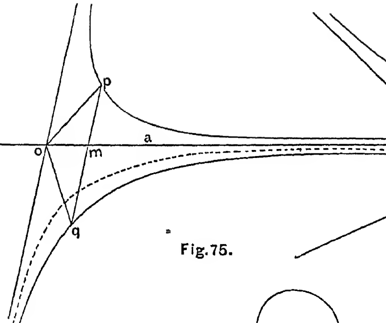


Fig. 75.

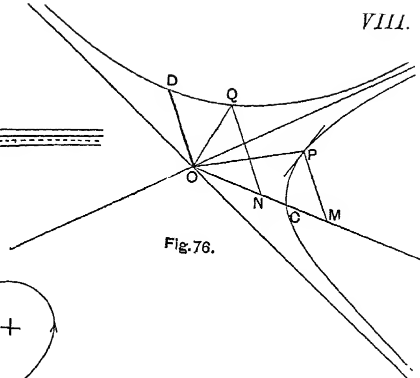


Fig. 76.

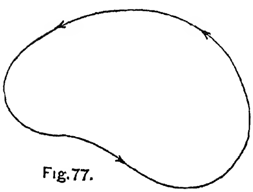


Fig. 77.

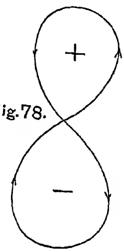


Fig. 78.

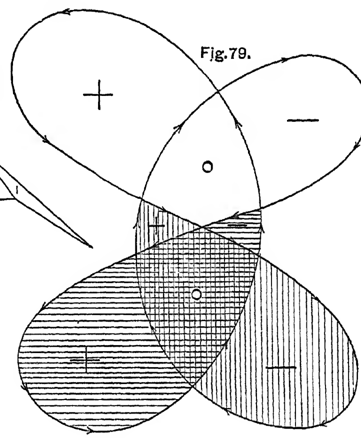


Fig. 79.

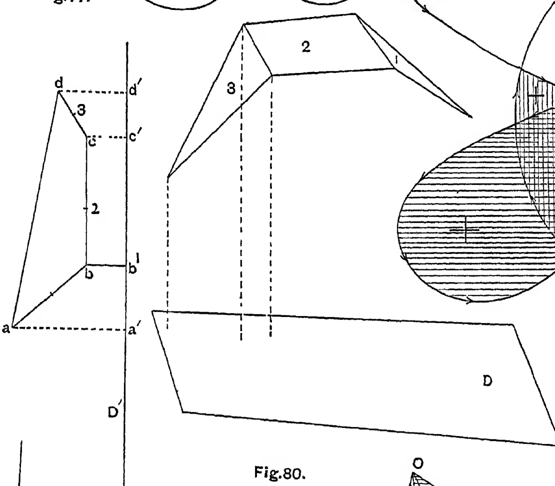


Fig. 80.

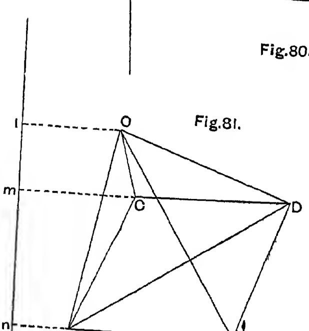


Fig. 81.

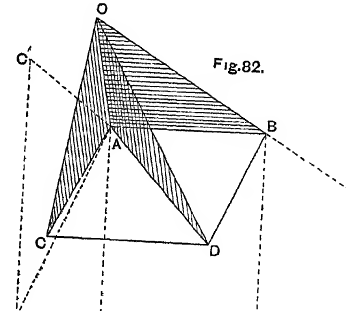


Fig. 82.

This however is not the same equation as (1). The signs of its terms are opposite to those of the terms in (1).

Grassmann called the vector-product of two lines their "outer product," because it has no existence unless one is outside the other.

The following proof of equation (2) was given by Sylvester. It shews on what geometrical fact the rule depends.

If OA and OB coincide they will enclose no parallelogram, and we shall have,

$$(4) \qquad V\alpha\alpha = 0.$$

This being true, try it on $V(\alpha+\beta)(\alpha+\beta)$.

This product is broken up into four parts,

$$0 = V\alpha(\alpha+\beta) + V\beta(\alpha+\beta) = V\alpha\alpha + V\alpha\beta + V\beta\alpha + V\beta\beta.$$

The first and last terms vanish, and hence

$$0 = V\alpha\beta + V\beta\alpha.$$

If we had assumed (2), equation (4) would follow at once from it, for we must have

$$V\alpha^2 = -V(\alpha^2).$$

This kind of multiplication has been called "polar multiplication." We see then that any kind of multiplication which is distributive and where the square of any quantity $=0$, must be polar.

We may now use the theorem about the geometrical representation of products by areas, to investigate the rotation of rigid bodies.

Let the angular velocity of a rigid body about the axis ab be ω. Any particle p of the body describes a circle round ab. At this moment p is moving perpendicular to the plane of the paper and its velocity (arc described in $1''$) is, $\omega \,.\, mp$.

If we measure off on ab a length ab proportional to ω, the velocity of p is twice the area of the triangle abp. [Fig. 84.]

It is necessary however to take into account the direction of motion of p, and to give a different sign to the triangle which represents the velocity.

Suppose ab measured so that, looking back from b to a, the motion of p appears to be counter-clockwise; p must be moving at this moment back from the paper. We make this convention because the sign of abp is negative, and therefore its representative vector must be measured downwards, or in the negative direction.

We have found, then,

$$\dot\rho = \omega \,.\, mp = 2 \,.\, abp = V\overline{ab} \,.\, \overline{ap} = V\omega\rho.$$

Similarly for every point in the space occupied by the body there is a certain vector representing the velocity of a particle at that point, and which is expressed in terms of its position-vector ρ. The aggregate of these vectors forms a velocity system.

We might have also an acceleration system; that is a definite acceleration belonging to every point of space. Both of these are vector-systems.

Now if we have to compound two vector-systems we shall compound the two vectors belonging to the systems respectively at every point of space. As an application of this we will compound the velocity systems corresponding to two rotations about a fixed axis. These rotations may be called "spins." They are completely defined when we know the rotation axis and the angular velocity.

The problem then is to find the resultant of two spins with angular velocities ω_1, ω_2, and with axes passing through a fixed point a.

We have $$\dot\rho_1 = V\omega_1\rho, \quad \dot\rho_2 = V\omega_2\rho$$

for any point of space which has the position-vector ρ.

Since the velocities are represented by the areas abp, acp we have for the resultant velocity ($adp = abp + acp$) [Fig. 85]

$$\begin{aligned}\dot\rho = \dot\rho_1 + \dot\rho_2 &= V\omega_1\rho + V\omega_2\rho \\ &= V(\omega_1+\omega_2)\,\rho.\end{aligned}$$

Hence a velocity system compounded of two spins is a velocity system with a spin which has the sum of the component angular velocities.

This way of writing the theorem of distributive vector multiplication is only a short-hand for the geometrical proof of the same proposition.

This expression for the resultant does not mean that the body has first a little turn about one axis and then about the other. The spin of the body is made up of two spins but it does not get those spins actually. We may easily extend this result to find the resultant of any number of angular velocity systems with axes passing through a fixed point.

We shall be led to consider another kind of product of two vectors :—the scalar product.

In the ancient geometry the product of these lines was represented by the rectangular parallelepiped which they contained.

We will extend this to the parallelepiped contained by any three lines meeting in a point, and we may then make the definition

$$OA\,.\,OB\,.\,OC = 6 \times \text{tetrahedron } OABC.$$

Now if we draw OM perpendicular to OAB and proportional in length to $2OAB$ [Fig. 86], we have

$$OA\,.\,OB\,.\,OC = \overline{OM}\,.\,\overline{OC}.$$

Now $$6\,.\,OABC = OM\,.\,ON = OM\,.\,OC\cos\phi.$$

Then, if the lengths of OA, OB, OC, OM are a, b, c, d we have

$$OA\,.\,OB\,.\,OC = c\,.\,d\,.\,\cos\phi.$$

We must now take account of signs.

The volume of $OABC$ shall be reckoned positive when the three points A, B, C, looked at from O are gone round counter-clockwise. The volume in the figure then is negative. It has been seen that the product of two vectors OA and OB is a directed quantity for it is an area, considered as to size and aspect, and may therefore be represented by a vector. Its magnitude is $ab\sin\theta$

and it is perpendicular to the plane of OA and OB. It is the vector-product. But now the product of three vectors has also been reduced to the product of two, viz. OM and OC. And since this is a volume, it can only have quantity not direction, and it is called the scalar product of two vectors. In the scalar product we consider one of the vectors OM as the area, the other as a vector. While in the vector-product both are considered as vectors.

This distinction between two kinds of vectors was first made by Maxwell. One kind of vector corresponds to a force, the other to a flow.

The product of two flows or of two forces will be a vector-product. The product of a force and a flow will be a scalar-product.

The point of view we have taken in considering an area as a product of two vectors of the same kind and a volume as a product of three vectors of the same kind, is that of Grassmann, not of Hamilton.

Since the volume represented by $OC \,.\, OM$ is to be negative we must draw OM so that looking back along it the rotation of OA to OB is in the negative direction. It is drawn right then in the figure. If the parallelepiped were rectangular OM would fall on OC, and supposing it of the same length we should have

$$S\alpha^2 = - \text{ squared length of } \alpha.$$

We may see at once that a scalar-product is not altered when we change the order of the factors. For interchanging OC and OM, the sign of ϕ is altered but not that of $\cos\phi$.

Therefore $S\alpha\beta = S\beta\alpha$.

We may define the scalar-product of two vectors geometrically as a solid, one of the vectors being an area; or physically as the product of three vectors of the same kind. Similarly the vector-product may be defined geometrically as an area considered in aspect and size or physically as the product of two vectors of the same kind.

Since the scalar-product of two vectors is one vector multiplied by the projection of the other upon its direction, we can at once deduce the distributive law for scalar multiplication, or

$$S\alpha\,(\beta+\gamma) = S\alpha\beta + S\alpha\gamma,$$

since the projection of $\beta+\gamma$ on α is the same as the sum of the projections on it of β and γ. [Fig. 87.]

Illustration of a scalar product in physics. The rate of doing work by a body moving with a velocity σ and acted on by a force $\varpi = S\sigma\varpi$.

Another illustration. The kinetic energy of a rotating body $= S \,.$ spin $\times$ moment of momentum. Each of these last quantities is a vector.

Again, if a plate is bent out of its plane or deformed in its plane,

$$\text{potential energy} = S \,.\, \text{strain} \times \text{stress}.$$

Recalling the double interpretation of the equality $2 \times 3 = 6$; viz. 1st, 2 and 3 multiplied together give 6; 2nd. 3 (operated on) by 2 gives 6; we will

NOTES OF LECTURES ON QUATERNIONS *.

We will define the signs + and − as indicating steps, whereby any magnitude may be increased or diminished, or by means of which we may move from one point of a progression to another.

Let the quantity which has the progressive values be measured along a line; but we shall suppose at first that the numbers stand for amounts of any kind of quantity, not necessarily length.

+			0	1	2	3	4	5
−3	−2	−1	0	+1	+2	+3		

Then we should have a zero value and successive values marked off, *above* the line, by the numbers 1 2 3 etc. Numbers standing alone mean amounts of the given quantity, while the sign + or − before them means a step is to be taken through successive values of that quantity so as to lead to a different value from that from which we set out.

Thus the equation

$$2+3=5$$

means that if we start with 2 things represented by 2 places on the line and step through 3 following places we get to the place indicated by 5.

But there is another way of considering this equation. Instead of marking off on our line a scale of numbers merely it may be a scale of steps as by the positive and negative numbers below the line.

And then our equation would be

$$+2+3=+5.$$

Or whatever number we set out from and take two steps to the right and then three again to the right it is the same as taking 5 at once to the right. Here are two interpretations of the equation.

Again consider this form of the same truth

$$5-3=2.$$

This may mean first—

Starting with a number 5 denoted by its place on the line, and taking three steps backward, to the left, we get to the number 2. Or it may mean, if we consider the scale as one of steps,

* [I am indebted to Miss E. Watson for these *Notes* of lectures delivered at University College, towards the close of 1877.]

If [Fig. 90]

$$q \cdot OA = OB, \text{ and } OB = OM + MB,$$

$$q = \frac{OM}{OA} + \frac{MB}{OA}$$

$$= \frac{OM}{OA} + \frac{MB}{OA'} \cdot \frac{OA'}{OA}$$

$$= a + b \cdot \frac{OA'}{OA},$$

where a and b are numbers, with positive or negative signs.

We want a symbol for the operator which turns a step through a right angle in the same plane. If i stands for this operator

$$q = a + bi:$$

i may be regarded as a handle perpendicular to the plane of OA and OB which turns OA through a right angle.

It is at once seen to be a property of i that if it is used twice on a vector it reverses that vector; or

$$i^2 = -1.$$

We can now use the operator i to establish some important results. First however we must find an expression for the rate of change of a product of two numbers, pq. Take their values p_2q_2 at the time t_2, and p_1q_1 at the time t_1, then for the mean rate of change of the product during the interval we have,

$$\frac{p_1q_1 - p_2q_2}{t_1 - t_2} = p_1 \cdot \frac{q_1 - q_2}{t_1 - t_2} + \frac{p_1 - p_2}{t_1 - t_2} \cdot q_2.$$

Making the beginning and end of the interval coincide, we get the actual rate of change of the product; that is

$$\partial_t pq = p\dot{q} + \dot{p}q \text{ .... (1).}$$

In getting out this result all we have assumed is that multiplication is distributive. Hence the result is true when p or q or both of them are vectors. Letting q only equal a vector ρ, we have

$$\partial_t (p\rho) = p\dot{\rho} + \dot{p}\rho \text{ .... (2).}$$

If both are vectors the result holds good for either the scalar or the vector-product; hence

$$\partial_t V(\alpha\beta) = V\alpha\dot{\beta} + V\dot{\alpha}\beta \text{ .... (3),}$$

$$\partial_t S(\alpha\beta) = S\alpha\dot{\beta} + S\dot{\alpha}\beta \text{ .... (4).}$$

We may apply equation (2) for one quantity a scalar and the other a vector to the problem of resolving the acceleration of a moving point along the tangent and the normal to its path.

The velocity of p, $\dot{\rho}$, is in the direction of the tangent at p, and its magnitude equals $\dot{s} = v$, where s is the length of the curve up to p, measured from a fixed point a. [Fig. 91.]

Hence $\frac{\rho}{s}$ is a vector of unit length parallel to pt. Let oc represent it; and denote it by ρ'.

Then we have

$$\dot{\rho}=\dot{s}\rho'=v\rho'.$$

Hence by equation (2) the acceleration of p is

$$\ddot{\rho}=\dot{v}\rho'+v\dot{\rho}'.$$

The first part is the acceleration parallel to the tangent. Its magnitude $=\dot{v}$.

The second term is v multiplied by the velocity of c. Since c is always at the end of a unit vector its path is a circle, of radius 1. Hence the velocity of c is always in a direction perpendicular to oc, and to pt, that is along the normal. Its magnitude = the angular velocity of oc, that is of the tangent pt.

The magnitude then of the normal component of the acceleration is the velocity of the point multiplied by the rate of turning round of the tangent.

But we know that,

$$\frac{\dot{\rho}'}{v}=\frac{\text{rate of turning round of tan } pt}{\text{velocity of } p}$$

$$=\text{rate of turning round of tangent per unit length of curve}$$

$$=\text{curvature.}$$

Hence the normal acceleration $=v^2\times$ curvature, curvature being measured by a line drawn along the inner normal to the curve.

Suppose a point or system of points rigidly connected, moving in a plane which also moves over a fixed plane. Each of these is to be a velocity-system consistent with rigidity. The characteristic of such a system is that the distance of any two points remains constant. This is the same as saying that the velocity of any point relatively to any other is perpendicular to the line joining them. For since ab is of constant length, and we suppose a fixed for a moment, b clearly moves perpendicular to ab.

If we compound with this system another velocity-system consistent with rigidity, for the same points a and b, the velocity of b relatively to a is a line perpendicular to ab, suppose it standing up perpendicular to the plane of the paper. Then the resultant of the two velocities of b is again in a direction perpendicular to ab. Hence the resultant of two velocity-systems each consistent with rigidity is a velocity-system consistent with rigidity. If we have any velocity-system of this nature in which some point a is moving with a certain velocity and combine with it a motion of translation of the whole system, equal and opposite in direction to the velocity of a, its velocity is destroyed and there results a spin about a as centre. Conversely, any velocity-system consistent with rigidity is made up of a translation and a spin round some point.

Now let us find an expression for the acceleration of any point P moving in a moving plane. At any moment the motion of P is a spin about some point a which is called the instantaneous centre.

We have then

$$\dot{P} = i \, . \, aP \, . \, \omega,$$

where ω is the angular velocity about a.

In general the instantaneous centre will move about; and then we find from equation (2) for the acceleration of P

$$\begin{aligned} \ddot{P} &= i \, . \, \dot{\overline{aP}} \, . \, \omega + i \, . \, \overline{aP} \, . \, \dot{\omega} \\ &= i \, (p - \dot{a}) \, \omega + i \, . \, \overline{aP} \, . \, \dot{\omega} \\ &= -aP \, . \, \omega^2 + i \, . \, \overline{aP} \, . \, \dot{\omega} - i \, . \, \dot{a} \, . \, \omega \quad \ldots\ldots\ldots\ldots \quad (1), \end{aligned}$$

where we have used the expression for a vector symbolically in terms of points, viz.

$$aP = p - a, \qquad \therefore \; \dot{\overline{aP}} = \dot{p} - \dot{a},$$

and have then substituted for $\dot{P}$ its value, $i \, . \, \overline{aP} \, . \, \omega$.

It is important to notice that $\dot{a}$ does not mean the velocity of the point a which is a point fixed in the plane, but the velocity of the instantaneous centre.

The first term in the expression for $\ddot{P}$ is the normal acceleration of P, supposing it to move uniformly round a. The second term is the tangential acceleration.

The third term is the acceleration of P at right angles to the velocity of the instantaneous centre.

We will apply this result to examine the motion of the instantaneous centre. Let the point P coincide with the instantaneous centre; calling it O, we have

$$\ddot{O} = - i \, . \, \dot{a} \, . \, \omega,$$

since $aP = aO$ vanishes.

Here $\ddot{O}$ is the acceleration of the point O, fixed in the moving plane, $\dot{a}$ is the velocity of the instantaneous centre in the fixed plane.

The problem is the same as that of determining the relative motion of the two planes. If we suppose them fixed alternately, all angular velocities of the first with regard to the second will be the opposite of the corresponding velocities of the second with regard to the first.

Hence if O_1 is a point in the other plane (supposed fixed before), we have, since ω is changed in sign,

$$\ddot{O}_1 = + i \, . \, \dot{a}_1 \, . \, \omega,$$

where $\dot{a}_1$ is now the velocity of the instantaneous centre in the plane which was moving before and is now considered as fixed.

But the acceleration of O in one plane with regard to a point O_1 considered as fixed in the other is the same as the acceleration of O_1 with regard to O considered as fixed, only in the reversed direction; that is

$$\ddot{O}_1 = - \ddot{O} \, ;$$

$$\therefore \; i \, . \, \dot{a}_1 \, . \, \omega = i \, . \, \dot{a} \, . \, \omega,$$

that is

$$\dot{a}_1 = \dot{a}.$$

The velocity of the instantaneous centre in the fixed plane is the same as the velocity of the instantaneous centre in the moving plane. Since these velocities are always in the same direction we see that the curves traced out by the instantaneous centre in the fixed plane and the moving plane respectively must touch. Further since the velocities of a and a_1 are equal in magnitude, we have $\dot{s}=\dot{s}_1$, where s, s_1, are the lengths of the two curves, loci of instantaneous centres, measured from any fixed point. Hence the curves must roll on each other without sliding, and the point P will lie on a roulette.

We shall be able to find the curvature of this roulette.

Let an be the normal to the two loci of instantaneous centres. [Fig. 92.]

The normal acceleration of P (along $\overline{aP}$), is the first term of equation (1) + the resolved parts of the other two terms. But the second part has no component along aP. Hence calling the magnitude of the velocity of a, $\dot{s}$, we have

$$\text{normal acceleration of } P=-\,\overline{aP}\,.\,\omega^2+\dot{s}\,.\,\omega\cos\theta.$$

Also

$$\text{velocity of } P=\overline{aP}\,.\,\omega.$$

And

$$\text{curvature}=\frac{\text{normal acceleration}}{(\text{velocity})^2}.$$

Hence

$$\text{curvature}=-\frac{1}{aP}+\frac{\dot{s}\,.\cos\theta}{\omega\,.\,aP^2}.$$

Now $\dfrac{\omega}{\dot{s}}$ = rate of turning round of any line (aP) in the moving plane per unit length of the curve of instantaneous centres.

Hence we have

$$(\text{curvature of roulette})=-\frac{1}{aP}+\frac{\cos\theta}{aP^2}\,\frac{1}{\dfrac{1}{r}-\dfrac{1}{r'}}.$$

There are two circles, loci respectively of points which have no normal, and no tangential acceleration. Their intersection is a point having no velocity at all.

We now go on to apply the symbol i to an exponential function.

We defined e^{pt}, as the result of making unity grow at logarithmic rate p for t seconds; and the rate of change of this quantity is got by multiplying itself by p.

Let us now examine what meaning we can give to e^{it}. This is a step OP in the plane, the rate of change of which is got by multiplying it by i, that is by turning it through a right angle.

Suppose OP was originally of length $1=OA$.

Then since the rate of change of OP is always perpendicular to its direction, P moves on a circle.

We can shew in a special case that e^{it} is a step in the plane making an angle of circular measure t with the fixed unit line OA.

Suppose $t=1$. Then we have

$$e^{i}=1+i+\frac{i^2}{2}+\frac{i^3}{2.3}+\frac{i^4}{2.3.4}+\ldots\ldots$$

We can construct the step e^{i} by operating on $OA=1$ by this series.

The series converges rapidly. The first five terms take us up to E. [Fig. 93.] Some point very near this gives the end of the step e^{i}. Now the angle subtended by the arc $AP=57^0$; E is therefore the point on the circle whose arc measured from $A=$ the radius. Since the radius was taken $=1$, the arc $AE=1=t$. Hence for this case ($t=1$) we have verified the meaning given to e^{it}.

From the figure we have, if now AE is any arc t,

$$OE=OM+ME,$$

or
$$e^{it}=\cos t+i\sin t.$$

Hence from the series for e^{it} we can get the series for the sine and cosine in powers of the angle.

Any step ρ of greater length than the radius, is thus expressed

$$\rho=re^{i\theta},$$

where r is the length of the step; θ the angle it makes with OA.

From equation (2) we can find the radial and transversal velocities and accelerations

$$\dot\rho=\dot re^{i\theta}+ir\dot\theta\, e^{i\theta},$$

$$\ddot\rho=(\ddot r-r\dot\theta^2)\,e^{i\theta}+i\,(2\dot r\dot\theta+r\ddot\theta)\,e^{i\theta}.$$

(radial) (transversal)

We will return to the general consideration of rectangular versors.

In one plane if the lengths of om, mb [Fig. 94] are x, y we have

$$ob=\left(\frac{om}{oa}+\frac{mb}{oa}\,i\right)oa$$

$$=(x+yi)\,oa \text{ for any versor.}$$

A rectangular versor is the operator which turns a vector through a right angle in its plane and stretches it. We may then represent it by a handle perpendicular to the plane in which the turning takes place and of length numerically equal to the ratio in which the length of the vector is to be altered. If the handle is of length 1 the vector is not stretched. We will now consider what is meant by the sum of two rectangular versors.

If we have

$$A\rho+B\rho=C\rho,$$

where A, B and C are rectangular versors, how can we represent C.

Let A and B lie in the same plane. Then we must take the vector operated on (ρ) perpendicular to each of these versors and therefore perpendicular to their plane.

Suppose it is of length 1.

Operating on it by the two versors successively we see that A turns it down into the plane of the paper and increases its length to A, B also turns it down into the same plane and increases its length to B, $A\rho$ is of course perpendicular to A, $B\rho$ perpendicular to B. The result of both operations is then $A\rho+B\rho$ [Fig. 95]; if now we ask what versor would be required in order to bring ρ into this position and to give it this length, the answer clearly is—a versor perpendicular to $A\rho+B\rho$ and equal to it in length; that is C, the diagonal of the parallelogram formed by A, B.

Hence $A+B=C$,

and the sum of two rectangular versors is formed in the same way as the sum of two vectors.

To find the sum of three rectangular versors not in the same plane,

$$A+B+C,$$

we must add $A+B$ by the preceding rule and then add the resulting rectangular versor to C by the same rule. The way in which we take the pairs is indifferent. We see that it is necessary to take the operations in instalments if we are to give any interpretation to successive steps. For if A, B, C are not all in the same plane, there is no vector on which they can operate simultaneously.

Of course the preceding result is independent of the length of ρ being unity. We may multiply each term of the equation by any number and the result will still hold good.

We now go on to consider the product of two rectangular versors. We assume them each of length unity. Let them be A and B inclined at an angle θ. [Fig. 96.]

The vector to be operated on must be placed so that after A has operated on it B will be able to operate on it. It must therefore be in the plane of AB perpendicular to A. After both operations it will be in the position ρ' perpendicular to B and in the same plane. The effect of the product is then a versor not rectangular, which turns ρ through the supplemental angle of θ.

Since BA turns oa into ob, we have,

$$BA=\frac{ob}{oa}=\frac{om}{oa}+\frac{mb}{oa}=(-\cos\theta+I\sin\theta),$$

where I is a unit vector perpendicular to the plane of AB. If these versors have any lengths x, y, ob will of course be xy times the length of oa, and we shall have

$$BA=(-\cos\theta+I\sin\theta)\,xy.$$

Now we have seen that $-\cos\theta\,.\,xy$ is the scalar-product of B, A, regarded as vectors. We saw also that $I\sin\theta\,.\,xy$ was their vector-product. We shall now say that these are respectively the scalar and vector *parts* of the product and write

$$BA=S\,.\,BA+V\,.\,BA.$$

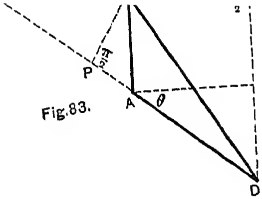


Fig.83.

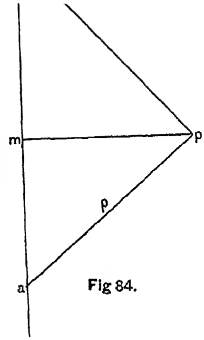


Fig 84.

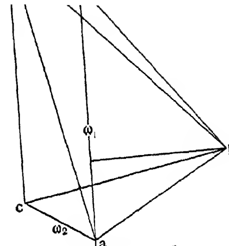


Fig.85.

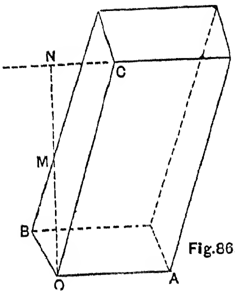


Fig.86.

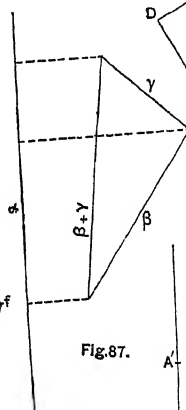


Fig.87.

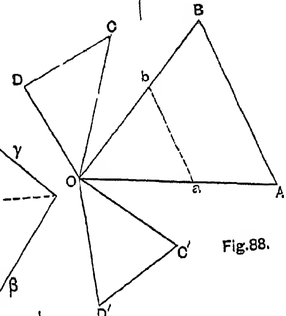


Fig.88.

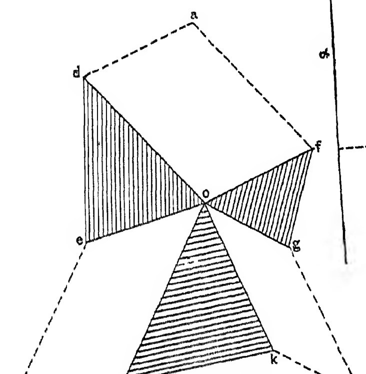


Fig.89.

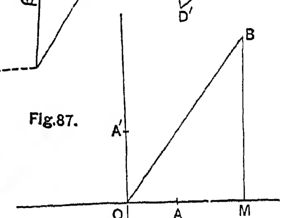


Fig.90.

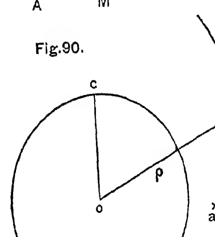


Fig.9

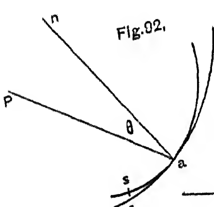


Fig.92.

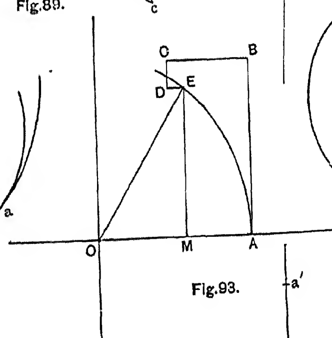


Fig.93.

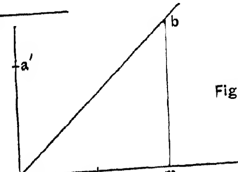


Fig 9

In order that $I \sin\theta . xy$ may represent the vector part of the product we need only agree to measure I so that looking back along it B is turned into A by a positive rotation.

In the case where the two versors are at right angles to each other the scalar part of the product vanishes and we can at once trace the analogy between the double interpretation allowable here and that which we gave to equations of multiplication in arithmetic.

We said that when two factors are multiplied together, either they may both be operators or the first may be an operator and the second a step. Similarly, when quantities can be measured in any direction in a plane, an equation of multiplication for two directions at right angles to each other may be written in two ways

$$BA = V . BA, \quad B\alpha = V . B\alpha.$$

In the first equation both factors are versors; in the second, the first only is a versor, the second is a vector.

We saw that in arithmetic the rule for finding the product was the same for both interpretations, e.g.

$$(-3)(+2) = -6.$$

Rule. Multiply the numbers together and remember that like signs give $+$, unlike signs $-$. The interpretation is either—

The product of the operations of first tripling and reversing, and next doubling is equivalent to the operation of sextupling and reversing; or,

The effect of the operation of tripling and reversing on the step $+2$ is to turn it into the step -6.

Similarly if B and A, or B and α are at right angles to each other, we have the same rule for finding the product in each case, viz. measure off the product of their lengths along a vector at right angles to their plane, and drawn so that looking back along it, we go from B to A, counter-clockwise.

The second factor may be either a vector or a versor, but the first *must* be a rectangular versor; and in both cases the directions must be at right angles to each other.

If however B and A are not at right angles the double interpretation is no longer possible. We must not regard A as a vector α. For a versor can operate only on vectors perpendicular to it. In this case both B and A must be rectangular versors. Then also the scalar part of the product would not vanish. We should have

$$BA = V . BA + S . BA.$$

The rule found for the addition of rectangular versors enables us to represent any one in terms of three units at right angles to each other.

If i, j, k, are unit lines at right angles to each other, then any vector or rectangular versor may be represented by

$$xi + yj + zk,$$

where x, y, z are components of its length.

To find the product of two such rectangular versors, first in one plane, take

$$(xi+yj)(ai+bj)=xi(ai+bj)+yj(ai+bj).$$

The right-hand side expresses the effect of performing the two operations successively. This is equal to

$$xai^2+xbij+yaji+ybj^2.$$

Since i or j operating twice reverse the direction of a vector, we have

$$i^2=-1, \qquad j^2=-1.$$

Also $ij=-ji$. For since these two lines are at right angles there is only a vector part in their product, and we have seen that the effect of changing the order of the factors in a vector product is merely to change the sign.

Hence the whole product is equal to

$$-(xa+yb)+(xb-ya)ij.$$

Taking now three such vectors i, j, k in space, at right angles to each other, if k is drawn so that looking back along it i is turned by positive rotation into j, we have, since the lengths are all $=1$

$$ij=k=-ji.$$

Similarly $$jk=i=-kj,$$

$$ki=j=-ik.$$

We can then form a multiplication table of versors.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

If we call $i^2=-1$, etc. $ij=ji$, etc. the primary rules and

$ij=k$, $jk=i$, $ki=j$ the secondary rules,

we may explain these last by means of the primary rules, if we regard i, j, k as binary products of quantities which fulfil the primary rules.

Let $a^2=-1$, $b^2=-1$, $c^2=-1$, $ab=-ba$, $bc=-cb$, $ca=-ac$.

Then suppose $i=bc$, $j=ca$, $k=ab$.

We shall find that i, j, k satisfy the primary and also the secondary rules.

For $i^2=bc\,bc=-bb\,cc=-1$, etc.

And $$ij = bc\,ca = bc^2a = -ba = ab = k, \text{ etc.}$$

Thus the secondary rules may be considered as bound up in the primary rules, if we agree to let i, j, k stand for these binary products. We may suppose a, b, c are points or stars at an infinite distance in three directions at right angles to each other, then the step ab will clearly be a rectangular versor.

Since any versor not rectangular is the sum of a scalar and a vector part and any vector may be expressed in terms of i, j, k, we are led to consider quantities of the form

$$\omega + xi + yj + zk,$$

which may also be written

$$m(\cos\theta + I\sin\theta),$$

where $$m^2 = x^2 + y^2 + z^2.$$

This is called a *quaternion* since it involves four numbers. It is the operation of turning one vector into another in the plane perpendicular to I, that is to the rectangular versor $xi + yj + zk$. This is its most general form. We may consider however that we have already studied quaternions in which the versors were in the same plane and in the same direction.

If the quaternion q is represented by

$$q = \omega + xi + yj + zk = \omega + \rho,$$

its conjugate is

$$Kq = \omega - xi - yj - zk = \omega - \rho.$$

Hamilton calls the part ρ the vector part of the quaternion. Strictly speaking it can only be a rectangular versor. For a quaternion, and therefore each of its parts, is an operation on a vector. It is convenient however to regard a rectangular versor as a vector. They are subject to the same laws.

The stretching power of Kq and also its plane are the same as those of q, but the angle through which it turns a vector is in the opposite direction to the angle through which q turns it.

If $$q = \frac{OB}{OA} = \frac{OM}{OA} + \frac{MB}{OA} \quad \text{[Fig. 97]}$$

we shall have (by definition)

$$Kq = \frac{OM}{OA} - \frac{MB}{OA} = \frac{OM}{OA} + \frac{MB'}{OA} = \frac{OB'}{OA}.$$

The effect of $q \,.\, Kq$, or of $Kq \,.\, q$ is merely to lengthen the line OA. For the turning successively through equal angles in opposite directions brings OA back to the same direction.

And in fact, the expressions for q and Kq in terms of ω and ρ give $q \,.\, Kq$ a scalar quantity

$$q \,.\, Kq = (\omega + \rho)(\omega - \rho) = \omega^2 - \rho^2 = \omega^2 + x^2 + y^2 + z^2.$$

The fundamental property of the conjugate of a quaternion is that it is another quaternion such that being multiplied by the first it gives a scalar.

We will now prove a proposition about the multiplication of quaternions.

The conjugate of the product of two quaternions is the product of their conjugates in the reversed order.

For two quaternions r and q,

$$K(qr) = Kr \, . \, Kq.$$

If we consider only the turning part of the quaternion it may be represented by an arc of a great circle on a sphere.

Each arc [Fig. 98] represents the operation of turning the radius to one end of it from the centre of the sphere, through its length to the other end. These are of course rectangular versors. If the two triangles are equal, we see at once that

$$AC = K(qr) = Kr \, . \, Kq \quad \text{.................................} (1).$$

We may see that the use of this equation leads to a right result, if we form the product, $qr \, . \, K(qr)$. This ought to be a scalar, from our first notions of a conjugate.

In fact if the equation (1) is true we have

$$q \, . \, r \, . \, K(qr) = q \, . \, r \, . \, Kr \, . \, Kq = q \, . \, Kq \, . \, r \, . \, Kr,$$

and this is a scalar.

We may at once extend equation (1) to any number of quaternions. We have

$$K(qrs) = Ks \, . \, Kqr = Ks \, . \, Kr \, . \, Kq.$$

And generally for n quaternions a similar equation holds good.

If all the quaternions are vectors (really rectangular versors) we have a particular case of the proposition

$$K(\alpha_1 \alpha_2 \ldots \alpha_n) = (-)^n \, \alpha_n \, . \, \alpha_{n-1} \ldots \alpha_2 \, . \, \alpha_1.$$

This product is in general a quaternion.

The two equations

$$S \, . \, \alpha\beta = S \, . \, \beta\alpha, \quad V \, . \, \alpha\beta = - V \, . \, \beta\alpha,$$

are particular cases of the equation

$$\alpha\beta = K\beta\alpha.$$

If we multiply together a quaternion, a vector and the conjugate of the quaternion we get a vector.

For the result is a quantity whose square is a scalar; and a quaternion squared, $(\omega + \rho)^2$, does not give a scalar unless either ω vanishes or ρ vanishes. $\{(\omega + \rho)^2 = \omega^2 + 2\omega\rho + \rho^2$. The middle term is a vector$\}$.

Forming now the square of $q \, . \, \rho \, . \, Kq$,

$$q \, . \, \rho \, . \, Kq \, . \, q \, . \, \rho \, . \, Kq = qKq \, . \, q\rho^2 \, . \, Kq = (q \, . \, Kq)^2 \, \rho^2.$$

Hence $q \, . \, \rho \, . \, Kq$ is a vector.

We will now examine the meaning of $Kq.r.q$, where r is any rectangular versor.

If the lines here drawn [Fig. 99] represent great circles on a sphere, we see that AC is $Kq.r.q$. That is the result of putting r between a quaternion and its conjugate is to slide r along the great circle of q through twice the length of the arc q; or to make r rotate round the pole of q through twice the angle of q.

In supposing the arcs drawn on a sphere, that is assuming the versors do not stretch, we have really made no restriction. Whatever the stretching powers are, the stretched vectors from the centre come out at the same points of the sphere.

A quaternion is an operator which turns any one vector into another. We will now consider an operator which turns any two vectors α and β into two others γ and δ. Hamilton called this a linear and vector function. We will denote it by ϕ. Then

$$\phi\alpha=\gamma,$$

$$\phi\beta=\delta,$$

$$\phi(l\alpha+m\beta)=l\gamma+m\delta.$$

The last equation expresses what is meant by the function being linear.

The effect of a quaternion, operating on all the vectors in a plane, for example a picture, would be to turn the whole picture through a certain angle and to increase its size, so that the result is a similar picture.

The effect of the function ϕ would be that while all parallel lines remain parallel to one another, their distances as well as their directions are altered. They are all altered in the same proportion. We may express this by saying "All parallel vectors are multiplied by the same quaternion, but the quaternions are different for different vectors."

A circle would become an ellipse. For the equation of a circle may be put into the form

$$\rho=\alpha\cos\theta+\beta\sin\theta \qquad \text{(1)},$$

if α and β are equal and at right angles.

Operating on this with ϕ we get

$$\phi\rho=\phi\alpha\cos\theta+\phi\beta\sin\theta,$$

or

$$\phi\rho=\gamma.\cos\theta+\delta.\sin\theta \qquad \text{(2)},$$

where γ and δ are in general not equal and not perpendicular. But this is the equation of an ellipse.

Putting θ proportional to the time, the first equation is that of uniform motion in a circle; the second that of harmonic motion in an ellipse.

If the lines which correspond to the axes of the circle are at right angles to each other, they give the axes of the ellipse: for parallel lines remain parallel.

The tangents at the ends of one axis must be parallel to the other axis.

The whole change may be represented thus. Take two lines at right angles; lengthen each in a certain proportion. Thus we get an ellipse. Then turn it through a certain angle.

When two directions at right angles remain so we have a pure strain. $\phi\rho$ is then called a pure function. There is no rotation.

For ϕ a pure function we can always get an ellipse such that the function of any semi-diameter is perpendicular to the conjugate diameter and equal to it in length.

We will suppose

$$\phi\alpha = i\beta, \quad \phi\beta = -i\alpha.$$

Then we have, if oq is the semi-conjugate diameter to op,

$$\begin{aligned}\phi\,(op) &= \phi\,(om) + \phi\,(mp)\\ &= \frac{om}{oa}\cdot\phi\alpha + \frac{mp}{ob}\cdot\phi\beta\\ &= \frac{nq}{ob}(-i\beta) - \frac{on}{oa}\,i\alpha\\ &= -i\,(nq + on) = -i\,.\,oq \qquad\qquad \text{(B)}.\end{aligned}$$

The linear function of a vector is best represented physically, as the result of a strain, undergone by a body. Then the function of any vector is the new, that is the strained vector. Or it may represent the displacement of the point at the end of a vector from the origin; and then the new vector will be the old one + the function. If the body undergoes a homogeneous strain, that is one which is the same at every point, the whole change consists of two parts. The body is pulled out in different proportions in three directions at right angles to each other and then turned through a certain angle. The first effect is pure strain, the second is rotation. The pure strain changes a circle into an ellipse, if we consider first a plane sheet, with only two dimensions.

We can now shew that the linear function of a vector may in certain cases be completely represented by an ellipse, called the *strain* ellipse. Any vector namely is in the direction of some diameter of this ellipse and we can find its function at once in this way. Take the conjugate diameter and draw a line perpendicular to it and equal to it in length. This is the function or strained vector.

For the pure strain consists in pulling out two lines at right angles to each other in different proportions and either positively or negatively. That is two diameters of a circle in these rectangular directions are lengthened or shortened, in certain proportions, while their directions are unaltered.

If these extensions are such that two lines oa, ob at right angles to each other have their lengths interchanged, this is represented by the equations

$$\phi\,(oa) = -i\,.\,ob, \quad \phi\,(ob) = i\,.\,oa,$$

where the sign $-$ occurs in the first, because we suppose the rotation from oa to ob to be positive. Each vector is to be simply stretched; not reversed.

Then all we have to do is to construct an ellipse on these lines as axes and the property in question being true for the axes is at once proved for all the diameters.

We have secured that

$$\phi(oa) = -i(ob),$$
$$\phi(ob) = i(oa).$$

Then we have the function of any other diameter [Fig. 100],

$$\begin{aligned}\phi(op) &= \phi(om) + \phi(mp)\\ &= \frac{om}{oa}\cdot\phi(oa) + \frac{mp}{ob}\cdot\phi(ob)\\ &= \iota\left(-\frac{om}{oa}\cdot ob + \frac{mp}{ob}\cdot oa\right)\\ &= -i(nq + on)\ \{\text{see (B) p. 510}\} = -i\,.\,oq.\end{aligned}$$

Hence pure strain of a body is this:—Every line (diameter of the strain ellipse) is changed into one equal in length to the conjugate diameter and perpendicular to it.

An exactly similar proof applies to the hyperbola where we start with the supposition

$$\phi(oa) = i\,.\,ob \text{ and } \phi(ob) = i\,.\,oa,$$

that is we suppose oa is reversed in direction. Then

$$\phi(op) = i\,.\,oq.$$

We may include both cases in the following theorem:

if $$\phi(oa) = \pm i\,.\,ob \text{ and } \phi(ob) = i\,.\,oa,$$

then $$\phi(op) = i\,.\,oq.$$

The upper sign applies to the hyperbola; the lower to the ellipse.

There is an important difference between the two strains. The elliptic strain is the only one which occurs in nature. Lines are not reversed in strain. Hyperbolic strain is however the same as elliptic strain, followed by turning the body round through an angle of continuation. Hyperbolic strain is an ideal case if we are merely representing a strain by the function ϕ.

If however the linear function of a vector represents the displacement of a point at the end of that vector, both the ellipse and the hyperbola can be used. The displacement of a point may be in the positive or negative direction along the vector to that point.

We shall find that if the displacements of points round a fixed point o are all outwards from, or inwards towards that point we must use an ellipse. If some are outwards and some inwards we must use a hyperbola. Of course all we have to do is to examine whether the two principal displacements along oa, ob are outwards or inwards.

Taking, as before, two rectangular vectors oa and ob such that

$$\phi(oa) = i\,.\,ob,\ \phi(ob) = i\,.\,oa,$$

then if e and f are the elongations along the two axes we have

$$\phi\,(oa)=e\,.\,oa,\quad \phi\,(ob)=f\,.\,ob.$$

Hence writing a, b for oa, ob, respectively,

$$ea=b,\quad fb=a.$$

Hence $$\frac{e}{f}=\frac{b^2}{a^2}.$$

The squares of the axes are inversely proportional to the elongations.

If both elongations are of the same sign, the squares and axes are of the same sign; this is the case in the ellipse. If the elongations are of different sign so are the squared axes, and this is the case of a hyperbola.

If we add to the stretching of pure strain, a rotation round some axis, we get the whole effect of a homogeneous strain, that is a strain in which all parallel lines remain parallel though their directions and distances are altered. This turns a circle into an ellipse, in every plane which cuts the strained body. Hence a sphere in the body is changed into an ellipsoid.

The whole effect of the strain being to pull the body out in three directions at right angles to each other and to turn it round; the axes of the ellipsoid are in these three directions; they are the shortest and the longest lines and one at right angles to them.

Any three diameters of the sphere at right angles to each other become three conjugate diameters of the ellipsoid. For parallel planes are to remain parallel and we know that the tangent plane at the end of a diameter of the sphere is parallel to the plane containing the other two diameters at right angles to it. This property must also belong to the corresponding diameters of the ellipsoid. Hence the three diameters perpendicular to each other in the sphere are changed into three diameters of the ellipsoid such that the tangent plane at the end of any one is parallel to the plane containing the other two. And this is the definition of conjugate diameters.

The strain ellipsoid in space gives the linear and vector function ϕ of any vector in a similar way to that in which ϕ is given by the ellipse in a plane.

If op, namely, is the original position of a vector, its new position is represented by the conjugate area, $\phi\,(op)$ is perpendicular to this area and proportional to it in magnitude. Then a vector oq is the strained position of the vector op.

If however we use ϕ to represent the displacement we shall want a surface bearing the same relation to the ellipsoid that the hyperbola bears to the ellipse.

These are got by making the hyperbola and its conjugate hyperbola revolve round the major axis and then squeezing both in one direction perpendicular to this axis. Thus we get hyperboloids of one and two sheets respectively.

(The ellipsoid was got in the same way by making the ellipse rotate round the major axis and then squeezing it in a direction perpendicular to this.)

These surfaces are then got by homogeneous strain from surfaces of revolution; and the plane conjugate to a diameter remains parallel to the tangent plane at the end of it. If the strained body is shortened or lengthened along all the axes we must use the ellipsoid in order to get the function ϕ which represents displacement. If it is shortened along one or two axes and lengthened along the other we must use the hyperboloid. The displacement of any diameter *op* is represented by the conjugate area *oqr* which in the figure [Fig. 101] is an ellipse. Some conjugate sections however will cut the hyperboloid in hyperbolas. What meanings can we give to the conjugate area in this case? We must take the area of the ellipse on the same axes in order to keep the same rule for all possible diameters.

This is what is meant by a linear and vector function of a vector in space. Every vector is turned into its linear function which is a vector represented by the area conjugate to the original one. This is pure strain. To get homogeneous strain generally which includes rotation we must first get the linear vector function and then turn it round.

In the preceding investigation all the Geometry we require is given by the properties of the homogeneous strain itself.

If we take an elastic rod of square section and twist and bend it there is a certain strain at every point, which may be represented by a vector. For instance, if, before the twisting and bending, a body is moving along the rod with uniform velocity then the strain at any point is represented in magnitude and direction by the velocity of this body. The rod tends to untwist and unbend itself. The action at any point which balances this tendency is the stress. There is a bending and also a twisting couple which compounded give another couple and this is represented by a vector perpendicular to its plane. Hence at every point the stress and the strain can be represented by vectors. It is found that the stress is a linear vector function of the strain. It is moreover a pure function. There is no turning round.

As another example we may take an elastic plate bent in two different ways. One kind of bending would at any point change the plane surface into a surface bent like a sphere. The other kind tends to bend the plane like a saddle. These two strains have two corresponding stresses. The plate tends to flatten itself again from both kinds of bending.

Here again the stress is a pure function of the strain.

Another instance is furnished by a solid body in which one point is fixed. Rotating it, there is one axis about which the moment of momentum is a maximum. If ω is the vector representing the spin and the momentum is represented by μ, then

$$\mu = \phi(\omega), \text{ momentum} = \text{a pure function of spin.}$$

This pure function depends on an ellipsoid called the momentum ellipsoid.

We can only give a short sketch of a part of the theory in which little was done by Hamilton, but which has been worked out by Tait; the part that is which treats of the operator

$$\nabla = i\partial_x + j\partial_y + k\partial_z.$$

Suppose a scalar quantity having various values all over a plane. The height of the plane for instance is such a quantity. At any point in a sloping plane or on the side of a hill there is one direction in which the height increases most rapidly. If we draw a line from any point in the direction of this greatest increase and proportional to the rate of increase, this is called the slope of the height.

If z is the height, slope of $z = i\partial_x z + j\partial_y z$.

We must then set off lines representing the rate of increase in two directions at right angles to each other and take their resultant. This will give the magnitude and direction of the greatest increase of height, that is the slope of z. Suppose for example that going northwards [Fig. 102] we rise one foot in 30, and going eastwards, one in fifty, the resultant of the lines $\frac{1}{30}$ and $\frac{1}{50}$ is the slope.

Now take a scalar quantity u having different values at different points of space. The slope of u is then

$$i\partial_x u + j\partial_y u + k\partial_z u.$$

It is the rate of increase in the direction in which u increases fastest and is found in this way: measure off, in three directions at right angles to each other, lengths which represent rates of increase and find their resultant.

Physical examples of slopes.

If u = potential energy, ∇u = force.

If u = velocity potential, ∇u = velocity.

In these cases ∇ operates on a scalar. Tait investigated the effect of ∇ on a vector.

Let a vector σ have the constituents p, q, r. Then

$$\begin{aligned}\nabla\sigma &= (i\partial_x + j\partial_y + k\partial_z)(ip + jq + kr)\\ &= -(\partial_x p + \partial_y q + \partial_z r) + i(\partial_y r - \partial_z q) + j(\partial_z p - \partial_x r) + k(\partial_x q - \partial_y p).\end{aligned}$$

The quantity in the first part is a scalar; the remaining part is a vector.

The two parts have definite relations to the vector σ. This vector may mean the velocity of a fluid at any point.

Then $-S.\nabla\sigma$ represents the expansion, that is, the rate of change of unit volume. In general there is a certain spin at every point of a fluid. $V.\nabla\sigma$ represents twice this spin.

If σ has a different meaning the interpretations given to the two parts of its slope are of course different. Suppose it means the vector called by Faraday the "electrotonic state" or magnetic induction.

Then a law of electricity is that $S.\nabla\sigma$ equals 0. But $-S.\nabla\sigma$ is the expansion in a moving fluid. Maxwell therefore interprets the law thus:—there

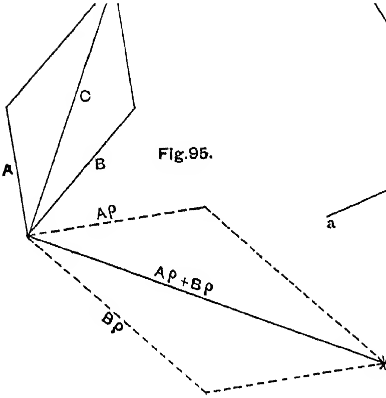


Fig.95.

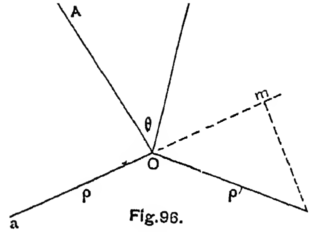


Fig.96.

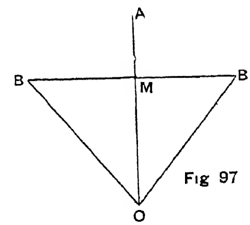


Fig 97

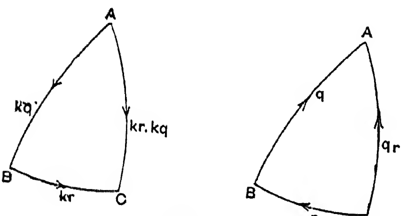


Fig.98.

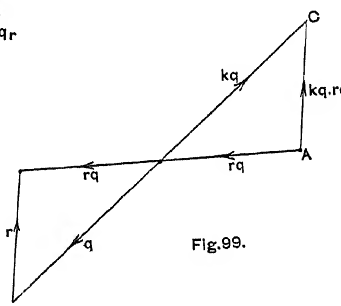


Fig.99.

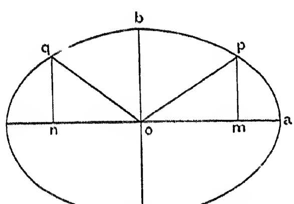


Fig.100.

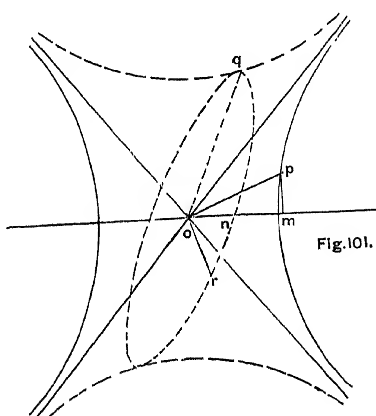


Fig.101.

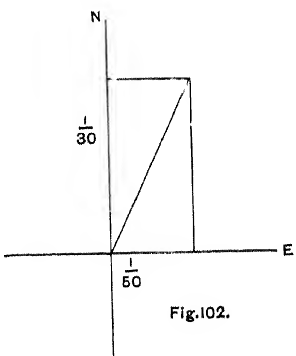


Fig.102.

is no expansion of the ether in electrical motions. But the analogy is not exact, for the electrotonic state σ represents a momentum, not a velocity. $V.\nabla\sigma$ here represents twice the magnetic induction.

The slope of the slope, $\nabla^2\sigma$ has no meaning for a fluid-motion, but for $\sigma=$ the electrotonic state it means electromotive force.

The equation

$$\nabla\sigma=S.\nabla\sigma+V.\nabla\sigma$$

is interpreted for σ any vector,

slope of a vector = convergence + curl,

$\nabla^2\sigma$ = the slope of the slope = the concentration.

Laplace's operator

$$-(\partial_x^2+\partial_y^2+\partial_z^2)=\nabla^2$$

is the square of

$$i\partial_x+j\partial_y+k\partial_z=\nabla.$$

Hamilton laid great stress on his having thus discovered the square root of this operator, which is of importance in many physical investigations. Its properties come out n a remarkable way from he properties of ∇.

SYLLABUS OF LECTURES ON MOTION*.

Division of the Subject.

The science which teaches how to describe motion accurately, and how to compound different motions together, without considering the circumstances under which motions take place, is called *Kinematic* (κινημα, motion).

The simplest kind of motion is that in which a body without changing its size or shape moves so that all straight lines in the body remain parallel to their original positions; this motion is called a *Translation*. As all parts of the body move alike, we may confine our attention to any one of them, however small; for this reason that part of Kinematic which treats of translations is often called the Kinematic of Particles.

A body which does not change its size or shape during the time considered is called a *rigid* body. That part of Kinematic which treats of motions in which there is no change of size or shape is called the Kinematic of Rigid Bodies.

A change of size or shape, considered without reference to change of position, is called a *strain*. The Kinematic of Strains teaches how to describe strains accurately, and how to compound them together. Bodies which change their size or shape are called *elastic;* and the corresponding branch of Kinematic is called the Kinematic of Elastic Bodies.

The science which teaches under what circumstances particular motions take place is called by one or other of two different names according to the view that is taken of it. If it is regarded as mainly based upon the Law of Force, and if its results are expressed in terms of force, it is called *Dynamic* (δυναμις, force); but if it is regarded as mainly based upon the Law of Energy, and if its results are expressed in terms of energy, it is called *Energetic* (ἐνεργεια). In either case it is divided into two parts; *Static*, which treats of those circumstances under which *rest* or *null motion* is possible, and *Kinetic*, which treats of circumstances under which actual motion takes place. Properly speaking, Static is a particular case of Kinetic which it is convenient to consider separately.

We may also make divisions between the Static and Kinetic of particles, rigid bodies, and elastic bodies; but the Static of particles and of rigid bodies

* [This syllabus appears to me to have been drawn up for lectures at University College, Michaelmas Term, 1873.]

is generally treated as one subject, while the Kinematic and Dynamic or Energetic of elastic bodies are grouped together as the science of *Elasticity*.

These divisions may be represented by the following scheme :—

Science of Motion.

$$\left.\begin{array}{l}\text{Kinematic,}\\ \text{Dynamic, or}\\ \text{Energetic, viz.}\end{array}\right\} \left\{\begin{array}{l}\text{Static,}\\ \text{Kinetic,}\end{array}\right\} \text{ of } \left\{\begin{array}{l}\text{Particles (Translations),}\\ \text{Rigid Bodies (Twists),}\\ \text{Elastic Bodies (Strains).}\end{array}\right.$$

Translations.

DEF. If two bodies A and B are in motion, the motion of B is said to be compounded of the motion of B relative to A, and the motion of A.

PROP. Translations represented by the sides of a parallelogram compound together into a translation represented by the diagonal.

DEF. A *vector* is a quantity having magnitude and direction. A translation is a particular kind of vector, and the composition of translations is equivalent to their addition as vectors; it satisfies the law

$$\alpha+\beta=\beta+\alpha.$$

DEF. Uniform rectilinear motion is that in which equal spaces are traversed in equal times.

Its equation is

$$\rho=\alpha+\beta t.$$

PROP. Two uniform rectilinear motions compound into a uniform rectilinear motion.

Harmonic Motion.

DEF. Uniform motion in a circle is that in which equal arcs are traversed in equal times.

DEF. If a point P move uniformly in a circle, and a perpendicular PM be always drawn from it to a fixed diameter AA' of the circle, the foot M of the perpendicular will oscillate to and fro in the diameter; this motion of the point M is called a *Simple Harmonic Motion*.

Its equation is

$$\rho=\alpha\cos(nt-\epsilon).$$

DEF. The radius of the circle is called the *amplitude* of the S. H. M.

DEF. The time which P takes to go once round the circle is called the *period* of the S. H. M.

DEF. The circular measure of the arc described by P from the era of reckoning till it came to the positive end of the diameter AA' is called the *epoch* of the S. H. M.

DEF. The portion of the whole period which has elapsed since the point M last passed through its middle position in the positive direction is called the *phase* of the S. H. M.

PROP. Two S. H. M. of the same period compound into a S. H. M. of that period.

The construction here made use of for compounding two S. H. M. is exemplified in the Tidal Clock of Sir W. Thomson. The clock has two hands whose lengths are proportional to the solar and lunar tides respectively, while their periods of revolution are made equal to the periods of these tides. A jointed parallelogram is constructed, having the hands of the clock for two sides; the height of that extremity of the parallelogram which is furthest from the centre will then be proportional to the height of the compound tide. For this purpose a series of horizontal strings at equal distances are stretched across the face of the clock, and the height is read off by running the eye along these to a vertical scale of feet in the middle.

DEF. The curve described by a point which has a uniform rectilinear motion compounded with a S. H. M. perpendicular to it is called a *harmonic curve*.

The composition of S. H. M. of different periods in the same line may be represented graphically by the super-position of harmonic curves; i.e. by drawing a curve whose height at any point is the sum of their heights.

PROP. Any S. H. M. may be resolved into two in the same line, differing in phase by a quarter period, and one of them having any given epoch.

PROP. S. H. M. on any number of different lines, having the same period and phase, compound into one having that period and phase.

PROP. Two S. H. M. on different lines, having the same period, but differing in phase by $\frac{1}{4}$, compound into harmonic motion in an ellipse (viz. an orthogonal projection of circular motion).

Its equation is

$$\rho = \alpha \cos(nt - \epsilon) + \beta \sin(nt - \epsilon).$$

PROP. Any number of S. H. M. having the same period compound into harmonic motion in an ellipse.

Two harmonic motions in different directions and with different periods produce a resultant which is best studied by wrapping round a cylinder of suitable size paper on which is traced a harmonic curve. The curve thus drawn on the cylinder may then be constructed in wire, and when turned round the axis of the cylinder will represent to an eye at a sufficient distance the curve of compound harmonic motion for varying values of the difference of phase of the simple motions. The simplest case is that in which the circumference of the cylinder is equal to the length of a wave of the harmonic curve; here the periods are equal, and the curve traced on the cylinder is merely an ellipse. The same result is produced by turning the cylinder round its axis while a pencil moves with simple harmonic motion up and down a generating line.

DEF. A motion which exactly repeats itself in the same place after a certain interval of time is called a *periodic* motion.

The resultant of any number of simple harmonic motions whose periods are commensurable is a periodic motion, its period being the least common multiple of their periods.

Fourier's Theorem. Every rectilinear periodic motion of period P may be resolved into a series of simple harmonic motions whose periods are P, $\frac{1}{2}P$, $\frac{1}{3}P$, etc.

Let $\phi(t)$ be the distance of the moving point from a fixed point on the line at a time t, then the periodicity of the motion is expressed by the fact that $\phi(t+P)=\phi(t)$, whatever t is. And the theorem asserts that in this case the quantities a, b can always be found so as to make true the following equation, where

$$\theta=\frac{2\pi t}{P}\ \phi(t)=\tfrac{1}{2}b_0+b_1\cos\theta+b_2\cos 2\theta+\ldots$$

$$+a_1\sin\theta+a_2\sin 2\theta+\ldots$$

The amplitudes and epochs of the several harmonic components may be represented as follows. Let a vertical cylinder revolve about its axis, while a pencil moves up and down one of its generating lines, so as to trace out a curve on the cylinder. If the motion of the pencil is periodic, and has a period equal to that of the cylinder or any exact multiple of it, this curve will return into itself and be a finite curve on the cylinder. Now let the pencil have the given periodic motion which it is required to resolve into harmonic constituents. When the cylinder revolves once in the period P, let the curve described be called C_1; when it revolves twice in that period let the curve be called C_2; when it revolves m times, let this curve be called C_m. And let a circle be drawn on the cylinder whose height is the mean height of the curve C_1; this will be called the mean circle.

If a plane be drawn through the axis of the cylinder, any curve traced on the cylinder may be orthogonally projected on that plane. It is necessary now to define the area between this projection and the line in which the plane is cut by the plane of the mean circle. Let AB be this line [Fig. 101], and let $PMQNP$ be the projection, where PMQ is projected from the near half of the cylinder, and QNP from the further half. Then for the *near* half, the area APM which is *below* AB must be considered negative, and the area MQB which is *above* it, positive. For the *further* half, QNB must be considered negative, and NPA positive. Thus the area is

$$-APM+MBQ-NBQ+APN$$

$$=MPN+MNQ=MPNQ.$$

The same rule is to be applied when the curve cuts itself or the line AB any number of times. Now it is found that for every closed curve traced on a cylinder, there is one plane through the axis such that the area of the projection on it is zero; and that for the plane at right angles to it the area is the greatest possible; while for an intermediate plane the area varies as the sine of the angle which it makes with the zero plane. It is thus possible to draw an ellipse upon the cylinder, the area of whose projection upon any plane whatever through the axis shall be the same as that of a given closed curve. Let the ellipse E_1 have

the same projected area as the curve C_1, E_2 half that of the curve C_2, E_m one-mth that of the curve C_m, and so on. If, while the cylinder revolves once on its axis during the period P, the pencil be made to follow the ellipse E_1, always remaining in the same vertical line, it will have a S. H. M. with the period P. If while the cylinder revolves m times during the period P, the pencil be made to follow the ellipse E_m, it will have a S. H. M. with the period $\frac{1}{m}P$. These motions are the harmonic components of the given periodic motion; and that motion may be reproduced by compounding them all together *.

Parabolic Motion.

PROP. If rectilinear motion in which the space passed over from the beginning is proportional to the square of the time occupied, be compounded with rectilinear motion, the resultant will be motion in a parabola.

Its equation is $$\rho = \alpha + \beta t + \gamma t^2.$$

Velocity.

DEF. If a body is in uniform rectilinear motion, and travels v centimetres in every second, the body is said to have at every instant a *velocity* of v centimetres per second, or simply a velocity v.

DEF. If a body undergo a translation whereby a point of it is carried in any manner by any path from A to B in t seconds, the body is said to have a *mean velocity* $\frac{AB}{t}$ in that interval of t seconds.

The two quantities here defined have magnitude and direction; they are *vectors*. A velocity may be expressed in terms of other units than centimetres per second; in feet or miles per second, leagues per hour, etc.; but when expressed as a number of centimetres per second, it is said to be given in *absolute measure*. In uniform rectilinear motion the mean velocity is the same in any interval whatever, and is equal to the instantaneous velocity at any instant; but the latter is a property which the body possesses *at an epoch* or point of time, while the former is a fact relating to its motion during an interval.

DEF. If any rectilinear motion of a point be compounded with a uniform motion of unit velocity at right angles to it, the curve traced out by the point is called the curve of positions for that rectilinear motion.

Lemma. PT is the tangent at a point P of a circle [Fig. 105]. Any angle being proposed, it is always possible to take a point Q on the circle so near to P that the chord of every arc pq included in PQ shall make with the tangent PT an angle less than the proposed angle.

Let C be the centre of the circle; make PCQ less than the proposed angle, and draw CM perpendicular to pq. Then PCM is the angle which the chord pq makes with PT, and it is always less than PCQ, therefore less than the proposed angle. Q.E.D.

* [Cf. *Dynamic*, p. 37, where it is said a proof of Fourier's Theorem will be given in the Appendix.]

Def. R, P are points on any curve, Q moves from R along the curve towards P; if when any angle is proposed, it is always possible to take Q so near to P that the chord of every arc pq included in PQ shall make with a certain line TP an angle less than the proposed angle; then the curve is said to have TP for a *tangent* at the point P [Fig. 106].

If S is a point on the other side of P and if Q moves from S towards P, there may be another line PT' such that an arc PQ may always be taken in which no chord shall be inclined to PT' so much as by a proposed angle. In this case we may speak of TP as the tangent *up to* P and of PT' as the tangent *on from* P. When TPT' is a straight line, the curve is said to be *elementally straight* or to have the property of *elemental straightness* at the point P; for the more it is magnified, the more will a portion containing P of given length in the magnified figure approach to the straight line TPT' in shape and position. For this, three conditions are necessary; there must be a tangent up to P, a tangent on from P, and these tangents must be in one straight line.

Prop. If the curve of positions of a rectilinear motion has a tangent at a point P, then it is possible to choose an interval ending at the instant corresponding to the point P so that the mean velocity in that interval (and in all intervals included in it) shall differ less than by a given amount from a certain quantity.

Let QP [Fig 107] be a portion of the curve of positions, PT the tangent at P; QN, PM parallel to the (vertical) rectilinear motion considered, and perpendicular to the (horizontal) uniform motion with which it is compounded; QR perpendicular to PM. Since the uniform motion has unit velocity, the number of units of length in NM is equal to the number of seconds in which the body has performed the vertical motion RP, and the mean velocity in the interval NM is therefore $\frac{RP}{NM}$. Now take AB a horizontal line equal to the unit of length, and draw AC, AD parallel to PT, PQ, meeting the vertical line through B in C, D. Then BD represents the mean velocity in the interval NM. Similarly if pq be any arc included in PQ (pm, qn perpendicular to NM), and if we draw Ad parallel to the chord pq, Bd will represent the mean velocity in the interval nm. Now it is possible by hypothesis to choose Q so near to P that the angle QPT, which is equal to CAD, shall be less than any proposed angle; and that the angle which any chord pq makes with PT, which angle is equal to CAd, shall be less than the proposed angle. Therefore it is possible so to choose N that for every interval included in NM the length Cd shall be less than a proposed amount; or so that the mean velocity shall differ from the velocity represented by BC by a quantity less than the proposed amount. Q. E. D. The quantity Bc or $\frac{MP}{TM}$ is then called the *instantaneous velocity* of the rectilinear motion at the instant M.

Def. Let Q, P be successive positions of a moving point, and let BD represent the mean velocity during an interval included in the passage from Q to P; then if it is always possible to find Q so near to P that for all intervals between Q and P the distance DC from D to a fixed point C shall be less than a proposed

length, the point at the instant of arriving at P is said to have an *instantaneou velocity* BC in magnitude and direction [Fig. 108].

PROP. If a moving point has a velocity, the curve described has a tangent in the same direction; and if a length equal to the arc RQ be measured off on a straight line as Q moves, this rectilinear motion will have a velocity whose magnitude is equal to that of Q.

PROP. If each of two motions has a velocity at a certain instant of time, the motion compounded of them has a velocity which is compounded of their velocities by the rule for addition of vectors.

Let AB and AC be the given velocities; complete the parallelogram $ABDC$ [Fig. 109]. Let also AB', AC' be the mean velocities during an interval which ends at the given instant; if the parallelogram $AB'D'C'$ be completed, we know that AD' is the mean velocity of the resultant motion. Now the interval may be so chosen that for it and all shorter ones included in it BB' and CC' are each less than half of any proposed length; and therefore DD', which is their vector-sum, less than the proposed length. Consequently AD is the velocity of the resultant motion at the given instant. Q. E. D.

It is to be noticed that in accordance with our definitions a motion may have one velocity *up to* a certain instant and another velocity *on from* that instant; or, as we may say, an *arrival* and a *departure* velocity. Such motions are for mathematical convenience supposed to take place in the theory of collisions; but it is believed that they do not occur in nature, and that the arrival and departure velocities are always identical. If a point has an arrival and a departure velocity at a given instant and if they are identical, its motion is said to be *elementally uniform;* for if a small portion of the path containing the position of the point at that instant be magnified to a definite length, and the times of traversing different parts of it be preserved in their proportions, then the smaller the portion taken, the nearer will the path approach to a straight line and the motion to uniform motion along it.

PROP. The velocity in the S. H. M.,

$$\rho = a\cos(nt-\epsilon)$$

is

$$\dot{\rho} = -na\sin(nt-\epsilon),$$

(when the position-vector of a point is called ρ, its velocity is denoted by $\dot{\rho}$).

The S. H. M. has a velocity, because its curve of positions has a tangent, being produced by unrolling an ellipse from a cylinder. Now uniform circular motion being compounded of two simple harmonic motions, its velocity is compounded of their velocities by the law of addition of vectors. Thus the velocity of P is compounded of the velocities of M and N; but these velocities are respectively perpendicular to the lines CP, CM, and MP, the vector CP being equal to $CM+MP$ [Fig. 110]. The velocities are therefore proportional to the lengths of these lines, and as the velocity of P is $n\,.\,CP$ along the tangent, the velocities of M and N are $n\,.\,MP$ and $n\,.\,CM$ along MC and CN respectively. But a length $n\,.\,MP = n\,.\,AC\sin PCM$ along MC is equal to $-na\sin(nt-\epsilon)$. Q. E. D.

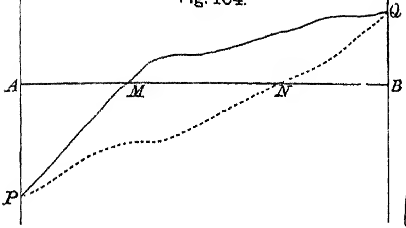


Fig. 104.

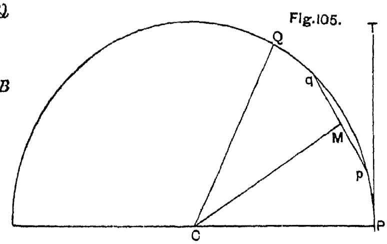


Fig. 105.

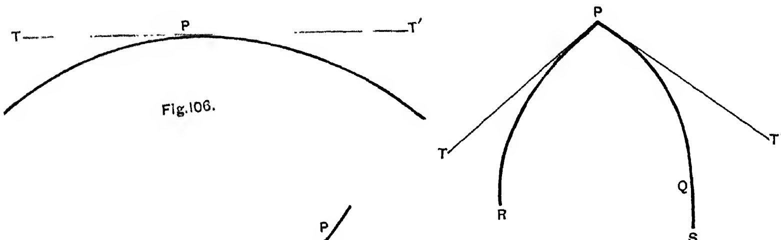


Fig. 106.

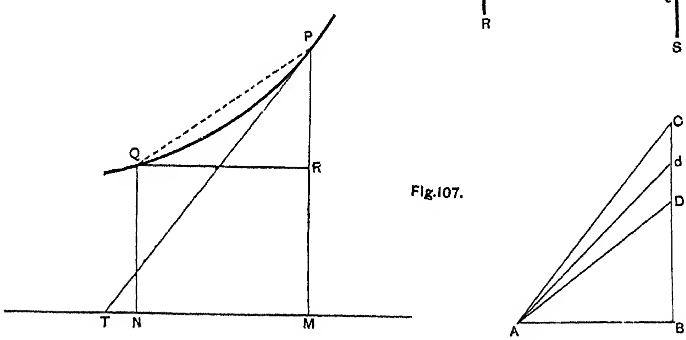


Fig. 107.

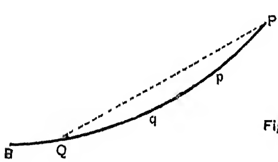


Fig. 108.

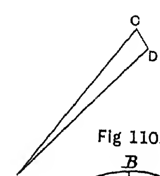


Fig. 109.

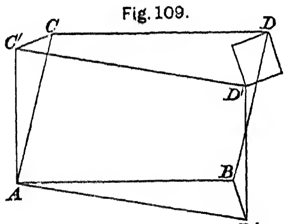
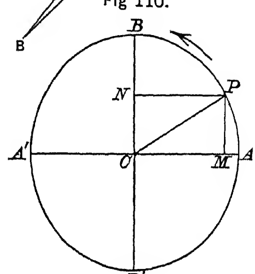


Fig 110.



face p. 522.

PROP. The velocity in the elliptic harmonic motion,

$$\rho = \alpha \cos (nt - \epsilon) + \beta \sin (nt - \epsilon),$$

is

$$\dot{\rho} = -n\alpha \sin (nt - \epsilon) + n\beta \cos (nt - \epsilon)$$

$$= n \left\{ \alpha \cos \left(nt - \epsilon + \frac{\pi}{2} \right) + \beta \sin \left(nt - \epsilon + \frac{\pi}{2} \right) \right\},$$

and is therefore proportional to the conjugate diameter.

PROP. The velocity in the parabolic motion

$$\rho = \alpha + \beta t + \gamma t^2$$

is

$$\dot{\rho} = \beta + 2\gamma t.$$

Let t_1, t_2, t_3, t be successive values of t, these quantities being therefore in ascending order of magnitude; ρ_1, ρ_2, ρ_3, ρ the corresponding values of ρ. Then the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = \beta + \gamma (t_2 + t_3).$$

Since t_2 and t_3 are intermediate between t_1 and t, this vector differs from $\beta + 2\gamma t$ less than $\beta + 2\gamma t_1$ does; that is, less than $2\gamma (t - t_1)$. Now it is possible so to choose t_1 that this shall be shorter than any proposed length γx; that is, it is possible to choose an interval ending at t, so that the mean velocity for every interval included in it differs from $\beta + 2\gamma t$ by less than a proposed amount. The same thing may be shewn for intervals beginning at t. Therefore the motion is elementally uniform and has $\beta + 2\gamma t$ for its velocity. Q. E. D.

PROP. In the motion whose equation is

$$\rho = \alpha t^n$$

(n a positive integer), the velocity is

$$\dot{\rho} = n\alpha t^{n-1}.$$

With the notation of the previous proposition, the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = \alpha \frac{t_3^n - t_2^n}{t_3 - t_2} = \alpha \left(t_3^{n-1} + t_3^{n-2} t_2 + \ldots + t_2^{n-1}\right).$$

Since t_2 and t_3 are intermediate between t_1 and t, this quantity differs from $n\alpha t^{n-1}$ less than $n\alpha t_1^{n-1}$ does; that is, less than $n\alpha (t^{n-1} - t_1^{n-1})$, which by proper choice of t_1 can be made less than an assigned quantity. Whence as before.

LECTURE NOTES*.

1. Geometry is a physical science. It is concerned with the sizes and shapes of objects, and the positions which they may occupy. The doctrines
Riemann. which it teaches on these subjects are derived from experience extended by hypotheses so as to become precise. The hypotheses by which geometrical experience is made precise are *three*.

2. *Hypothesis of Continuity*. When two adjacent portions of space differ in any way, e.g. the space occupied by a body and the space not occupied by it, the boundaries of the two portions appear—so far as we can examine—to be identical. By the hypothesis of continuity we assume that they really *are* identical, or that the surface is surface to both these portions and takes up absolutely no room. This definition is from *Arist.* Cat. 6. On continuity see *Boscovitch*, de Continuitatis lege.

Similarly when two adjacent portions of a surface are different, the boundary is by this hypothesis assumed to be common to both; it is called a line, and takes up absolutely no surface-room. And when two adjacent portions of a line are different, their common boundary, taking up no room of any kind upon them, is called a point. From this definition it follows that there is an infinite number of points between two points on a line, and an infinite number of lines between two lines on a surface, and an infinite number of surfaces between two surfaces in space.

3. *Hypothesis of Rigid Motion*. When we move bodies about they do not in general sensibly alter in size or shape. By this second hypothesis we assume that the appearance is accurate, and that a body may move from one position to another without undergoing any the very least alteration in size or shape. Or, that exactly the *same* geometrical relations may exist in two different portions of space.

Leibnitz. 4. *Hypothesis of Infinite Extent*. A surface which is of the same shape all over and of the same shape on both sides is called a plane. A line in a plane which is of the same shape all along and of the same shape on both sides is called straight. If a point travel along a straight line there appears to be no reason for supposing that it would ever come back to the same position from the other direction. The third hypothesis assumes that a straight line is actually of infinite extent in both directions, and that the point might travel on for ever

* [I am indebted to Mr A. B Kempe for the information that these *Notes* were given to students attending a course of lectures at Trinity College, Cambridge, in the year 1870. The page containing Articles 19, 20, 21, is missing from all the copies I have met with.]

without revisiting any of its former positions. According to this a plane and space itself are infinitely extended in every direction.

5. *Perpendicular and Parallel.* From the first two hypotheses it may be proved that straight lines can only intersect in two points. Those straight lines which meet therefore divide space into four regions, each of these is called an angle. If these are all of the same shape the lines are called perpendicular. By using now the third hypothesis we may shew that two lines meet in only one point. Two lines making equal angles with the same line can then not meet at all. They are called parallel. It appears probable from experience that only one line can be drawn through a fixed point parallel on given line. We may assume either that lines meet only in one point, and that this probable experience is accurately true, or we may assume the third hypothesis; either assumption involves the other.

6. *Quantity and Measuring.* A discrete assemblage of things, as of chairs, say, is estimated by counting the number of them. It is assumed that the number is the same in whatever order they are counted. From this assumption flow the theorems of addition and multiplication $a+b=b+a$, $ab=ba$, $a(b\pm c)=ab\pm ac$. We cannot however count the number of points in a piece of line. Yet we suppose it to have a certain magnitude; we speak of another piece as greater, or less, and greater or less in a certain degree. The degree in which one piece is greater than another is called the *ratio* of the two pieces. The ratio of $A+B$ to C is called the sum of the ratios of A to C and B to C. The ratio of A to C is called the product of the ratios of A to B and B to C. It is assumed that two things may each be broken up into any number of parts and the parts rearranged without altering their ratio. From this assumption flow as with numbers the theorems of addition and multiplication $a+b=b+a$, $ab=ba$, $a(b\pm c)=ab\pm ac$.

7. *Calculus of Ratios.* Every quantity is therefore measured by the ratio which it bears to some fixed quantity, called the unit. But between any two ratios is an infinite number of ratios; it is therefore impossible to tabulate all ratios, or to give them names. A ratio then can only be described approximately, as being very near to the ratio of two numbers, that is, of two quantities which have a common measure. On the assumption that two ratios which are always greater or less than the *same* numerical fractions are equal, it may be shewn, as in Euclid, that similar triangles have proportional sides.

8. *Analysis of Position on a Straight line.* On a straight line take a fixed point o, and a fixed length oa in a given direction from o; then the position of a point p on the line is known if we know the ratio of op to oa and the side of o on which p is, and *vice versa*. Let μ denote the ratio, then $op=\mu\,.\,oa$. μ may be regarded as a direction to perform the following operation: change the value of oa in the ratio of 1 to μ. The equation asserts that by performing this upon oa we attain the value of op (op and oa may be regarded as quantities of motion). Subtraction will mean motion in the contrary direction. Hence $-$ may be regarded as an abbreviation for $-+$ or *reversed addition*.

9. *Vectors. Ratios of Vectors.* The quantity of motion which carries a point from the position a to the position b is called the *vector ab*. The vector ab is said to be *equal* to the vector cd when b is on the same side of a that d is of c,

and at the same distance from it. *Addition* of vectors is then defined by the equation $ab+bc=ac$, or $ab+bc+ca=0$. The *ratio* of two vectors is that operation which changes the second into the first. The operation consists of two parts; a *tensor* or stretching part which merely alters the length of the vector or the quantity of motion, and a *versor* or turning part, which either *preserves* the direction of motion or *reverses* it. Let μ be the ratio of the quantities of motion in the vectors ab and cd; then if ab is in the same direction as cd we shall have $\frac{ab}{cd}=+\mu$; but if they are in different directions $\frac{ab}{cd}=-\mu$. Ratios of vectors are added and compounded or multiplied by the same rules as the ratios of magnitude (6). In the ordinary language of algebra ratios of quantities are called *signless numbers*, while ratios of vectors are called *numbers with signs to them.* The theorems $\alpha+\beta=\beta+\alpha$, $\alpha\beta=\beta\alpha$, $\alpha(\beta\pm\gamma)=\alpha\beta\pm\alpha\gamma$, are still true when $\alpha\beta\gamma$ are numbers with signs to them.

Analysis of Position on a Plane.

10. *Gauss's Plane of Numbers.* By the operation -1 a vector has its direction *suddenly* changed into the opposite one. We may however conceive the same result brought about by the continuous rotation of the vector in a plane through two right angles. The operation may then be *halved;* that is to say, the vector may be turned through a right angle. This operation of turning a vector through a right angle is denoted by the letter i. Thus if aa', bb' are two equal lines bisecting each other at right angles at the point o, $oa'=-oa$, and $ob'=-ob$; moreover $ob=i\,.\,oa$, $oa'=i\,.\,ob$, $ob'=i\,.\,oa'=-i\,.\,oa$, and $oa=i\,.\,ob'$. From this definition it appears that $i^2=-1$.

11. *Vectors in a Plane. Complex Numbers.* The *Equality* of vectors in a plane is thus defined: $ab=cd$ means that the line ab is parallel to cd and in the same direction, and that the length ab is equal to the length cd. Addition of vectors is then defined by the equation $ab+bc=ac$, or, as before, $ab+bc+ca=0$. The *ratio* of two vectors is that operation which changes the second into the first. But when we have defined the addition and composition of ratios as in (6) we may shew that the ratio of any two vectors is the sum of two ratios, one of which is a *signed number* (9) and the other is the product of i by a signed number. For let ab and ac be two vectors; then draw bm perpendicular to ac and ad parallel to bm so that in length $ad=ac$. Then

$$\frac{ab}{ac}=\frac{am+mb}{ac}=\frac{am}{ac}+i\,.\,\frac{mb}{ad}\ (\text{since } ad=i\,.\,ac).$$

Now $\frac{am}{ac}$ and $\frac{mb}{ad}$ are both signed numbers, or ratios of vectors on the same straight line; denote them by x and y. Then we have proved that the ratio of any two vectors on a plane is of the form $x+iy$, where x and y are signed numbers. The expression $x+iy$ is called a *Complex Number*, and may be denoted by a single letter z.

12. *Modulus and Argument.* The ratio of the lengths ab and ac (a signless number, 9) is called the *modulus* of the complex number which is the ratio of

these vectors The angle bac is called the *argument* of the same complex number. Thus if r, ϕ are the modulus and argument of z,

we have
$$z=x+iy=r(\cos\phi+i\sin\phi),$$
$$r^2=x^2+y^2, \quad \tan\phi=\frac{y}{x}.$$

13. *Addition.* To add together two complex numbers, we must add separately the parts which are not multiplied by i and the parts which are so multiplied. Thus $Oa=Om+ma$, and

$$ab=ap+pb;\ \text{now}\ Oa+ab=Ob=On+nb=\overline{Om+ap}+\overline{ma+pb}.$$

Hence, on substituting for these vectors their ratios to the fixed vector OI, we obtain the rule enunciated. From the fact that the opposite sides of parallelograms are equivalent vectors flows the theorem

$$Oa+ab=Oc+cb=ab+Oa, \text{ or } z+w=w+z.$$

14. *Multiplication.* To compound the ratios of OI to Oa, and of OI to Ob is to find a vector which bears the same relation to Ob that Oa does to OI. Let Oc be such a vector, then length $Oc=\dfrac{Oa\,.\,Ob}{OI}$, and angle $I\hat{O}c=I\hat{O}a+I\hat{O}b$. Hence to multiply two complex numbers, multiply their moduli and add their arguments. This rule shews that $zw=wz$. By altering the triangle Oab in a certain ratio (mod. z) and turning it through a certain angle (arg. z) we may shew that $z(u\pm v)=zu\pm zv$. $\left(u=\dfrac{Oa}{OI},\ v=\dfrac{ab}{OI}\right)$.

15. *Expansion of* F (x+y). By means of the three laws of addition and multiplication which we have now proved true for complex numbers we may New
shew that the Binomial Theorem with a positive integral exponent is true for these numbers; that is, that $\dfrac{(x+y)^n}{\underline{|n}}=\sum\dfrac{x^a}{\underline{|a}}\cdot\dfrac{y^b}{\underline{|b}}$, where a, b take all values consistent with the equation $a+b=n$. Now if $F(x)=ax^n+bx^{n-1}+\ldots+kx+l$, by applying this theorem to each term we may shew that

$$F(x+y)=F(x)+yF'(x)+\frac{y^2}{\underline{|2}}\,.\,F''(x)+\ldots$$ Tayl

where $F'(x)$, $F''(x)$ .. are rational integral functions of x; we shall return to consider the method of deriving these from $F(x)$.

16. *Transformation* z = F (x). *Similarity of smallest parts.* Let the complex number x be as before the ratio of the vector Ox to the vector OI in a certain plane, and let z be the ratio of the vector $O'z$ to the vector $O'I'$ in *another* plane. Then by the equation $z=F(a)$ (a rational integral function) a is determined as soon as x is known. Let x receive a small change and become $x'=x+y$, and let z consequently become $z'=z+v$. Then by (15) $u=y\,.\,F'(x)+$ terms containing y^2. Suppose y so small that it may be neglected in comparison with 1, and therefore y^2 may be neglected in comparison with y. In that case u is got from y by increasing it in the ratio mod. $F'(x):1$ and turning it through the angle arg. $F'(x)$.

Gauss. Thus the direction of zz' makes a constant angle with the direction of xx', and if we take three points $xx'x''$ very near to each other, the triangle formed by their corresponding points $zz'z''$ will be similar to $xx'x''$. So if any picture be drawn in the first plane, while x describes the lines of this picture z will describe the lines of a distorted copy; but the two pictures will be *similar in their smallest parts*, and any two lines in the one will cut at the same angle as the corresponding lines in the other.

Argand. 17. *The Equation* F (x) = 0 *has* n *roots.* By properly choosing x I say that we can make z come to the origin. For if not, there is some position of z which is the *nearest* to the origin that it can possibly have. Consider this position, and the corresponding position of x; and let λ be the corresponding value of $F'(x)$. If x move in any direction, z moves in a direction making an angle arg. λ with this; if therefore x move in a direction making an angle arg. λ with the line joining z to the origin, z will move straight towards the origin, contrary to the supposition that it could not get any nearer. Hence there is some value of x for which $F(x)=0$. Let α be this value, then we know that $F(x)=(x-\alpha)F_1(x)$. But now as before $F_1(x)$ must have a root, say β; then $F(x)=(x-\alpha)(x-\beta)F_2(x)$, and so on. Thus finally $F(x)$ has as many roots as dimensions.

Cauchy. 18. *Number of Roots in a given Closed Contour.* The argument of $F(x)$ is then the sum of the arguments of $\overline{x-\alpha}$, $\overline{x-\beta}$, ... Now if x describe any closed contour containing no roots, these arguments may increase or decrease but will ultimately resume their original value. But if α is within the contour the argument of $x-\alpha$ will increase by 2π. Hence the number of roots of $F(x)$ within the contour is the number of (2π)s added to its argument, when

[*a page is missing.*]

22. *Position on a Plane. Cartesian formulæ.* In the method of Descartes the position of a point is defined by its distances from two fixed straight lines X and Y called the *axes*; the distance from each axis is measured parallel to the other, as pn, pm in Fig. [111]. The signed numbers $\frac{np}{OI}$, $\frac{mp}{OI}$, are denoted by x and y, and called the *coordinates* of the point. The position of a straight line is defined by its distance p from the origin, measured perpendicular to the line, and by the angles α, β which this perpendicular makes with Y and X respectively. These are connected by the equation $\alpha+\beta=\omega$ (the angle between X and Y) which may also be written

$$\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\omega=\sin^2\omega.$$

23. *Expressions for the simplest geometric magnitudes.*

(1) Distance of points $(x_1\,y_1)$ $(x_2\,y_2)$

$$\delta^2=(x_1-x_2)^2+(y_1-y_2)^2+2(x_1-x_2)(y_1-y_2)\cos\omega.$$

(2) Area of triangle (x_1y_1) (x_2y_2) (x_3y_3)

$$\frac{2\Delta}{\sin\omega}=x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(3) Perpendicular distance of point (x, y) from line (p, α, β)

$$= +p - x\cos\alpha - y\cos\beta.$$

(4) Angle between the lines $(p\,\alpha\,\beta)$ $(p'\alpha'\beta')$; $\theta = \alpha - \alpha' = \beta' - \beta$, or

$$\cos\theta\sin^2\omega = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' - (\cos\alpha\cos\beta' + \cos\alpha'\cos\beta)\cos\omega$$

$$\sin\theta\sin^2\omega = \cos\alpha\cos\beta' - \cos\alpha'\cos\beta.$$

24. *Equations.* If we know that $x\cos\alpha + y\cos\beta - p = 0$, we know that the point (x, y) is on the line $(p\alpha\beta)$, thus all the values of x, y which satisfy this equation represent all the points on that line. The equation itself may then be said to represent all the points on the line; or, (less accurately) to represent the line. In general, the condition that (x, y) may be on a known curve is called the *equation of the curve.*

$$\left(\text{In the following } \omega \text{ is taken } = \frac{\pi}{2}\right).$$

If a point is on a

Circle, its distance from a fixed point (the centre) is constant,

say distance of (x, y) from (a, b) is equal to r,

$$\therefore\ (x-a)^2 + (y-b)^2 = r^2.$$

Parabola, its distance from a fixed point (focus) = dist. from fixed line (directrix),

say distance of (x, y) from (a, o) = dist. from $\left(a, \pi, -\frac{\pi}{2}\right)$,

$$\therefore\ (x-a)^2 + y^2 = (a+x)^2, \qquad \text{or } y^2 = 4ax.$$

Ellipse / *Hyperbola* } its distance from a fixed point $= e$ times dist. from fixed line, say dist. of (x, y) from $(ae, o) = e$ times dist. from $\left(\frac{a}{e}, o, \frac{\pi}{2}\right)$,

$$\therefore\ (x-ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2, \text{ or } \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

For the ellipse e is less than 1, and we write $a^2(1-e^2) = b^2$,

,, hyperbola e is greater than 1, ,, ,, $a^2(e^2-1) = b^2$.

25. *Reduction of Equation of the First Order.* The equation $lx + my + n = 0$ will be reduced to the form $x\cos\alpha + y\cos\beta - p = 0$ if we multiply it by a quantity R, provided that $R^2(l^2 + m^2 - 2lm\cos\omega) = \sin^2\omega$. From the value of R thus indicated we derive the formula,

$$\cos\alpha = \frac{l\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}},\ \cos\beta = \frac{m\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}},\ p = \frac{-n\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}}$$

whence by substitution

perpendicular from $x'y'$ on $lx + my + n = 0$ is $\dfrac{lx' + my' + n'}{\sqrt{(l^2+m^2-2lm\cos\omega)}}\sin\omega$;

angle between lines $lx + my + n = 0$, $l'x + m'y + n' = 0$,

$$\cos\theta = \frac{ll' + mm' - (lm' + l'm)\cos\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}\sqrt{(l'^2+m'^2-2l'm'\cos\omega)}}$$

$$\sin\theta = \frac{(lm' - l'm)\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}\sqrt{(l'^2+m'^2-2l'm'\cos\omega)}}.$$

26. *Grassmann Notation for points and lines.* A point is denoted by a single small letter, and a line by a single large letter. The symbol aB or Ba is taken to mean the result of substituting the coordinates of the point a in the equation of the line B. The symbol ab stands for the line joining the points a, b, and AB for the point of intersection of the lines A, B. Accordingly abc stands for the determinant formed with the coordinates of the points a, b, c, and ABC for the determinant formed with the coefficients of the lines A, B, C. $abc=0$ means that the three points are in a line, and $ABC=0$ means that the three lines meet in a point. Let a mean the point (x_1y_1), b the point (x_2y_2) and c the point (x_3y_3); also let A be the line $lx+my+n=0$, B the line $l'x+m'y+n'=0$, and C the line $l''x+m''y+n''=0$. Then $aB=Ba=l'x_1+m'y_1+n'$.

$$abc=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad ABC=\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \quad \begin{matrix}\text{Equation of} \\ ab \text{ is}\end{matrix} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

Coordinates of AB are given by

$$\frac{x}{mn'-m'n}=\frac{y}{nl'-n'l}=\frac{z}{lm'-l'm}.$$

[I have not met with any further *Notes*.]

ANALYSIS OF LOBATSCHEWSKY.

*Propositions common to the two theories**.

1. A straight line fits itself in all its positions: i. e. if we turn the surface containing it about two points of the line, the line does not move.

2. Two straight lines cannot cut in two points.

3. A straight line may be produced indefinitely.

4. Two lines perpendicular to the same line and in the same plane cannot meet each other.

5. One straight line must cut another if it has points on both sides of it.

6. Angles (plane or dihedral) vertically opposite are equal.

7. Straight lines making equal angles with a third straight line cannot meet.

8. In a triangle equal angles are opposed by equal sides, and conversely.

9. Greater angle opposed to greater side. In right-angled triangle, hypothenuse is greater than either of the other sides, and the two angles next it are acute.

10. Equalities of triangles.

11. A line perpendicular to two other lines not in the same plane with it is perpendicular to every other line in their plane.

12. The intersection of a sphere and a plane is a circle.

13. A line perpendicular to the intersection of two perpendicular planes, lying in one of them, is perpendicular to the other.

14. In a spherical triangle equal sides are opposed by equal angles, and conversely.

15. Two spherical triangles are equal when they have equal sides containing equal angles, or equal angles adjacent to equal sides.

* [*i.e.* to Euclid's and Lobatschewsky's.]

THE POLAR THEORY OF CUBICS.

1. Consider a net of conics B_2 (viz., $B_2^{(1)}\, B_2^{(2)}$.). All the conics b_2 which are harmonic of all these form a tangential net. The locus of points x whose polars in respect of the entire net B_2 meet in a point is a curve of the third degree H_3. So also the envelope of lines X whose poles in respect of the entire net b_2 are in a line is a curve of the third class h_3. These curves are defined by the equations

$$(xB_2 \,.\, xB_2' \,.\, xB_2'') = x^3 H_3,$$

$$(Xb_2 \,.\, Xb_2' \,.\, Xb_2'') = X^3 h_3.$$

Let y be the point of intersection of the polars of x, and Y the line on which the poles of X lie. Then y is clearly a point on H_3, and Y is a tangent to h_3. x, y are called corresponding points of H_3, and X, Y are corresponding tangents of h_3.

2. The points x, y and the lines X, Y satisfy the equations

$$xyB_2 = 0, \qquad XYb_2 = 0,$$

where B_2, b_2 are any conics of their respective nets. These shew that the point-pair xy belongs to the net b_2 and that the line-pair XY belongs to the net B_2; for each net includes all conics that are harmonic of all conics of the other. Again, since the whole net B_2 is harmonic of the points xy, and since *one* conic of the net can be drawn through any two points, there must be one conic containing the line xy as half of it; and this line must consequently be a tangent to h_3. So likewise the point XY must be on H_3. We have then these theorems:—

If a conic b_2 *breaks up into two points* x, y, *these are points on* H_3, *and the line* xy *is a tangent to* h_3.

If a conic B_2 *breaks up into two lines* XY, *these are tangents to* h_3 *and the point* XY *is on* H_3.

3. Let a be a point subtended in involution by $b_2\; b_2'\; b_2''$. Then $\lambda\mu\nu$. can be so chosen that

$$a\,(\lambda b_2 + \mu b_2' + \nu b_2'') \equiv 0,$$

which means that a is one of the two points into which $\lambda b_2 + \mu b_2' + \nu b_2''$ breaks up; or that a is a point on H_3. Thus *all points on* H_3 *are subtended in involution by the net* b_2.

In a similar way we should prove that *all tangents to* h_3 *are cut in involution by the net* B_2. Or *we may define* H_3 *as the locus of points subtended in involution by the* b_2, *and* h_3 *as the envelope of lines cut in involution by the* B_2.

4. Let x be the intersection of XY, and y its corresponding point. Let also $x'y'$ be a pair of corresponding points very near to xy. Then $x'y'$ is a point-pair belonging to the net b_2, and XY is a line-pair belonging to the net B_2. But every conic of the net b_2 is harmonic of every conic of the net B_2. Therefore xx', X, xy', Y is a harmonic pencil. Now xx' is the tangent to H_3 at x, and xy' is the same as xy. Hence

The three tangents to h_3 *from a point* x *on* H_3, *together with the tangent to* H_3 *at* x, *form a harmonic pencil.*

The three points in which H_3 *cuts a tangent* X *to* h_3, *together with the point of contact of* X, *form a harmonic range.*

5. There are three cubics C_3 C_3' C_3'' which have H_3 for their hessian, and three curves of the 3rd class c_3 c_3' c_3'' which have h_3 for their hessian. These correspond in pairs, so that we have

$$C_3c_3=0,\ C_3'c_3'=0,\ C_3''c_3''=0.$$

I attend only to the pair C_3 c_3.

We have now also

$$C_3h_3=24S,\ c_3H_3=24s,\ H_3h_3=T=t,$$

where S, T are the fundamental invariants of C_3, and s, t of c_3.

The conics B_2 are the first polars of C_3, and the b_2 are the first polars of c_3. In fact, x being any point and X any line,

$$xC_3=B_2,\ B_2c_3\equiv 0,\ \ B_2h_3=x,$$

$$Xc_3=b_2,\ \ b_2C_3\equiv 0,\ \ b_2H_3=X.$$

Suppose B_2 and b_2 to break up; then we have the theorems:

The mixed polar in regard to h_3 *of two of its corresponding tangents is a point on* H_3.

The mixed polar in regard to H_3 *of two of its corresponding points is a tangent to* h_3. (Cayley.)

We may here take the mixed polars in either case with regard to any syzygetic cubic.

The equations $B_2h_3=x$, &c., are virtually given by Salmon. *Conics*, 5th ed., p. 349.

6. The condition that xC_3 shall touch X is of the second order in x, X and the coefficients of C_3. Denote it by $(\overline{xC_3}^2,\ X^2)$, then if $X=yz$, we have

$$(\overline{xC_3}^2,\ X^2)=xy^2C_3\,.\,xz^2C_3-\overline{xyzC_3}|^2.$$

Regard X as fixed; then x describes a conic, the (second) polar envelope of X, which

(a) is the locus of points whose first polars touch X,

Cayley. (b) is the envelope of second polars of points on X,

Steiner. Cremona. (c) is the locus of poles of X in regard to first polars of its points,

Cayley. (d) breaks up when X touches h_3.

Salmon, l. c. Let $\widehat{xC_3}^2$ denote the tangential form of the first polar of x; then $\widehat{xC_3}^2 H_3 = x^2C_3 . S$; or the first polar in regard to the cubic of a point on the cubic is harmonic of the first polar of that point in regard to the hessian.

The condition that X shall pass through an intersection of xC_3, x^2C_3 (that is, through a double conjugate of x) is, of course,

$$xC_3(x^2C_3)^2X^2=0.$$

X being fixed, this is a quintic curve whose equation may also be written

$$E_2C_3+\lambda L^2H_3=0,$$

where E_2 is the second polar envelope of X as above. This equation indicates the properties of the curve.

7. Operating with the mixed concomitant of (6') [?] on h_3 we obtain a new contravariant q_5 of the fifth order in the coefficients of C_3, which is thus seen to be the envelope of a line whose polar envelope is harmonic of its first polar in respect of h_3.

8. Two point-cubics 1 2 3 and 1′2′3′ are said to be harmonic of one another when

$$11'.22'.33'+12'.23'.31'+13'.21'.32'+13'.22'.31'$$
$$+12'.21'.33'+11'.23'.32'=0,$$

or when $$\Sigma 11'.22'.33'=0.$$

If 1 2 3 is harmonic of 1′2′3′ and 1 coincides with 1′, it must have the same harmonic in regard to 2 3, 2′3′.

If three of the six points coincide, a fourth coincides with them.

Every point-cubic is harmonic of itself.

The envelope of a line cut harmonically by two cubic curves is of the third class. This envelope is the same for any two cubics through the nine points of intersection.

Two syzygetic cubics cut every line harmonically. To a *net* of cubics will thus correspond a net of curves of the third class, one for every pencil belonging to the net.

The condition that X may be cut harmonically by C_3 and $xC_3 . x^2C_3$ is the vanishing of the mixed concomitant $a_x c_x^2(abu)^2(bcu)$.

If ϵ_3 is the harmonic envelope of the pencil $\lambda C_3+\mu D_3$, the tangents to ϵ_3 from a point of intersection are the three flexes at that point.

ON PFAFFIANS.

I.

Consider $2n$ alternate units $\alpha_1\alpha_2\ldots\alpha_{2n}$, and form a linear function of their binary products, viz.

$$\Sigma p_{hk}\alpha_h\alpha_k \qquad [h,\ k=1,\ 2\ \ldots 2n.$$

Since to every term in this $p_{hk}\alpha_h\alpha_k$ there corresponds a term $p_{kh}\alpha_k\alpha_h$, the complete coefficient of $\alpha_h\alpha_k$ is $p_{hk}-p_{kh}$, and the whole expression involves only these differences; hence it is convenient to make $p_{kh}=-p_{hk}$, and then

$$\Sigma p_{hk}\alpha_h\alpha_k = 2\Sigma' p_{hk}\alpha_h\alpha_k,$$

where the Σ' signifies that the summation is to be extended to those values only for which $k>h$.

The n^{th} power of $\Sigma' p_{hk}\alpha_h\alpha_k$ is $\Pi\alpha=\alpha_1\alpha_2\ .\ \alpha_n$ multiplied by a numerical constant and a function of the p which is called a *Pfaffian of the* n[th] *order;* viz. it is

$$\frac{1}{N}\Sigma \pm p_{12}p_{34}\ldots p_{2n-1,\,2n},$$

where the suffixes are to be permuted in all possible ways, and signs prefixed according to the rule for determinants. Each term occurs $N=\Pi n$ times and accordingly the result is divided by this number. The equation may then be written

$$(\Sigma' p_{hk}\alpha_h\alpha_k)^n = \Pi_n\,.\,P_n\,.\,(\alpha)^n.$$

The Pfaffian of the first order is p_{12}, or generally any constant. For the second order we have

$$(p_{12}\alpha_1\alpha_2+p_{23}\alpha_2\alpha_3+p_{13}\alpha_1\alpha_3+p_{14}\alpha_1\alpha_4+p_{24}\alpha_2\alpha_4+p_{34}\alpha_3\alpha_4)^2$$
$$=2\,(p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23})\,\alpha_1\alpha_2\alpha_3\alpha_4,$$

so that
$$P_2=p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}.$$

It is sometimes convenient to denote this by P_{1234}, and so for higher orders.

Observe that these P are the concomitants of a linear complex in n dimensions.

This being so, we have for any n units $\alpha_1\alpha_2 \,.\, \alpha_n$

$$(\Sigma' p_{hk}\alpha_h\alpha_k)^2 = 2\Sigma P_{hklm}\alpha_h\alpha_k\alpha_l\alpha_m,$$

$$(\Sigma' p_{hk}\alpha_h\alpha_k)^3 = 6\Sigma P_{hklmnr}\alpha_h\alpha_k\alpha_l\alpha_m\alpha_n\alpha_r,$$

and generally

$$(\Sigma' p_{hk}\alpha_h\alpha_k)^m = \Pi_m \,.\, \Sigma P_m\,(\alpha)^m. \qquad [2m < n.$$

We may also form products such as $(\Sigma' p_{hk}\alpha_h\alpha_k)(\Sigma' q_{hk}\alpha_h\alpha_k)\ldots$; these give rise to what may be called mixed Pfaffians, the relation of which to the preceding may be easily recognized.

II.

Let there be two sets of n variables $\alpha_1\alpha_2\ldots\alpha_n$ and $\beta_1\beta_2\ldots\beta_n$. If we form the product of the n linear functions,

$$\varpi_1 = \Sigma p_{h1}\alpha_h,\ \varpi_2 = \Sigma p_{h2}\alpha_h,\ \ldots \varpi_n = \Sigma p_{hn}\alpha_h,$$

it is known that $\Pi\varpi = |p| \,.\, \Pi\alpha$. Hence if we now form the lineo-linear function $\Sigma\varpi_h\beta_h = \Sigma p_{hk}\alpha_h\beta_k$, we shall find

$$(\Sigma p_{hk}\alpha_h\beta_k)^n = (\Sigma\varpi_h\beta_h)^n = (-)^{\frac{1}{2}n(n-1)}\,\Pi n \,.\, |p| \,.\, \Pi\alpha \,.\, \Pi\beta,$$

that is, the nth power of a lineo-linear function of two sets of n units is the product of the units into a numerical multiple of the determinant formed by the coefficients.

Now suppose that $p_{hk} = -p_{kh}$, $p_{hh} = 0$; then

$$\Sigma p_{hk}\alpha_h\beta_k = \Sigma' p_{hk}(\alpha_h\beta_k - \alpha_k\beta_h) \qquad [h < k \text{ under } \Sigma';$$

or we have a linear function of the binary determinants formed with the α and the β. Under these circumstances the determinant $|p|$ is skew symmetrical; or *the* n*th power of a linear function of the binary determinants of the* α, β *is* $\Pi n \,.\, \Pi\alpha \,.\, \Pi\beta \times$ *by a skew symmetrical determinant.*

These binary determinants are symmetrical in regard to the α and the β; for $\alpha_h\beta_k - \alpha_k\beta_h = \alpha_h\beta_k + \beta_h\alpha_k = (h, k)$ suppose. We may easily prove the following theorems, viz.,

$$(h, k)^2 = -2\alpha_h\alpha_k\beta_h\beta_k,$$

$$(h, k)\,(h, l) = +\alpha_h\beta_h\,(\alpha_k\beta_l + \alpha_l\beta_k).$$

Hence with $n = 3$ we have

$$-\tfrac{1}{2}\{p_{12}(1, 2) + p_{13}(1, 3) + p_{23}(2, 3)\}^2 = (p_{12}\alpha_1\alpha_2 + p_{13}\alpha_1\alpha_3 + p_{23}\alpha_2\alpha_3)$$
$$\times\,(p_{12}\beta_1\beta_2 + p_{13}\beta_1\beta_3 + p_{23}\beta_2\beta_3).$$

But with $n \gtreqless 4$,

$$-\tfrac{1}{2}\{\Sigma' p_{hk}(h,\ k)\}^2 = (\Sigma' p_{hk}\alpha_h\alpha_k)\cdot(\Sigma' p_{hk}\beta_h\beta_k) - 2\Sigma P_{hklm}(\alpha_h\alpha_k\beta_l\beta_m + \ldots 6 \text{ terms}).$$

Here the coefficient of $2P_{hklm}$ is worthy of attention. Let us write

$$\left|\begin{matrix}\alpha & \alpha & \beta & \beta \\ h & k & l & m\end{matrix}\right| = (h,\ k,\ l,\ m)$$

$$= \alpha_h\alpha_k\beta_l\beta_m + \alpha_l\alpha_m\beta_h\beta_k + \alpha_h\alpha_m\beta_k\beta_l + \alpha_k\alpha_l\beta_h\beta_m + \alpha_h\alpha_l\beta_m\beta_k + \alpha_m\alpha_k\beta_h\beta_l,$$

then

$$(h,\ k,\ l,\ m)^2 = 6\alpha_h\alpha_k\alpha_l\alpha_m\beta_h\beta_k\beta_l\beta_m$$

$$(h,\ k,\ l,\ m)\ (h,\ k,\ l,\ n) = \alpha_h\alpha_k\alpha_l\beta_h\beta_k\beta_l(\alpha_m\beta_n + \alpha_n\beta_m) \qquad \{\text{say } [m,\ n]$$

$$(h,\ k,\ l,\ m)\ (h,\ k,\ n,\ r) = \alpha_h\alpha_k\beta_h\beta_k\,(lmnr)$$

$$(h,\ k,\ l,\ m)\,(h,\ n,\ r,\ s) = \alpha_h\beta_h\,[klm,\ nrs],$$

where $[klm,\ nrs] = \Sigma\,(\alpha_n\alpha_l\alpha_m\beta_k\beta_r\beta_s + \alpha_k\alpha_r\alpha_s\beta_n\beta_l\beta_m)$, the klm being permuted among themselves and the nrs among themselves, according to the rule of signs.

In the same way we find

$$\tfrac{1}{6}\left(\Sigma' P_{hk}\cdot\left|\begin{matrix}\alpha & \beta\\ h & k\end{matrix}\right)^4 = (\Sigma' p_{hk}\alpha_h\alpha_k)^2\cdot(\Sigma' p_{h'k'}\beta_{h'}\beta_{k'})^2 - 2\,(\Sigma' p_{hk}\alpha_h\alpha_k)\,(\Sigma' p_{h'k'}\beta_{h'}\beta_{k'})\right.$$

$$\times\,\Sigma' P_{lml'm'}\cdot\left|\begin{matrix}\alpha & \alpha & \beta & \beta\\ l & m & l' & m'\end{matrix}\right. + 4\Sigma' P_{hklmh'k'l'm'}\cdot\left|\begin{matrix}\alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta\\ h & k & l & m & h' & k' & l' & m'\end{matrix}\right.,$$

$$\tfrac{1}{36}\left(\Sigma' p_{hk}\cdot\left|\begin{matrix}\alpha & \beta\\ h & k\end{matrix}\right)^6 = (\Sigma' p_{hk}\alpha_h\alpha_k)^3\cdot(\Sigma' p_{h'k'}\beta_{h'}\beta_{k'})^3 - 3\,(\Sigma' p_{hk}\alpha_h\alpha_k)^2\,(\Sigma' p_{h'k'}\beta_{h'}\beta_{k'})^2\right.$$

$$\times\,\Sigma' P_{lml'm'}\cdot\left|\begin{matrix}\alpha & \alpha & \beta & \beta\\ l & m & l' & m\end{matrix}\right. + 9\,(\Sigma' p_{hk}\alpha_h\alpha_k)\,(\Sigma' p_{h'k'}\beta_{h'}\beta_{k'})\cdot\Sigma' P_8\,(\alpha^4\beta^4) - 27\Sigma' P_{12}\,(\alpha^6\beta^6).$$

To write the general formula more conveniently let us abbreviate thus:

$$p_{\alpha^2} = \Sigma' p_{hk}\alpha_h\alpha_k,\quad p_{\alpha\beta} = \Sigma' p_{hk}\cdot\left|\begin{matrix}\alpha & \beta\\ h & k\end{matrix}\right|;\ \text{then}$$

$$(p_{\alpha\beta})^{2n} = A_1 p_{\alpha^2}{}^n p_{\beta^2}{}^n - A_2 p_\alpha{}^{n-1}\cdot p_\beta{}^{n-1}\cdot\Sigma P_4\,(\alpha^2\beta^2) + A_3 p_\alpha{}^{n-2} p_\beta{}^{n-2}\,\Sigma P_8\,(\alpha^4\beta^4) - \text{etc.}$$

where the A are numerical multipliers.

ANALYSIS OF CREMONA'S TRANSFORMATIONS.

Quadric transformation [3].

		1	1
1	1		1
1	1	1	

$\mu\,(\alpha\beta\gamma)$ gives $\overline{2\mu-\Sigma\alpha}\,(\mu-\beta-\gamma,\ \ldots)$.

Jacobian is three lines bc, ca, ab.

[Cr. 3.] Cubic transformation $[41]=[3](001)$.

	a_2	a	b	c	d
2	1	1	1	1	1
1	1	1			
1	1		1		
1	1			1	
1	1				1

$$\mu\,(\alpha_2\alpha\beta\gamma\delta) \text{ gives } \overline{3\mu-2\alpha_2-\Sigma\alpha}\,\left(\begin{array}{l|}2\mu-\alpha_2-\Sigma\alpha, \\ \mu-\alpha_2-\alpha,\ \ .\end{array}\right)$$

Jacobian is conic a_2abcd and lines a_2a, a_2b, a_2c, a_2d.

$[41](2,\ 0111)=[3]$.

[Cr. 4.] 1. Quartic transformation $[601]=[3](001)(002)$.

	a_3	a	b	c	d	e	f
3	2	1	1	1	1	1	1
1	1	1					
1	1		1				
1	1			1			
1	1				1		
1	1					1	
1	1						1

$$\mu\,(\alpha_3\alpha\beta\gamma\delta\epsilon\zeta) \text{ gives } \overline{4\mu-3\alpha_3-\Sigma\alpha}\,\left(\begin{array}{l}3\mu-2\alpha_3-\Sigma\alpha, \\ \mu-\alpha_3-\alpha\ldots\end{array}\right.$$

Jacobian is cubic $a_3{}^2abcdef$ and lines a_3a, a_3b, ...

$[601](3,\ 011111)=[3]$.

[Cr. 4.] 2. Quartic transformation $[330]=[3](000)$.

	a_2	b_2	c_2	a	b	c
2	1	1	1		1	1
2	1	1	1	1		1
2	1	1	1	1	1	
1		1	1			
1	1		1			
1	1	1				

$$\mu\,(\alpha_2\beta_2\gamma_2\alpha\beta\gamma) \text{ gives } \overline{4\mu-2\Sigma\alpha_2-\Sigma\alpha}\,\left(\begin{array}{l}2\mu-\Sigma\alpha_2+\alpha-\Sigma\alpha\ldots \\ \mu-\Sigma\alpha+\alpha\ldots\end{array}\right.$$

Jacobian is conics $a_2b_2c_2bc$, ... and lines b_2c_2, c_2a_2, a_2b_2.

$[330](222,\ 011)=[3]$.

Quintic Transformations.

[Cr.] 5.1. [8001]=[3](001)(002)(003).

	a_4	a	b	c	d	e	f	g	h
4	3	1	1	1	1	1	1	1	1
1	1	1							
1	1		1						
1	1			1					
1	1				1				
1	1					1			
1	1						1		
1	1							1	
1	1								1

$\mu\,(\alpha_4\alpha\beta\gamma\delta\epsilon\zeta\eta\theta)$ gives $\overline{5\mu-4\alpha_4-\Sigma\alpha}\;(4\mu-3\alpha_4-\Sigma\alpha \mid \mu-\alpha_4-\alpha,\ldots)$

Jacobian is quartic $a_4^{\,3}abcdefgh$ and the lines a_4a, . .

In De Jonquières' transformation of the $\overline{n+1}|^{\text{th}}$ order $[2n, \ldots 1]\,\mu\,(\alpha_n,\ \alpha\beta\gamma\ldots)$ gives $\overline{(n+1)\,\mu-n\alpha_n-\Sigma\alpha}\,(n\mu-\overline{n-1}\alpha_n-\Sigma\alpha,\ \mu-\alpha_n-\alpha,\ \ldots)$,
and the Jacobian is $\overline{n}|^{\text{ic}}\ a_n^{\,n-1}a\beta\gamma$. and the $2n$ lines a_na, $a_n\beta$, ...

[8001](4, 01111111) = [3].

[Cr.] 5.2. [3310]=[3](001)(001).

	a_3	a_2	b_2	c_2	a	b	c
3	2	1	1	1	1	1	1
2	1	1	1	1	1		
2	1	1	1	1		1	
2	1	1	1	1			1
1	1	1					
1	1		1				
1	1			1			

$\mu(\alpha_3\alpha_2\beta_2\gamma_2\alpha\beta\gamma)$ gives $\overline{5\mu-3\alpha_3-2\Sigma\alpha_2-\Sigma\alpha}\;(3\mu-2\alpha_3-\Sigma\alpha_2-\Sigma\alpha \mid 2\mu-\alpha_3-\Sigma\alpha_2-\alpha,\ldots \mid \mu-\alpha_3-\alpha_2,\ \ldots)$

Jacobian is cubic $a_3^{\,2}a_2b_2c_2abc$
3 conics $a_3a_2b_2c_2a$, .
3 lines a_3a, .

[3310](3, 222, 011) = [3].

[Cr.] 5.3. [0600]=[3](000)(111).

	a_2	b_2	c_2	d_2	e_2	f_2
2		1	1	1	1	1
2	1		1	1	1	1
2	1	1		1	1	1
2	1	1	1		1	1
2	1	1	1	1		1
2	1	1	1	1	1	

$\mu\,(\alpha_2\beta_2\gamma_2\delta_2\epsilon_2\zeta_2)$ gives $\overline{5\mu-2\Sigma\alpha_2}\,(2\mu-\Sigma\alpha_2+\alpha_2,\ \ .)$.

Jacobian is six conics $b_2c_2d_2e_2f_2$, &c.

[0600](022222) = [0600]

[0600](002222) = [04040000] = Cr. 9. 4

Sextic Transformations.

[Cr.] 6.1. [10, 0001]=[3](001)(002)(003)(004). See 5.1.

[Cr.] 6.2. [14200]=[3](000)(011).

	a_3	b_3	a_2	b_2	c_2	d_2	a
3	1	2	1	1	1	1	1
3	2	1	1	1	1	1	1
2	1	1		1	1	1	
2	1	1	1		1	1	
2	1	1	1	1		1	
2	1	1	1	1	1		
1	1	1					

$\mu\,(\alpha_3\beta_3\alpha_2\beta_2\gamma_2\delta_2\alpha)$ gives $\overline{6\mu-3\Sigma\alpha_3-2\Sigma\alpha_2-\alpha}$
$(3\mu-\alpha_3-2\beta_3-\Sigma\alpha_2-\alpha,\ \ldots \mid 2\mu-\alpha_3-\beta_3-\Sigma\alpha_2+\alpha_2,\ . \mid \mu-\alpha_3-\beta_3)$

Jacobian is 2 cubics
$a_3^{\,2}b_3a_2b_2c_2d_2a$; $a_3b_3^{\,2}a_2b_2c_2d_2a$
4 conics $a_3b_3b_2c_2d_2$, etc.
1 line a_3b_3.

[14200](33, 2222, 0) = [3].

[Cr.] 6. 3, 4. [41300]=[3](001)(000); [34010]=[3](000)(002).

	a_3	b_3	c_3	a_2	a	b	c	d
4	2	2	2	1	1	1	1	1
2	1	1	1	1	1			
2	1	1	1	1		1		
2	1	1	1	1			1	
2	1	1	1	1				1
1		1	1					
1	1		1					
1	1	1						

$\mu\,(\alpha_3\beta_3\gamma_3\alpha_2\alpha\beta\gamma\delta)$ gives $\overline{6\mu-3\Sigma\alpha_3-2\alpha_2-\Sigma\alpha}$

$$\left(\begin{array}{l}4\mu-2\Sigma\alpha_3-\alpha_2-\Sigma\alpha,\\ 2\mu-\Sigma\alpha_3-\alpha_2-\alpha,\ ..\\ \mu-\beta_3-\gamma_3,\ ..\end{array}\right)$$

$\mu\,(\alpha_4\alpha_2\beta_2\gamma_2\delta_2\alpha\beta\gamma)$ gives $\overline{6\mu-4\alpha_4-2\Sigma\alpha_2-\Sigma\alpha}$

$$\left(\begin{array}{l}3\mu-2\alpha_4-\Sigma\alpha_2-\beta-\gamma,\ .\\ 2\mu-\alpha_4-\Sigma\alpha_2,\\ \mu-\alpha_4-\alpha_2\end{array}\right)$$

Jacobian of 6. 3 is the quartic $a_3{}^2b_3{}^2c_3{}^2a_2abcd$, 4 conics $a_3b_3c_3a_2a$, etc. and three lines b_3c_3, c_3a_3, a_3b_3.

Jacobian of 6. 4 is 3 cubics $a_4{}^2a_2b_2c_2d_2bc$, etc., conic $a_4a_2b_2c_2d_2$, and 4 lines a_4a, a_4b, ...

[41300].

Septic Transformations.

[Cr.] 7. 1. [12,00001]=[3](001)(002)(003)(004)(005). See 5. 1.

[Cr.] 7. 2. [232100]=[3](000)(001).

	a_4	a_3	b_3	a_2	b_2	c_2	a	b
4	2	2	2	1	1	1	1	1
3	2	1	1	1	1	1	1	
3	2	1	1	1	1	1		1
2	1	1	1		1	1		
2	1	1	1	1		1		
2	1	1	1	1	1			
1	1	1						
1	1		1					

[Cr.] 7. 3. [034000]=[3](001)(001)(111).

	a_3	b_3	c_3	d_3	a_2	b_2	c_2
3	2	1	1	1	1	1	1
3	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
3	1	1	1	2	1	1	1
2	1	1	1	1	1		
2	1	1	1	1		1	
2	1	1	1	1			1

[Cr.] 7. 4, 5. [503100]=[3](001)(002)(001); [350010]=[3](001)(001)(003).

	a_4	a_3	b_3	c_3	a	b	c	d	e
5	3	2	2	2	1	1	1	1	1
2	1	1	1	1	1				
2	1	1	1	1		1			
2	1	1	1	1			1		
2	1	1	1	1				1	
2	1	1	1	1					1
1	1	1							
1	1		1						
1	1			1					

Octavic Transformations.

[Cr.] 8. 2. [3230100] = [3](001)(001)(002).

	a_5	a_3	b_3	c_3	a_2	b_2	a	b	c
5	3	2	2	2	1	1	1	1	1
3	2	1	1	1	1	1	1		
3	2	1	1	1	1	1		1	
3	2	1	1	1	1	1			1
2	1	1	1	1		1			
2	1	1	1	1	1				
1	1	1							
1	1		1						
1	1			1					

[Cr.] 8. 3. [1322000] = [3](001)(001)(011).

	a_4	b_4	a_3	b_3	a_2	b_2	c_2	a
4	2	2	2		1	1	1	1
4	2	2		2	1	1	1	1
3	2		1	1	1	1	1	
3		2	1	1	1	1	1	
2	1	1	1	1	1			
2	1	1	1	1		1		
2	1	1	1	1			1	
1	1	1						

[Cr.] 8. 4. [0070000] = [3](001)(001)(111)(222) = [7. 3](222).

	a_3	b_3	c_3	d_3	e_3	f_3	g_3
3	2	1	1	1	1	1	1
3	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
3	1	1	1	2	1	1	1
3	1	1	1	1	2	1	1
3	1	1	1	1	1	2	1
3	1	1	1	1	1	1	2

[Cr.] 8. 5. [3600010] = [3](000)(002)(004) = 6. 4)(004).

[Cr.] 6. [6013000] = [3](001)(002)(000) = 4. 1)(000).

	a_6	a_2	b_2	c_2	d_2	e_2	f_2	a	b	c
4	3	1	1	1	1	1	1		1	1
4	3	1	1	1	1	1	1	1		1
4	3	1	1	1	1	1	1	1	1	
3	2	1	1	1	1	1	1			
1	1	1								
1	1		1							
1	1			1						
1	1				1					
1	1					1				
1	1						1			

[Cr.] 8. 7. [0520100] = [3](000)(111)(002) = 5. 4)(002).

8. [2051000] = [3](001)(000)(112) = 6 . 3)(112).

	a_5	a_3	b_3	a_2	b_2	c_2	d_2	e_2
4	2	2	2	1	1	1	1	1
3	2	1	1		1	1	1	1
3	2	1	1	1		1	1	1
3	2	1	1	1	1		1	1
3	2	1	1	1	1	1		1
3	2	1	1	1	1	1	1	
1	1	1						
1	1		1					

[I have printed the above *Analysis* as it is given in a Note-Book, adding on the "Cr.," herein copying Prof. Cayley, who, in his paper "*on the Ration transformation between two spaces*" (Proceedings of L. Math. Society, Vol. III pp. 127—180), writes :—"Prof. Clifford calculated in this way the followi table, shewing how any transformation of an order not exceeding 8 can] expressed by means of a series of quadric transformations ; the symbols Cr. &c., refer to the order and number of Cremona's tables." There is furth reference to Prof. Clifford's work in connexion with this subject in § 68 a § 69. Prof. Cremona's Memoir, *Sulle trasformazioni Geometriche delle figu piane*, is in the Mem. di Bologna, t. ii., 1863 and t. v., 1865. I have compar the two lists, and have been able to verify the results given above when they a given also in Prof. Cayley's list. I add Cr. 8 . 1.

[14,000001] = [3](001)(002)(003)(004](005)(006).

and Cr. 8. 9 (due to Cayley, see Proc. l. c. p. 143),

= [3303000] = [3](000)(000)(000).

I may refer also to papers by Mr S. Roberts, *On Prof. Cremona's Transformatio between two planes and Tables relating thereto* (Proc. L. M. S., Vol. IV. pp. 121–139), and by Mr T. Cotterill, *On a correspondence of Points, such that a curve c the* n*th order in one plane corresponds to a curve of the 4*n*th in another plan with three multiple points of the order* n *on the line of intersection of the plane and three other multiple points of the order* 2n. To this last paper are appende some combined observations, due to Profs. Cayley and Clifford (Proc. L. M. S Vol. II., p. 123.)]

BITANGENT CIRCLES OF A CONIC.

1. If two conics have double contact, any point on the chord of contact has the same polar in respect of them.

For the polar of O, a point on AB, must pass through C where the tangents at A and B intersect, and also through P the harmonic of O in respect of A, B. [Fig. 112.]

Cor. If one of the conics is a circle, and the point O at infinity, CP is a diameter of both conics which bisects chords at right angles to itself; i.e. an axis. Hence

If a circle have double contact with a conic, its centre must lie on one of the axes and the chord of contact must be perpendicular to that axis.

2. Through three points to draw a conic having double contact with a given conic.

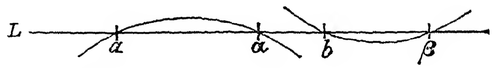
Let O, P, Q be the points: let OP meet the given conic in AB, OQ in CD. Let X, Y be the points which divide harmonically both AB and OP, X', Y' those which divide harmonically both OQ and CD. Then XY', XX', YY', YX' are the four positions of the chord of contact. For suppose X to be a point on the chord, then by the last proposition it must have the same polar in respect of both conics; and therefore the same conjugate in respect of AB and OP. [Fig. 113.]

We see thus that the conics, having double contact with the given one, which pass through OP divide themselves into two systems; viz., those whose chords of contact pass through X and those whose chords of contact pass through Y. In the particular case in which O, P are the points at infinity on a circle, X and Y become the points at infinity on the axis of the conic, and so

Circles having double contact with a given conic divide themselves into two systems, and according as the chord of contact is parallel to one axis or the other; and two circles of each system can be drawn through a given point.

3. The locus of a point whose powers in respect of two given conics are in a fixed ratio is a conic passing through their points of intersection.

Let L be an arbitrary line which cuts the two conics in $a\alpha$ and $b\beta$ respectively.



It is required to find a point x on L such that

$$xa \, . \, x\alpha = m \, . \, xb \, . \, x\beta.$$

Let two points p, q be so related that

$$\frac{pa}{pb} = m \frac{q\beta}{q\alpha},$$

then these two points uniquely determine each other, and therefore there are two positions at which they coincide, or two positions of the point x; that is, two points of the locus are on the arbitrary line L.

Therefore the locus is of the second order, and it passes through the intersections of the conics because two quantities which are in a fixed ratio must vanish simultaneously.

It follows conversely that if three conics pass through the same four points, the ratio of the powers of any point on one in respect of the other two is constant. For let U, V, W be the conics, and consider a point m on W. The powers of m in respect of U, V are in a certain ratio. Now the locus of a point whose powers in respect of U, V are in that ratio is a conic through the intersections of U, V and of course also through m. Thus it has five points in common with W and must therefore coincide with it.

In particular, let W be made up of the two chords L, M of U and V. Then the power of any point on U in respect of V is in a constant ratio to the product of its distances from the lines L, M. [Fig. 114.]

If we suppose L, M to coincide, then U and V will have double contact; and we learn that in this case the power of any point on U in respect of V is in a constant ratio to the square of its distance from L. [Fig. 115.]

Now the power of any point in respect of a circle is the squared tangent from the point to the circle: hence we see that

The tangent from any point of a conic to a bitangent circle is in a constant ratio to the perpendicular on the chord of contact.

4. If we suppose the point on the curve to go to infinity in the direction CP, the ratio becomes ultimately $\frac{CM}{CP}$, and is therefore the same for all circles of this system. For circles of the other system it is $\frac{PM}{CP}$. These two ratios may be called ϵ and ϵ'. [Fig. 116.]

5. The sum or difference of the tangents drawn from any point of a conic to two bitangent circles of the same system is constant.

Let A, B be the circles, L, L' their chords of contact. Then

$$PM = \epsilon \, . \, PA,$$

and $$PN = \epsilon \,.\, PB;$$

$$\therefore \quad MN = \epsilon\,(PA \pm PB)$$

according as P is between L and L' or outside of them. [Fig. 117.]

(The radical axis of the circles is clearly midway between their chords of contact.

Since the four points of contact are symmetrically situated in respect of the axis, a circle will pass through them having its centre on the axis. This is called the circle of contact.)

6. The product of the tangents drawn from any point of a conic to two bitangent circles of the same system is equal to the square of the tangent drawn from the same point to their circle of contact.

For the conic, the circle of contact, and the pair of chords of contact, make three conics through the same four points: and therefore the power of any point on the conic in respect of the circle of contact bears a constant ratio to the product of its distances from the chords of contact, and therefore a constant ratio to the product of the tangents drawn to the two circles By considering the points at infinity on the conic we see that this ratio must be one of equality.

OF POWER-COORDINATES IN GENERAL.

The equation of a circle contains *three* disposable constants, the equation of any circle may therefore be put into the form

$$aX+bY+cZ+dW=0 \quad \ldots\ldots \quad (1),$$

where X, Y, Z, $W=0$ are the equations of four circles. And any equation of the first order in $XYZW$ represents a circle.

The main object of the following paper is to discuss the general equation of the *second* order in $XYZW$. Now if we write

$$\left.\begin{aligned} X &\equiv \overline{x-\alpha_1}^2+\overline{y-\beta_1}^2-r_1^2 \\ Y &\equiv \overline{x-\alpha_2}^2+\overline{y-\beta_2}^2-r_2^2 \\ Z &\equiv \overline{x-\alpha_3}^2+\overline{y-\beta_3}^2-r_3^2 \\ W &\equiv \overline{x-\alpha_4}^2+\overline{y-\beta_4}^2-r_4^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2),$$

then the equation (1) has an obvious geometrical meaning; viz. X means the squared tangent from the point (xy) to the first circle, or, as it is called, the *power* of that point in respect of the circle; and we have learned that if the squared tangents drawn from a point to four fixed circles satisfy an equation of the first degree, the locus of that point is a circle. We thus come to regard the quantities $XYZW$ as a sort of coordinates; and a set of values of the coordinates may represent a point.

But a point is determined by *two* coordinates; and therefore in order that a set of values of these *four* quantities may represent a point, they must satisfy two equations identically. I shall now prove that one of these equations is non-homogeneous and of the first degree, while the other is homogeneous and of the second degree.

In fact, if we write A for the determinant

$$\begin{vmatrix} 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \\ 1 & \beta_4 & \gamma_4 \end{vmatrix},$$

and B, C, D for the three similar determinants that can be formed from the other triads of the four circles, we shall have

$$AX+BY+CZ+DW \equiv (1234) \quad \ldots\ldots\ldots\ldots \quad (3),$$

where (1234) denotes the determinant

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \alpha_1^2+\beta_1^2-r_1^2 \\ 1 & \alpha_2 & \beta_2 & \alpha_2^2+\beta_2^2-r_2^2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3^2+\beta_3^2-r_3^2 \\ 1 & \alpha_4 & \beta_4 & \alpha_4^2+\beta_4^2-r_4^2 \end{vmatrix}.$$

This is the non-homogeneous equation of the first degree.

If we choose to concern ourselves only with the ratios of the quantities $X\ Y\ Z\ W$, then the equation

$$AX+BY+CZ+DW=0$$

will represent the line (or, as we may say, circle) at infinity. I shall call the form $AX+BY+CZ+DW$ the *first absolute*, and denote it by the symbol ∞.

Again, we have in the equations (2) $X\ Y\ Z\ W$ exhibited as linear functions of x^2+y^2, x, y, 1. It is therefore possible, by solving the equations, to exhibit x^2+y^2, x, y, 1 as proportional to linear functions of X, Y, Z, W. That is to say, we must have

$$\frac{x^2+y^2}{P}=\frac{x}{Q}=\frac{y}{R}=\frac{1}{S},$$

where P, Q, R, S are four linear functions of X, Y, Z, W. But

$$(x^2+y^2)\cdot 1=(x)^2+(y)^2,$$

therefore

$$PS=Q^2+R^2,$$

a homogeneous equation of the second degree in X, Y, Z, W which must be identically satisfied if X, Y, Z, W are the coordinates of a point. The form $PS-Q^2-R^2$ I shall call the *Second Absolute*, and denote by the symbol ϕ.

The question now naturally arises; supposing that the coordinates $XYZW$ do not satisfy the equation $\phi=0$, and therefore do not represent a *point*, what do they represent? This question I proceed to answer.

Let δ be the distance between the centres of two circles, r and r_1 their radii. The quantity $\delta^2-r^2-r_1^2$ I call the *power of one circle in respect of the other.* If one of them becomes a point, so that $r_1^2=0$, the power is δ^2-r^2, which is the squared tangent from that point to the other circle; so that this definition agrees with the previous use of the word *power.* The power is also equal to $-2rr_1\cos\theta$, where θ is the angle of intersection of the circles; for, in the figure, ACB is this angle (namely the angle through which (A) must be turned about the point C, in order that its concavity may coincide with the concavity of (B)), and we have at once

$$\delta^2=r^2+r_1^2-2rr_1\cos\theta,$$

which establishes the equivalence in question. [Fig. 118.]

Let us now examine the effect of supposing X, Y, Z, W to denote the powers of a *circle* in respect of the four fixed circles.

First, the equation (3) is still satisfied; for the coordinates $XYZW$ of the *circle* are got from the coordinates $X_1\ Y_1\ Z_1\ W_1$ of its *centre* by subtracting r^2 from each of them; we have then only to prove that

$$(A+B+C+D)\,r^2\equiv 0,$$

which is obvious, for $A+B+C+D$ is the coefficient of (x^2+y^2) on the left-hand of the equation, and the right-hand is constant.

Next, what is meant by an equation of the first order? Let us write, a little more generally,

$$a_1X=a_1(x^2+y^2)+2b_1x+2c_1y+d_1,$$

and let the equation of our new circle be

$$0=a(x^2+y^2)+2bx+2cy+d,$$

then we shall have

$$\delta^2=\left(\frac{b}{a}-\frac{b_1}{a_1}\right)^2+\left(\frac{c}{a}-\frac{c_1}{a_1}\right)^2,$$

$$r^2=\frac{b^2+c^2-ad}{a^2},$$

$$r_1^2=\frac{b_1^2+c_1^2-a_1d_1}{a_1^2},$$

and therefore

$$\delta^2-r^2-r_1^2=\frac{ad_1+a_1d-2bb_1-2cc_1}{aa_1}\ \dots\dots\dots\dots\dots \quad (4)$$

Now if the power of two circles vanishes, they cut at right angles; for the power is $-2rr_1\cos\theta$, and $\cos\theta=0$ means that θ is a right angle. We see thus that the condition for two circles to cut at right angles is

$$ad_1+a_1d-2bb_1-2cc_1=0,$$

which is linear in the coefficients of each. And further, any linear relation among the coefficients of a circle expresses that it cuts some fixed circle at right angles. For the relation

$$la+mb+nc+sd=0$$

expresses that the circle

$$a(x^2+y^2)+2bx+2cy+d=0$$

cuts at right angles the circle

$$s(x^2+y^2)-mx-ny+l=0.$$

It follows that *if the coordinates of a circle satisfy an equation of the first degree, the circle cuts at right angles a certain fixed circle.* For the equation

$$lX+mY+nZ+sW=0$$

implies a linear relation among $a\ b\ c\ d$, since $X\ Y\ Z\ W$ are proportional to linear functions of these quantities by the equation (3). And this circle is precisely the one that we before represented by this equation, viz. when we considered therein X, Y, Z, W as functions of x, y determined by equations (2). For the circle is cut orthogonally by all circles whose coordinates satisfy the equation; but if these circles become points, they must be points on the circle which they cut orthogonally.

These results I sum up as follows:

The coordinates of a circle are four quantities proportional to its powers in respect of four fixed circles not having the same radical centre.

The coordinates of all circles which cut at right angles a given circle C satisfy a homogeneous equation of the first degree, which is called the equation of the circle C.

The coordinates of all *points* satisfy a homogeneous equation of the second order.

The Orthogonal System.

The following theorem includes about one-third of the metrical properties of points, lines, and circles.

If there are five circles, 1 2 3 4 5, and five other circles, 1′ 2′ 3′ 4′ 5′, and if we form the determinant whose constituents are the powers of the first set of circles in respect of the second set, arranged as in a multiplication table so as to be represented in the umbral notation by the symbol $\frac{1\ 2\ 3\ 4\ 5}{1'\,2'\,3'\,4'\,5'}$, this determinant vanishes identically.

For if we multiply together the two matrices

$$\begin{matrix} 1, & 2b_1, & 2c_1, & d_1 \\ 1, & 2b_2, & 2c_2, & d_2 \\ 1, & 2b_3, & 2c_3, & d_3 \\ 1, & 2b_4, & 2c_4, & d_4 \\ 1, & 2b_5, & 2c_5, & d_5 \end{matrix} \qquad \begin{matrix} d_1', & -b_1', & -c_1', & 1 \\ d_2', & -b_2', & -c_2', & 1 \\ d_3', & -b_3', & -c_3', & 1 \\ d_4', & -b_4', & -c_4', & 1 \\ d_5', & -b_5', & -c_5', & 1 \end{matrix}$$

we shall form a determinant which, by the theory of [matrices], vanishes identically; while its constituents are of the type

$$d' + d - 2bb' - 2cc',$$

which, as we have already seen, means the power of the circles

$$x^2 + y^2 + 2bx + 2cy + d = 0,$$
$$x^2 + y^2 + 2b'x + 2c'y + d' = 0.$$

Reserving for the present the further discussion of this theorem and of a similar one, I proceed to consider a particular case. Let us take for the first four circles of each set the fundamental circles $X\,Y\,Z\,W$, and let the coordinates of the circles 5 and 5′ be represented by x, y, z, w and x', y', z', w'. Then the identity

$$\frac{X\ Y\ Z\ W\ 5}{X\ Y\ Z\ W\ 5'} = 0$$

becomes

$$\begin{vmatrix} -2r_1^2, & (XY), & (XZ), & (XW), & x \\ (XY), & -2r_2^2, & (YZ), & (YW), & y \\ (XZ), & (YZ), & -2r_3^2, & (ZW), & z \\ (XW), & (YW), & (ZW), & -2r_4^2, & w \\ x', & y', & z', & w', & (5\,5') \end{vmatrix} = 0.$$

By this equation the power of two circles is expressed in terms of their coordinates. Let the expression be denoted thus:

$$(5\,5') = (*\between x, y, z, w \between x', y', z', w'),$$

then clearly the *equation* of the circle whose *coordinates* are $x\,y\,z\,w$ is

$$0=(*\between x,\,y,\,z,\,w\between X,\;Y,\,Z,\,W).$$

And further, if we suppose the circles 5, 5′ to coincide in a circle of radius r, we have

$$-2r^2=(*\between x,\,y,\,z,\,w)^2,$$

which gives the radius of a circle in terms of its coordinates. It follows that the Second Absolute is

$$\Phi=(*\between X\,Y\,Z\,W)^2.$$

By dividing the first four rows and columns of the determinant by r_1, r_2, r_3, r_4 respectively, we may reduce this to the simpler form

$$\begin{vmatrix} 1 & \cos XY & \cos XZ & \cos XW & \frac{X}{r_1} \\ \cos XY & 1 & \cos YZ & \cos YW & \frac{Y}{r_2} \\ \cos XZ & \cos YZ & 1 & \cos ZW & \frac{Z}{r_3} \\ \cos XW & \cos YW & \cos ZW & 1 & \frac{W}{r_4} \\ \frac{X}{r_1} & \frac{Y}{r_2} & \frac{Z}{r_3} & \frac{W}{r_4} & 0 \end{vmatrix}$$

which it is worth stating is an identical relation connecting the equations of any four circles.

It is obvious that all these formulæ will be immensely simplified if we take for our fundamental circles four circles cutting each other orthogonally. In this case the quantities (XY), (XZ)...&c. are all zero; the Second Absolute becomes

$$\Phi\equiv\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{W^2}{r_4^2},$$

the radius of $(x\,y\,z\,w)$ is

$$\tfrac{1}{4}\left(\frac{x^2}{r_1^2}+\frac{y^2}{r_2^2}+\frac{z^2}{r_3^2}+\frac{w^2}{r_4^2}\right);$$

the power of $(x\,y\,z\,w)$ in respect of $(x'\,y'\,z'\,w')$ is

$$-\tfrac{1}{2}\left(\frac{xx'}{r_1^2}+\frac{yy'}{r_2^2}+\frac{zz'}{r_3^2}+\frac{ww'}{r_4^2}\right),$$

and consequently the equation of $(x\,y\,z\,w)$ is

$$\frac{x}{r_1^2}\,X+\frac{y}{r_2^2}\,Y+\frac{z}{r_3^2}\,Z+\frac{w}{r_4^2}\,W=0.$$

I shall simplify these expressions still further by taking as coordinates of a circle not the powers $X\,Y\,Z\,W$ themselves, but the powers each divided by the radius of the corresponding fundamental circle, viz. the quantities

$$\frac{X}{r_1},\;\frac{Y}{r_2},\;\frac{Z}{r_3},\;\frac{W}{r_4}.$$

Calling these quantities X_1 Y_1 Z_1 W_1, we find for the absolute merely

$$X_1^2+Y_1^2+Z_1^2+W_1^2,$$

and the equation of $(l\,m\,n\,s)$ is

$$lX_1+mY_1+nZ_1+sW_1=0.$$

This step, it will be seen, is precisely analogous to the step from *areal* to *trilinear* coordinates in the geometry of straight lines. It must be remembered that the only speciality which has been introduced is that our four fundamental circles form an orthogonal system. One of them is therefore imaginary.

The radius of the circle

$$lX_1+mY_1+nZ_1+sW_1=0$$

vanishes when

$$l^2+m^2+n^2+s^2=0.$$

It thus appears that a point (or circle of no radius) is a circle which satisfies the analytical condition of touching the Second Absolute. This remark will be found useful in the sequel.

Equation of an Anallagmatic.

The general equation of the second order in power-coordinates represents an *anallagmatic* curve of the fourth order, i.e. a quartic curve having a double point at each of the circular points at infinity. For if in the expression

$$(*\between X,\ Y,\ Z,\ W)^2$$

we substitute for X, Y, Z, W their values in terms of x^2+y^2, x, y, 1, it becomes

$$a\,(x^2+y^2)^2+L\,(x^2+y^2)+U,$$

where L is of the first degree in x, y, and U of the second. Now this equated to zero is the general equation of an anallagmatic quartic; and it contains one constant less than the former, which is thus (if we bear in mind the existence of the Second Absolute) just sufficiently general to represent all such anallagmatic curves.

I say, "if we bear in mind the existence of the Second Absolute" for this reason. The equation $\Phi=0$ is satisfied identically by the coordinates of every *point*. Now if we take $\Theta=0$ for an equation of the second order in X, Y, Z, W, the equation

$$\Theta+\lambda\Phi=0$$

must represent exactly the same curve as $\Theta=0$; for the equation is satisfied by the coordinates of all *points* which satisfy this latter equation. Out of the nine apparently arbitrary constants, therefore, in the expression Θ, one is at our disposal independently of the determination of the curve; and the real number of constants is therefore eight.

In fact, if we regard $X\,Y\,Z\,W$ as the coordinates of a point in space, then the Second Absolute represents a quadric surface; and every anallagmatic curve is represented in the first instance by another quadric surface, but in the second instance (viz., when we remember the remarks just made) by the curve of inter-

section of this surface with the Second Absolute Now a quadri-quadric curve depends upon eight constants only, and so fitly represents a general anallagmatic curve.

Bitangent Circles.

Consider therefore any quadric Θ and the second absolute Φ. We know that by a linear transformation it is possible in one and only one way to reduce these simultaneously to the canonical form. That is to say, there is precisely one set of four circles $X\ Y\ Z\ W$ such that when we express Θ and Φ in terms of them they take the forms

$$\Phi \equiv X^2 + Y^2 + Z^2 + W^2,$$

$$\Theta \equiv \frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2}.$$

The first of these indicates that the circles $X\ Y\ Z\ W$ are mutually orthogonal. They may be called the *principal circles* of the curve. (Moutard.)

The quantities a^2, b^2, c^2, d^2, are not necessarily positive; they are written in this form for the sake of subsequent convenience.

It is now possible, with the aid of the second absolute, to express the equation of the curve in terms of any three of the four principal circles. In fact, eliminating W between the equations $\Theta = 0$, $\Phi = 0$, we have

$$d^2\Theta - \Phi \equiv \frac{X^2}{a^2}(d^2 - a^2) + \frac{Y^2}{b^2}(d^2 - b^2) + \frac{Z^2}{c^2}(d^2 - c^2) = 0,$$

an equation which, if X, Y, Z were trilinear coordinates, would represent a conic referred to a self-conjugate triangle.

A circle which satisfies the analytical condition of touching this curve is a bitangent circle of the anallagmatic. For if

$$x \equiv lX + mY + nZ,$$

$$y \equiv l'X + m'Y + n'Z,$$

$$u \equiv l_1X + m_1Y + n_1Z,$$

where x and y are two such tangents, then we know the equation of the curve can be reduced to the form

$$xy = ku^2,$$

where it is clear that, for instance, the circle x touches the curve at the two points where it meets u. All these circles are orthotomic of W; for since XYZ are orthotomic of W, any circle $lX + mY + nZ$ is so. Hence we may enunciate the following propositions:—

The bitangent circles of an anallagmatic curve arrange themselves into four systems, all the circles of each system cutting orthogonally one of the principal circles.

The four points of contact of two circles x, y, of the same system lie on a circle u (circle of contact), and the product of the tangents drawn from any point of the curve to the circles x, y is in a constant ratio to the square of the tangent drawn to the circle u.

Two bitangent circles of a given system can be drawn through an arbitrary point.

If two bitangent circles be drawn through a point on u, their circle of contact is coaxal with x, y.

Further, let x, y, z be any three bitangent circles of the system W, where

$$z \equiv l''X + m''Y + n''Z,$$

then we know from the theory of conics that the equation of the anallagmatic can be written in the form

$$\alpha\sqrt{x} + \beta\sqrt{y} + \gamma\sqrt{z} = 0,$$

where α β γ are constants. Hence

The tangents drawn from any point of the curve to three bitangent circles of the same system satisfy a linear relation.

A focus of the curve is a bitangent circle of evanescent radius. Since the two quadrics

$$\frac{X^2}{a^2}(d^2 - a^2) + \frac{Y^2}{b^2}(d^2 - b^2) + \frac{Z^2}{c^2}(d^2 - c^2) = 0,$$

$$X^2 \qquad + Y^2 \qquad + Z^2 \qquad = 0,$$

have just *four* common tangents, and since a circle of evanescent radius is one satisfying the analytical condition of touching the absolute, it follows that:

There are four foci on each principal circle.

The distances of any point on the curve from three foci of the same system are connected by a linear relation.

Any two foci of the same system have a contact circle, such that the product of the distances of any point of the curve from the two foci is proportional to the power *of the point in respect of this circle.*

Confocal Curves.

The equation

$$\frac{X^2}{a^2+\theta} + \frac{Y^2}{b^2+\theta} + \frac{Z^2}{c^2+\theta} + \frac{W^2}{d^2+\theta} = 0$$

(in which θ is a variable parameter) represents a series of confocal anallagmatics: for it is readily observed that the equations for determining the foci involve only the *differences* of the quantities a^2, b^2, c^2, d^2. In fact, the foci are common bitangents of the curve and of the absolute; now the equation above written obviously represents a curve touching all the bitangents common to

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2} = 0,$$

and

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

Two confocals cut at right angles. For let $X_1Y_1Z_1W_1$ be a point of intersection. The bitangent circles of the W-system at that point are

$$\frac{XX_1}{a^2}(d^2-a^2)+\frac{YY_1}{b^2}(d^2-b^2)+\frac{ZZ_1}{c^2}(d^2-c^2)=0,$$

$$\frac{XX_1}{a^2+\theta}(d^2-a^2)+\frac{YY_1}{b^2+\theta}(d^2-b^2)+\frac{ZZ_1}{c^2+\theta}(d^2-c^2)=0,$$

which cut at right angles if

$$\frac{X_1^2}{a^2(a^2+\theta)}(d^2-a^2)^2+\frac{Y_1^2}{b^2(b^2+\theta)}(d^2-b^2)^2+\frac{Z_1^2}{c^2(c^2+\theta)}(d-c^2)^2=0.$$

But this equation follows readily from the three

$$\frac{X_1^2}{a^2}+\frac{Y_1^2}{b^2}+\frac{Z_1^2}{c^2}+\frac{W_1^2}{d^2}=0 \qquad \text{(A)},$$

$$\frac{X_1^2}{a^2+\theta}+\frac{Y_1^2}{b^2+\theta}+\frac{Z_1^2}{c^2+\theta}+\frac{W_1^2}{d^2+\theta}=0 \qquad \text{(B)},$$

$$X_1^2+Y_1^2+Z_1^2+W_1^2=0. \qquad \text{(C)}.$$

It is worth while to write down the ratios of $X_1Y_1Z_1W_1$ which are deducible from these. They are

$$kX_1=\frac{aa'}{\sqrt{(\overline{a^2-b^2}\,.\,\overline{a^2-c^2}\,.\,\overline{a^2-d^2})}},$$

$$kY_1=\frac{bb'}{\sqrt{(\overline{b^2-c^2}\,.\,\overline{b^2-d^2}\,.\,\overline{b^2-a^2})}},$$

and so on, where a'^2 has been written for shortness instead of $a^2+\theta$.

The conditions that a circle

$$lX+mY+nZ+sW,$$

shall be a bitangent of the system W to the curve (A) are $s=0$ and

$$\frac{l^2a^2}{d^2-a^2}+\frac{m^2b^2}{d^2-b^2}+\frac{n^2c^2}{d^2-c^2}=0.$$

Call this $U=0$, and let

$$V\equiv\frac{l^2}{d^2-a^2}+\frac{m^2}{d^2-b^2}+\frac{n^2}{d^2-c^2},$$

then the equation $U=0$ can be written in either of the forms

$$\frac{U-Va^2}{a^2}=\frac{U-Vb^2}{b^2}=\frac{U-Vc^2}{c^2}.$$

I shall now prove that the quantities $U-Va^2$, $U-Vb^2$, $U-Vc^2$ are proportional to the products of the powers of $(lmns)$ in respect of pairs of foci of the systems X Y Z respectively.

In fact, if we write

$$P^2=(a^2-b^2)(c^2-d^2),$$

$$Q^2=(a^2-c^2)(d^2-b^2),$$

$$R^2=(a^2-d^2)(b^2-c^2),$$

then it is easily seen that the coordinates of the sixteen foci are

four of the X-system $0\ ,\ \pm P,\ \pm Q,\ \pm R,$

four of the Y-system $\pm P,\ 0\ ,\ \pm R,\ \pm Q,$

four of the Z-system $\pm Q,\ \pm R,\ 0\ ,\ \pm P,$

four of the W-system $\pm R,\ \pm Q,\ \pm P,\ 0\ .$

Now, for instance,

$$U - Va^2 \equiv \frac{m^2(b^2-a^2)}{d^2-b^2} + \frac{n^2(c^2-a^2)}{d^2-c^2},$$

$$\equiv \frac{(mP+nQ)(mP-nQ)}{(d^2-b^2)(d^2-c^2)},$$

$$\equiv \frac{(-mP-nQ)(-mP+nQ)}{(d^2-b^2)(d^2-c^2)},$$

which verifies the assertion.

From these equations we deduce the following propositions:—

Any bitangent circle of the W-system is so related to the four foci of the X-system that the product of its powers in respect of a certain two of them is equal to the product of its powers in respect of the other two.

These products in respect of the X- Y- and Z-systems have constant ratios for all bitangent circles of the W-system.

[The following Notes, "Theory of Powers," preceded the above paper in the Note-book, but appear to have no connection with it. They are very fragmentary and, in places, apparently inaccurate, but it has been thought desirable to print them almost as they were left by the writer. It is not easy to see how the equations in 1 and 4 are got, nor how the other equation in I. contains a linear relation between the powers of a point with respect to a, &c.]

Let a, b, c be three points in a right line: the rectangle or product $ab \, . \, ac$ is called the power of the point a in respect of the point-pair bc.

1. Let a, α; b, β; c, γ be pairs of points dividing harmonically the length XY, then (m, n, p, o being the middle points respectively)

$$a\alpha + XY = [2]\, m \, . \, o,$$

$$b\beta + XY = [2]\, n \, . \, o,$$

$$c\gamma + XY = [2]\, p \, . \, o,$$

multiply by $\overline{n-p}$, $\overline{p-m}$, $\overline{m-n}$ and add; then

$$\overline{n-p} \, . \, a\alpha + \overline{p-m} \, . \, b\beta + \overline{m-n} \, . \, c\gamma = 0.$$

Now the origin is arbitrary, and the distances $\overline{n-p}$, $\overline{p-m}$, $\overline{m-n}$ are constant: hence

The powers of any point whatever, in respect of three pairs in involution, satisfy a certain linear relation.

2. Let a, b correspond uniquely to α, β respectively, and let X, Y be the double points of the correspondence.

Then because any two corresponding points make the same anharmonic ratio with the double points, $(a\alpha XY)=(b\beta XY)$, or

$$\frac{a\alpha \,.\, XY}{aX \,.\, \alpha Y}=\frac{b\beta \; XY}{bX \,.\, \beta Y}.$$

Also because the anharmonic ratio of four points = anharmonic ratio of corresponding points,

$$(abXY)=(\alpha\beta XY),$$

or

$$\frac{ab \,.\, XY}{aY \,.\, bX}=\frac{\alpha\beta \; XY}{\alpha Y \,.\, \beta X}.$$

Multiply these equations together; then

$$\frac{a\alpha \,.\, ab}{aX \,.\, bY}=\frac{b\beta \,.\, \alpha\beta}{\beta X \,.\, \beta Y}, \text{ or } \frac{a\alpha \,.\, ab}{\beta\alpha \,.\, \beta b}=\frac{aX \,.\, aY}{\beta X \,.\, \beta Y};$$

that is, if two points be taken one from each system of an unique correspondence, the ratio of their powers in respect of the points corresponding to them is equal to the ratio of the powers in respect of the double points.

If A, B, C are three lines through a point, $\sin AB \,.\, \sin AC$ is called the power of A in respect of the pair BC.

It will follow in a similar manner that the corresponding proposition is true about an unique correspondence of lines.

3. Let now pOp', qOq' be two chords of a conic, meeting in O. Let the lines qq', qp, $p'q'$, $p'p$ be called a, α, b, β respectively; it is clear that the directions a, b correspond uniquely to α, β and that the double lines are the asymptotes. Now

$$\frac{Oq}{Op}=\frac{\sin Opq}{\sin Oqp}=\frac{\sin b\beta}{\sin ab},$$

and

$$\frac{Oq'}{Op'}=\frac{\sin Op'q'}{\sin Oq'p'}=\frac{\sin \alpha\beta}{\sin a\alpha};$$

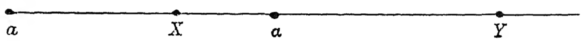
$$\therefore \quad \frac{Oq \,.\, Oq'}{Op \,.\, Op'}=\frac{\sin\beta b \,.\, \sin\beta\alpha}{\sin ab \,.\, \sin a\alpha}=\frac{\sin\beta X \,.\, \sin\beta Y}{\sin aX \,.\, \sin aY},$$

by the previous proposition. [Fig. 119.]

Whence the ratio of the product of segments of chords drawn through any point is independent of the position of the point, and depends only on the direction of the chords relative to the asymptotes. Also the product of segments of any chord drawn through a point, divided by square of parallel semidiameter, is independent of the direction of the chord: this I call the *power* of the point in respect of the conic. Further

The square of any semidiameter is inversely proportional to the products of the sines of the angles it makes with the asymptotes.

4. Properties of Harmonic Section.



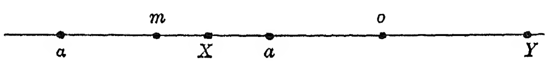
If a, α are harmonic of X, Y, we have by definition

$$\frac{aX}{aY} = -\frac{\alpha X}{\alpha Y}, \text{ or } aX \cdot \alpha Y + \alpha X \cdot aY = 0;$$

$$\therefore aX \cdot \alpha Y + \alpha X \cdot aY = (a\alpha + \alpha Y)\, aY + (\alpha a + aX)\, \alpha Y$$
$$= a\alpha\, (aY - \alpha Y) = a\alpha^2.$$

Therefore if two points are harmonic in respect of X, Y the sum of their powers in respect of X, Y is equal to the square of their distance.

Hence the sum of squares of reciprocals of two conjugate diameters of a conic varies as squared sine of angle between them.



This proposition may also be got from the one concerning the middle points. Thus

$$a\alpha + XY = 2mo,$$

whatever be the origin. Take a and α successively for origin, then

$$aX \cdot aY = 2am \cdot ao = a\alpha \cdot ao,$$

$$\alpha X \cdot \alpha Y = 2\alpha m \cdot \alpha o = -a\alpha \cdot \alpha o,$$

$$\therefore aX \cdot \alpha Y + \alpha X \cdot aY = a\alpha\,(ao - \alpha o) = a\alpha^2,$$

also $$aX \cdot aY \cdot \alpha X \cdot \alpha Y = -a\alpha^2 \cdot ao \cdot \alpha o = -a\alpha^2 \cdot oX^2$$
$$= -\tfrac{1}{4} \cdot a\alpha^2 \cdot XY^2:$$

that is, product of powers of a, α in respect of XY = squared distance of a, α $\times$ squared distance of X, Y. Hence product of conjugate diameters into sine of angle between them is constant.

Further, by division,

$$\frac{1}{aX \cdot aY} + \frac{1}{\alpha X \cdot \alpha Y} = -\frac{4}{XY^2},$$

therefore sum of squares of conjugate diameters is constant.

Theory of the Linear Relations.

XY is divided harmonically by a and ∞, m is any other point. Then

$$mX \,.\, mY = ma^2 - aX^2 = ma^2 + aX \,.\, aY,$$

or
$$mX \,.\, mY \,.\, \overline{a\infty}^2 = \overline{ma}^2 \,.\, X\infty \,.\, Y\infty + aX \,.\, aY \,.\, \overline{m\infty}^2 :$$

the equation is now homogeneous in each point, and therefore projective. For ∞ we may write α, and a, α will be any two harmonics of X, Y. Then

$$\overline{a\alpha}^2 \,.\, mX \,.\, mY = \overline{ma}^2 \,.\, \alpha X \,.\, \alpha Y + \overline{m\alpha}^2 \,.\, aX \,.\, aY \quad \ldots\ \ldots\ \ldots\ldots \text{(1)}.$$

Let m be at infinity, then

$$a\alpha^2 = \alpha X \,.\, \alpha Y + aX \,.\, aY.$$

Applying (1) to the three points m, $m+n$, n, we have

$$\tfrac{1}{2}a\alpha^2\,(mX \,.\, nY + nX \,.\, mY) = ma \,.\, na \,.\, \alpha X \,.\, \alpha Y + m\alpha \,.\, n\alpha \,.\, aX \,.\, aY.$$

Hence if m, n are conjugate of X, Y, say $m, n = b, \beta$, then

$$\frac{ab \,.\, a\beta}{\alpha b \,.\, \alpha\beta} = \frac{aX \,.\, aY}{\alpha X \,.\, \alpha Y}.$$

But if on the other hand m, n coincide with X, Y respectively,

$$-\tfrac{1}{2}a\alpha^2 \quad XY^2 = 2aX \,.\, aY \,.\, \alpha X \,.\, \alpha Y,$$

or
$$a\alpha^2 \,.\, XY^2 = -\,4aX \,.\, aY \,.\, \alpha X \,.\, \alpha Y.$$

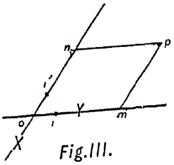


Fig.111.

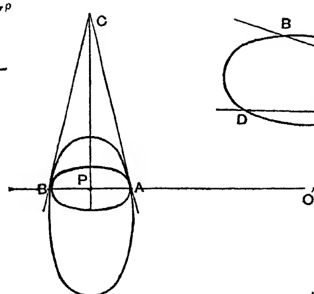
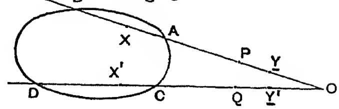


Fig.112.

Fig.113.



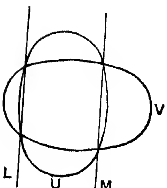


Fig.114.

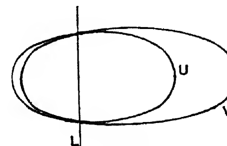


Fig.115.

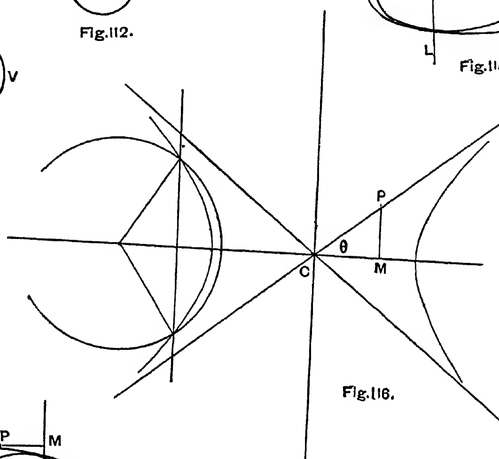


Fig.116.

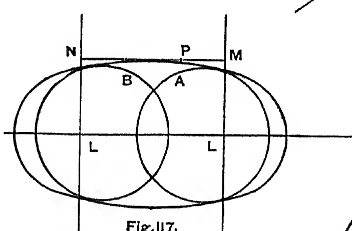
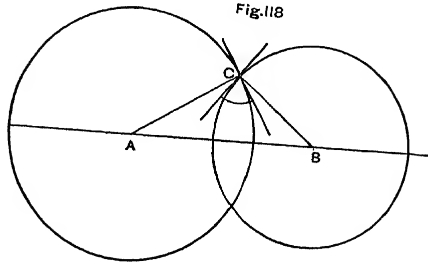


Fig.117.

Fig.118



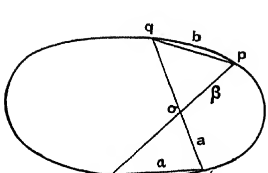


Fig 119.

REVIEWS.

I.

A Budget of Paradoxes. .. By Augustus De Morgan (Reprinted, with the Author's additions, from the Athenæum). London: Longmans. 1872 *.

It is an opinion current among librarians, that there is no such thing as trash: that the most foolish unconnected flysheet treating of nothing at all should in all cases be preserved and bound up with other such flysheets, not in view of any possible future investigator to whom it may be as gold among quartz, but because it is right that this thing should be done. No doubt a very great good comes of this absolute universalism in the conscience of all kinds of collectors. But still, for the purposes of the outer world, it remains true that there are books and books. It is obvious of some kinds of literary or scientific work that if *A. B.* had not done it, *C. D.* would have taken his place; and that at no loss to the world, even though *C. D.* were a person of mean capacities. The bricklayer who was to lay a certain course of bricks may fall off a ladder and yet the house be no worse in the end; while the skilled mason who carves a gargoyle may leave something which represents not merely his day's wages but himself (invaluable) so long as stone shall last, and therein something which no other man could exactly produce. The book before us is essentially a gargoyle. It is by very far the most individual book of the age—individual, not merely in its own singularity as a book, but as presenting with a marked degree of clearness and exactness the personality of one who was never quite a man among men, but always a man among other men.

The paradoxes herein treated of are those set forth by people ignorant of mathematics, who think themselves qualified to shew such as are not ignorant where they have gone astray. It might well have been conceived that a large book on such a subject would have been the dullest of the dull; that it would appeal only to mathematical readers, and even to them only for so long a time as the follies exposed in it were of recent interest. No anticipation could be more thoroughly wrong. The fault of the book is in the direction of a too incessant playfulness; it is excellent grotesque, which is only to be borne because it is so clearly an outwork of the beautiful. For while the pretenders here slaughtered are for the most part indeed nobodies, whose only use is for an example and a warning; while these jokes (with some notable exceptions) are small jokes, and such as we like chiefly in our idiotic moods; while even the character which so clearly shews through these pages, great and lovable

* [From *The Academy*, Vol. IV. No. 78. *August 15th*, 1873.]

as it is, is yet rather singular than pre-eminent, a study for comparative psychology rather than an ideal for the world to come; while all this is true, the book is an endeavour and a stretching forth towards right thinking and a protest against wrong thinking which is of infinite solemnity and weight to us of this present time. For we have no right to conclude that these paradoxers upon whom the *Budget* has fallen have been sinners above all that prate, we ought rather to learn that except we mend our ways we shall all likewise perish.

The word *paradox* is unfortunate; it includes under one name a rare thing and a common thing, and it brings upon the rare thing which is good some of the discredit that belongs to the common thing which is bad. "A paradox is something which is apart from general opinion, either in subject-matter, method, or conclusion." The "general opinion" must be that of people who have an opinion; not of all people indiscriminately, including those who have never considered the subject. The common form of paradox consists in ignorance of the subject-matter, powerlessness in the method, or incapacity to understand the conclusion. The rare form of paradox is an addition to the reasonable part of general opinion, which happens to contradict some of the unreasonable part. The older use of the word was strictly impartial, and it might be applied without any want of respect; De Morgan says the change came in the seventeenth century. It is certain that at present the epithet is a disparaging one; the overplus of wrong thoughts included under it has slowly sapped the moral constitution of the word, and it now sometimes stands in the way of a right appreciation of the nobler form of paradox.

For as it is hardly possible to lay too great a stress on the weight and worthiness of thought which diverges from the general opinion on account of its greater strength, which by its continual work in the world has in fact built up the present mind of man; so it is before all things necessary here to distinguish carefully from it that other divergence which comes of weakness and goes to destruction. It is true in all departments of human action that reform is the most precious and sacred prerogative of a citizen; but in order to have that prerogative one must be a citizen, not an alien; and one must act like a citizen in a legitimate and constitutional way. A man who should find an error in the value of π—even in the six hundredth place—would have all honour paid him as a true reformer by the brotherhood; but to this two things are necessary: he must not be ignorant of trigonometry, and he must work out the calculation. The belief of the weak paradoxer, on the contrary, is that things can be done by a flash; that a discovery is to start from his ignorant and untried mind like Pallas from the brain of Zeus. We know, of course, that the great discoveries—the true and noble paradoxes—have always come from men who by long prenticeship have so far mastered the tools forged by their fathers that they were not tied down to one particular way of using them; we know that Jove's head cannot crack with Minerva unless he have previously swallowed Metis. The time taken by distant discoveries—gravitation for instance, is foreshortened by perspective; but we have good cases immediately before our eyes. In Maxwell's theory of Electricity we have as instructive an example of the paradox of

right thinking as might well be; a conclusive victory over rival doctrines won by twenty years' patient proving (and improving) of the weapons wherewith previous battles had been gained; a testimony to all time that genius is a capacity for taking an infinite amount of the right sort of trouble. But your paradoxer of the *Budget* will master by a *coup-d'état* the republic of science, which allows no masters, but proved comrades only; he will climb by the back stairs into the house of knowledge, that has no back stairs. If there be any reward in the penal incurable blindness that follows such sacrilege, verily he has his reward.

And here is another important difference between the two kinds of heretics. The strong heretic is so because his ideas are living and plastic, and have an internal motion whereby they adapt themselves continually to new work; so that no man is so perfectly open to conviction as he is. But the weak heretic is so from the very narrowness of his range, which cannot grasp even established demonstration; he is hermetically sealed against all possible argumentative germs that might bring into his mind the lower forms of life.

In drawing this sharp distinction between two habits of mind, however, we must not forget what the *Budget* is specially calculated to impress upon us in a terrible and alarming manner; the exceedingly gradual transition from one to the other, and the possible coexistence of both in the same person in regard to different subjects. De Morgan has some very good remarks on the value of a study of logic in helping us to extend the habits of right thinking which we have got by practice in one subject over the whole range of our knowledge. A good specialist who is also a good logician can hardly be betrayed into gross paradoxy out of his proper range; for his special knowledge will make him cautious about facts, and his logic about conclusions. No man could have greater advantages in this respect than the author of the *Budget*, who had himself made important additions to logic, and was an excellent mathematician. And yet—this is the solemn warning of the book—he has in one case fallen into a sin to which we are all tempted, whether by the uncompromising precepts of theological systems, or by the insidious seductions of scientific text-books; the sin of making assumptions and then hiding from ourselves that they are assumptions and that we have no right to believe in them. Apropos of "From Matter to Spirit," he says that he refers certain phenomena "*either* to unseen intelligence *or something which man has never had any conception of.*" This apparently suspended judgment involves and hides the assumption that the said phenomena cannot possibly be referred to certain well known and commonly conceived things—the art of the conjuror, and the delusion of contagious excitement. This enormous assumption is, of course unconsciously, introduced and hidden under a brilliant display of candid impartiality and cautious scepticism. We point to this, not as throwing a stone thereat; but desiring that it should indicate the great and serious importance of the *Budget of Paradoxes*. To sum up, this is a book that should be read by those who care about circle-squarers and all manner of jokes, mathematical and other; by those who care to make the acquaintance of Augustus de Morgan, which it is well worth while to do; but above all by those who care to be led into right thinking and warned from wrong.

II.

A Treatise on some New Geometrical Methods, containing Essays on the Geometrical Properties of Elliptic Integrals, Rotatory Motion, the Higher Geometry, and Conics derived from the Cones; with an Appendix to the First Volume. In Two Volumes. Vol. II. By James Booth LL.D , F.R.S., F.R.A.S., &c., Vicar of Stone, Buckinghamshire. (*pp.* xxxii. + 440) *London: Longmans & Co., Paternoster Row.* 1877*.

If Rip van Winkle, instead of being an idle scapegrace, had been a most original and accomplished geometer; and if, instead of sleeping on the mountains for twenty years, he had from time to time applied himself in a sheltered cave to mathematical pursuits; he might, on rejoining his neighbours in the valley, have produced such a treatise as this. In the meantime, those neighbours have grown a great deal wiser; they know a great deal more than they did when he left them to live in solitude. For one thing, they have learned to appreciate what he did before he went away. And accordingly their first impulse, when they recognise the veteran explorer, is to think how very delighted he will be with what they can teach him. Those who read these pages on the representation and application of elliptic integrals, based on no later authority than the treatise of Verhulst, will think with sympathetic delight of the pleasure with which the author will read the *Fundamenta Nova*, and the setting forth and completion of that theory in Cayley's treatise, to say nothing of the works of Abel and the all-embracing method of Riemann. When he speaks of a hyperelliptic integral as a thing at present wholly beyond the powers of analysis, we at once think of a heap of volumes and memoirs which we must give him to revel in; and we wish we could see his face as he took in the discoveries of Gopel and Rosenhain, of Weierstrass, Hermite, and Königsberger. And so in reading these chapters about the "Higher Geometry" and "Conics," we feel that we should like to introduce him to Reye's *Geometrie der Lage*, and to the admirable volumes of Lindemann; wondering whether he would recognise his own children in their present developed and systematized condition.

But, on examining a little more closely, we find that this is a one-sided view of the matter; and that, instead of thinking of all the beautiful things that we can teach *him*, it would be more profitable to pay attention to many beautiful things which he can teach *us*. Those, indeed, whose knowledge of mathematics is derived from Tripos text-books, and who are accustomed to think of an elliptic differential as a thing whose "integral cannot be found†,"

* [From the *Educational Times*, June, 1877.]

† Cheyne, *The Earth's Motion of Rotation*, 1867, p. 3. Such language is very common; but when it is considered that the θ-series is as legitimate an algebraic form as the exponential of a continuous quantity, it will, we think, appear more proper to say that elliptic differentials have been integrated in finite terms; especially as they are now a part of the university curriculum.

will suppose that they are here reading the latest developments of an obscure subject, and may profit largely by every paragraph. But the freshness and originality of treatment which are here to be found will prove an effective stimulus to those who have endeavoured to keep up with the rapid progress of science. The geometrical applications of Elliptic and Abelian functions which have chiefly interested later geometers are those in which the function of a complex argument is used to represent the aggregate of the real and imaginary points of a curve. Such applications bear upon the projective theory of curves, and it is rarely that any metrical properties emerge from them. But Dr Booth considers the representation of functions of a real argument by the rectification and quadrature of curves; and it is well worth while to be recalled for a time to this aspect of the elliptic functions, which is, of course, the most important for physical purposes. The study of these chapters in the light of later ideas and with the help of later notation cannot fail to be highly profitable. With a small example of this we shall conclude a very meagre notice.

Of the two modes of representation of elliptic integrals which Dr Booth uses, that by the arcs of quadriquadric curves is undoubtedly the more important; but considerable interest attaches to the expression of elliptic integrals of the first kind by the arcs of the negative pedal of an ellipse in regard to the centre. If arc $AP=s$, $ZP=t$, $AC=a$, angle $ACZ=\phi$, then

$$\phi=\text{am}\left(\frac{s-t}{a}, e\right),$$

the modulus e being the eccentricity of the ellipse. The quantity $s-t$ is called the *residual arc*, and a method is given for finding a curve whose residual arc shall represent any given integral. In fact, $s-t=\int p\,d\phi$, if $p=CZ$; so that the tangential polar equation of the curve which represents $\int f(\phi)\,d\phi$ by its residual arc is $p=f(\phi)$. There are two special cases of interest. When $e=0$, the ellipse becomes a circle, and the negative pedal coincides with the circle itself; in this case $t=0$, and the equation becomes $a\phi=s$. If, while CA is kept constant, e is made to increase from 0 to 1, the ellipse will lengthen out until it becomes the pair of tangents at the extremities of the minor axis. The negative pedal then becomes two parabolas, and it is clear that we need only attend to one of them. The elliptic amplitude becomes a function which is called by Cayley the Gudermannian (*Elliptic Functions*, p. 56), and which may be thus defined:—If $\cos\phi\cos(iu)=1$, then $\phi=\text{gd}\,u$. It follows that $iu=\text{gd}(i\phi)$; other formulæ are

$$\tan\phi=i\sin(iu),\quad\text{whence}\quad u=\int_0 \sec\phi\,d\phi=\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right).$$

We have

$$\tan\text{gd}(u+v)=i\sin(iu+iv)=i\sin(iu)\cos(iv)+i\sin(iv)\cos(iu)$$
$$=\tan\text{gd}\,u\sec\text{gd}\,v+\tan\text{gd}\,v\sec\text{gd}\,u.$$

Dr Booth writes this formula thus:—

$$\tan(\phi \perp \chi) = \tan\phi \sec\chi + \tan\chi \sec\phi;$$

so that
$$\phi \perp \chi = \text{gd}(\text{gd}^{-1}\phi + \text{gd}^{-1}\chi);$$

similarly,
$$\phi \top \chi = \text{gd}(\text{gd}^{-1}\phi - \text{gd}^{-1}\chi).$$

Observe that the operations $\perp$, $\top$ satisfy the commutative and associative laws.

This notation appears convenient for some purposes, and the reader will find a number of very interesting developments obtained by means of it. It seems worth considering whether a similar notation might not be applied with advantage to the more general function $\text{am}\, u$, which, like $\text{gd}\, u$, has no proper addition-theorem.

PROBLEMS AND SOLUTIONS *.

1362. For every point A on a rectangular hyperbola, there exists a straight line BC, passing through the centre, such that if through any other point D on the curve, lines be drawn parallel to the asymptotes, cutting BC in B, C, the intercept BC subtends a right angle at A. [March, 1863, solved May, 1863. The proposer's solution, bearing date March 30, 1863, is given Reprint, Vol. XXXII. p. 32.]

Let O be the centre of the hyperbola; join OA, and draw BOC perpendicular to it. Let DB, DC, parallel to the asymptotes, cut them in F, E; and draw AM perpendicular to either of the asymptotes.

Then we have $CO : OB = CE : ED$,

$$\therefore \; CO^2 : CO \,.\, OB = OE \,.\, EC : OE \,.\, ED \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But, by similar triangles OCE, OAM,

$$CO^2 : OE \,.\, EC = OA^2 : OM \,.\, MA \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

therefore, comparing (1) and (2), since

$$OE \,.\, ED = OM \,.\, MA, \quad OA^2 = CO \,.\, OB,$$

or BAC is a right angle.

The property holds for *any* hyperbola, but the line BC does not always pass through the centre; if it cut the asymptotes in P and Q, the angles APQ, AQP, are each equal to the angle between the asymptotes, and to BAC. I have not been able to find a geometrical construction for the line.

1373. Given a circle (C) and *any* point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that, if LEM be drawn cutting the circle in L, M, we may always have $AE^2 = LE \,.\, EM \pm BA \,.\, AD$. [Proposed by T. T. Wilkinson, F.R.A.S. Solution sent 1863, printed in Reprint, Vol. XXXII. p. 22.]

* [From the *Educational Times* and from *Mathematical Questions with their Solutions from the Educational Times*. This latter work, which is cited under the title "Reprint," contains also many papers and solutions not published in the *Educational Times*. Professor Clifford contributed solutions to two questions which appeared in a weekly journal, *The Key*. They will be found in Vol. II. No. 34, August 22nd, p 124, No. 35, August 29th, 1863, p. 140 (this last solution is given in full). The solutions are of no special interest.]

1. For the upper sign, when A is *without* the circle, and the lower, when A is *within*, E lies on the polar of A. [TET': AG perpendicular to CEG.]

For
$$AE^2 = AG^2 + GE^2 = AC^2 - CG^2 + GE^2$$
$$= AC^2 - CG^2 + (CG^2 + CE^2 - 2CE \cdot CG)$$
$$= AC^2 + CE^2 - 2CT^2$$
$$= BA \cdot AD - LE \cdot EM,$$

when E is within the circle, as in the figure; when E is without the circle, $AE^2 = LE \cdot EM + BA \cdot AD$. By interchanging A and E, we get the second case.

2. For the same; E may obviously lie on a circle with CA as diameter.

3. For the upper sign, when A is *within* the circle, and the lower, when A is *without*; bisect CA in P, and with P as centre describe a circle whose radius is $\sqrt{(r^2 - 3CP^2)}$; E may lie on this circle. For, if EQ be perpendicular to AC,

then
$$r^2 - 3CP^2 = PE^2 = PQ^2 + QE^2,$$
or
$$2r^2 - \tfrac{3}{2}AC^2 = CE^2 + AE^2 - 2AP^2,$$
therefore
$$AE^2 = (r^2 - CE^2) \pm (r^2 \mp CA^2) = LE \cdot EM + BA \cdot AD.$$

4. In the same cases, E may evidently lie on a straight line through A perpendicular to CA.

1378. A *tangent* to an ellipse is a *chord* of a *concentric* circle, whose radius is equal to the distance between the ends of the axes of the ellipse; shew that the straight lines which join the ends of the chord to the centre are conjugate diameters. [April, 1863, solved June, 1863. Proposer's solution sent May 14th, 1863, printed in Reprint, Vol. XXXII. p. 31.]

Let the equations to the *ellipse*, the *circle*, and the *chord*, be respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 + y^2 = a^2 + b^2, \quad \frac{xh}{a^2} + \frac{yk}{b^2} = 1 \ldots \ldots \ldots \; (1, 2, 3).$$

Then the equation
$$x^2 + y^2 = (a^2 + b^2)\left(\frac{xh}{a^2} + \frac{yk}{b^2}\right)^2 \ldots\ldots\ldots\ldots (4)$$

represents the two straight lines passing through the origin and the intersections of (2) and (3). If these are conjugate diameters, we must have

$$-\frac{b^2}{a^2} = \frac{b^4}{a^4} \cdot \frac{a^4 - (a^2 + b^2) h^2}{b^4 - (a^2 + b^2) k^2},$$

which may easily be shewn to be the case, since (h, k) is on the ellipse, and therefore $a^2k^2 + b^2h^2 = a^2b^2$.

If we change b^2 into $-b^2$, we obtain a similar theorem for the hyperbola; but the conjugate diameters will be imaginary, if

$$\frac{h^2}{a^4} + \frac{k^2}{b^4} < \frac{1}{a^2 - b^2}.$$

1379. [If a curve of the third order have a double point A, and be cut by any straight line in B, C, D; and if, when ABC is taken as triangle of reference, the tangents at A are represented by the equation

$$P\beta^2+Q\beta\gamma+R\gamma^2=0,$$

and the tangents at B, C, by the equations

$$P\alpha+N\gamma=0, \text{ and } M\beta+R\alpha=0;$$

shew that the equation to the straight line AD is

$$N\beta+M\gamma=0,$$

and find the equation of the curve. Proposed by * * *, April, 1863, solved June, 1863.]

Since $P\beta^2+Q\beta\gamma+R\gamma^2=0$ and $P\alpha+N\gamma=0$ are the tangents at the points where $\gamma=0$ meets the curve, its equation must be of the form

$$(P\beta^2+Q\beta\gamma+R\gamma^2)(P\alpha+N\gamma)=\gamma^2\phi;$$

also of the form

$$(P\beta^2+Q\beta\gamma+R\gamma^2)(M\beta+R\alpha)=\beta^2\chi.$$

It will be found that the equation

$$\alpha(P\beta^2+Q\beta\gamma+R\gamma^2)+\beta\gamma(N\beta+M\gamma)=0 \;\ldots\ldots\ldots\ldots\ldots\; (1)$$

is of both these forms. For clearly the lines $P\beta^2+Q\beta\gamma+R\gamma^2=0$ only meet the curve at the point $(\beta\gamma)$, and the lines $P\alpha+N\gamma=0$, $M\beta+R\alpha=0$, *touch* the curve at the points $(\alpha\gamma)$, $(\alpha\beta)$. It is now obvious that the other point where α meets the curve is in the line $N\beta+M\gamma=0$, which is therefore the line AD.

We may notice that the tangents drawn from D to the curve are represented by

$$(Q\pm 2\sqrt{PR})\,\alpha+N\beta+M\gamma=0 \;\ldots\ldots\ldots\ldots\; (2);$$

there are only two because there is a double point. The lines drawn from A to their points of contact are represented by $\beta\sqrt{P}\pm\gamma\sqrt{R}=0$; hence these form an *harmonic* pencil with AB, AC.

The equation to the tangent at D is

$$N^2M\beta+M^2N\gamma=(M^2P+N^2R-MNQ)\,\alpha \;\ldots\ldots\ldots\ldots\; (3),$$

and that to the line joining A with the other point where (3) meets the curve is $MP\beta+NR\gamma=0$; hence the condition that D may be a *point of inflexion* is

$$PM^2=RN^2.$$

1387. Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be *opposite* intersections of the tangents, shew that OA and OB are equally inclined to the tangent at O to the ellipse. [May, 1863, corrected to the above form in July, 1863, solved September, 1863: Reprint, Vol. I. pp. 19, 33.]

We use *rectangular tangential* coordinates (Ferrers, *Tril. Co.*, p. 130; Salmon, *Higher Plane Curves*, p. 2). It is easily shewn that the sum of the squares of the reciprocals of the intercepts made by any tangent to a circle on two diame-

ters at right angles is constant. Hence the equation to a circle whose centre is the origin is

$$\xi^2 + \eta^2 = c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The points $\xi = 0$, $\eta = 0$, are at an infinite distance, one on each of the axes; and $k = 0$ (where k is a constant) represents the origin. From this it follows that the equation

$$\xi^2 + bk\xi + ck\eta + dk^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(where the k may be left out at pleasure) represents a conic touching the axis of ξ at the origin. For if we seek the tangents drawn from $k = 0$ to the curve, we find that they both coincide with the line $k\xi$, that is with the axis of ξ. Now if we put

$$\xi^2 + \eta^2 - c^2 \equiv S, \quad \xi^2 + b\xi + c\eta + \delta \equiv T,$$

it is clear that the equation $S + \lambda T = 0$ represents an envelope of the second class, touching all the common tangents of S and T. The discriminant of this equation is of the third degree in λ; hence there are three values of λ for which $S + \lambda T = 0$ represents two points. But in every case the coefficient of $\xi\eta$ is zero; which is just the condition that the line joining the origin to the two points (which are evidently opposite intersections of the common tangents) should be equally inclined to the axis of ξ. For if $a\xi + b\eta = 1$ be the equation of a point, (a, b) are its ordinary rectangular coordinates, and $(b : a)$ is the tangent of the angle which the line joining it to the origin makes with the axis of ξ. Hence if two points (a, b) and (c, d) are equally inclined to the point ξ, we must have

$$\frac{b}{a} = -\frac{d}{c}, \text{ or } ad + bc = 0;$$

but $(ad + bc)$ is the coefficient of $\xi\eta$ in the product $(a\xi + b\eta - 1)(c\xi + d\eta - 1)$. The theorem is therefore proved.

It will be observed that the discriminant being of the third degree in λ, must always have one real root; but there will be four real common tangents only when the conic is an ellipse cutting the circle in four points.

It appears therefore that *any* two conics have two *real* intersections of real or imaginary common tangents, corresponding to the centres of similitude of two circles.

By projection we may shew that "If a straight line A join the poles of B with respect to two conics, then the lines joining AB to a pair of opposite intersections of common tangents, form, with A, B, an harmonic pencil."

And by reciprocation,—"If a point A be the intersection of the polars of B with respect to two conics, and AB be cut by a pair of common chords in C, D, then $ACBD$ is an harmonic range."

[A solution of this "elegant theorem," "included as a particular case in the known theorem—'given three conics inscribed in the same quadrilateral, the tangents from any points to these conics form a pencil in involution'"—is given by Prof. Cayley in the same volume of the Reprint on page 33.]

1389. [A curve of the third order, consisting of three symmetrical branches, is drawn so as to touch the sides of an equilateral triangle at their middle points.

These three points are joined so as to form a new equilateral triangle. Shew that if PA, PB, PC be the perpendiculars from any point P on the curve upon the sides of one equilateral triangle, and PD, PE, PF the perpendiculars from the same point on the sides of the other equilateral triangle, then the ratio vol. $PA \cdot PB \cdot PC$: vol. $PD \cdot PE \cdot PF$ is constant, wherever P be taken on the curve. Proposed by * * *. Reprint, Vol. I. p. 10, *E. T.* July, 1863.]

The general equation to a cubic touching $B+C=0$, $C+A=0$, $A+B=0$, where they meet $A=0$, $B=0$, $C=0$, is evidently

$$ABC = k(B+C)(C+A)(A+B) \;\ldots\ldots\ldots \quad (1).$$

In the case supposed, let $B+C \equiv \alpha$, $C+A \equiv \beta$, $A+B \equiv \gamma$, then (1) becomes

$$(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)(-\alpha+\beta+\gamma) = k\alpha\beta\gamma,$$

which expresses the property in question.

The asymptotes are parallel to $\alpha\beta\gamma$, and there are three points of inflexion, all at an infinite distance.

The proof holds if the cubic touch the sides of any triangle in three points, such that the lines joining them to the opposite vertices meet in a point.

1399. From a point A two chords are drawn meeting a conic section in four points B, joined also by four straight lines a. These intersect two and two in two points P lying on the polar of A. At the points B are drawn four tangents b, which intersect in six points, two of which are on the polar of A, and the others lie two and two on the two straight lines AP. These tangents intersect the original chords in four points, which may be joined by four straight lines intersecting by pairs in the points P. The lines a and b intersect in eight points C, which may be joined by twenty lines c; four of these pass through A, and the others may be divided into groups of four. Each group has six intersections, two of which lie on the polar of A, and the others lie two and two on lines through A. Any two groups intersect in eight points, having properties like those of the points C. [June, 1863. Reprint, Vol. I. p. 14, *E. T.* August, 1863.]

Take the chords through A for the sides β, γ of the triangle of reference, one of the straight lines a as the side α; and let another of them be represented by

$$l\alpha + m\beta + n\gamma \,(=\delta) = 0.$$

Let the equation to the conic be

$$\alpha\delta = k\beta\gamma \;\ldots\ldots\ldots \quad (1),$$

thus the four lines a are

$$\left.\begin{array}{l} a_1 \ldots\ldots\ldots \alpha = 0 \\ a_2 \ldots\ldots\ldots \delta = 0 \\ a_3 \ldots\, l\alpha + m\beta = 0 \\ a_4 \ldots\, l\alpha + n\gamma = 0 \end{array}\right\} \;\ldots\ldots\ldots \quad (2).$$

The polar of A is evidently

$$2l\alpha + m\beta + n\gamma = 0 \;\ldots\ldots\ldots \quad (3).$$

The equations to the tangents b are

$$\left.\begin{aligned} b_1 \; &..m\alpha - k\gamma = 0 \\ b_2 \; &.\; n\alpha - k\beta = 0 \\ b_3 \; &.m\delta + lk\gamma = 0 \\ b_4 \; &.n\delta + lk\beta = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

For instance, assume $\delta = p\gamma$ for the equation to b_3; then from (1) we find that this meets the curve where it meets $p\alpha = k\beta$; but as b_3 is a *tangent*, this must coincide with a_3, or $mp = -lk$; and thus b_3 becomes

$$m\delta + lk\gamma = 0.$$

The intersections of $b_1 b_2$ and of $b_3 b_4$ lie on $m\beta - n\gamma = 0$; of $b_1 b_4$ and of $b_2 b_3$ on $m\beta + n\gamma = 0$; and of $b_1 b_3$ and $b_2 b_4$ on $2l\alpha + m\beta + n\gamma = 0$.

The equations of the four lines joining intersections of the lines b with β and γ are

$$\left.\begin{aligned} mn\delta + lk\,(m\beta + n\gamma) &= 0 \\ mn\alpha - k\,(m\beta + n\gamma) &= 0 \\ mn\,(l\alpha + n\gamma) - lk\,(m\beta - n\gamma) &= 0 \\ mn\,(l\alpha + m\beta) + lk\,(m\beta - n\gamma) &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The first pair meet $2l\alpha + m\beta + n\gamma = 0$ where it meets $m\beta + n\gamma = 0$; the second pair where it meets $m\beta - n\gamma = 0$.

The eight points C may be represented as follows:

$$\begin{array}{cccccccc} a_4 b_3, & a_3 b_4, & a_2 b_2, & a_1 b_3, & a_3 b_1, & a_4 b_2, & a_1 b_4, & a_2 b_1 \\ E & F & G & H & K & L & M & N. \end{array}$$

With this notation, the five groups of lines C are

(EK, FL, GM, HN); (EF, GH, KL, MN); (EG, HL, KM, NF);

(EM, FH, NL, GK); (EM, FG, ML, KH).

All the lines of the first group pass through A, and have for equations

$$\left.\begin{aligned} EK, \ldots\ldots\ldots\; & m^2\beta + kl\gamma = 0 \\ FL, \ldots\ldots\ldots\; & n^2\gamma + kl\beta = 0 \\ GM, \ldots\; & n^2\gamma + (mn + kl)\,\beta = 0 \\ HN, \ldots & m^2\beta + (mn + kl)\,\gamma = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

which may be easily verified.

As an example of the others, take the equations of the lines of the fourth groups:

$$\left.\begin{aligned} &EM, \ldots (l^2k^2 + 2klmn)\,(l\alpha + n\gamma) = (lk + mn)\,(m\delta + kl\gamma)\,n \\ &FH, \ldots (l^2k^2 + 2klmn)\,(l\alpha + m\beta) = (lk + mn)\,(n\delta + kl\beta)\,m \\ &NL, \ldots\, k\,(mn + kl)\,\delta = mn^2\,(m\alpha - k\gamma) \\ &GK, \ldots\, k\,(mn + kl)\,\delta = m^2 n\,(n\alpha - k\beta) \end{aligned}\right\} \ldots\ldots\ldots\ldots (7).$$

The intersections of EM and FH, and of NL and GK, lie on $m\beta - n\gamma = 0$; those of EM and NL, and of FH and GK, on $m\beta + n\gamma = 0$; those of EM and GK, and of NL and FH, on $2l\alpha + m\beta + n\gamma = 0$; and so with each of the other groups.

I was wrong in saying that *any* two groups intersect in eight points, &c.; this is true of the last four, for it will be found that any two of these form two quadrilaterals, the vertices of one resting on the sides of the other, two diagonals of each passing through A, and the others being identical $(2l\alpha+m\beta+n\gamma=0)$; hence, by the converse of the first part of the equation, a conic may be inscribed in one so as to circumscribe the other, and the preceding reasoning applies. But the first group is an exception; there are only four new points formed by combining it with any of the others, and these may be joined by four lines meeting two and two in the polar of A.

All these theorems may be proved very readily by projecting the conic into a circle whose centre is the projection of A.

1393. [A shell formed of two equal paraboloids of revolution, having a common axis, is fixed with its vertex downwards, and axis vertical; and a heavy uniform rod of given length rests within it, in a vertical plane through the axis. Compare the pressures on the lower surface of the shell. Proposed by Mr J. R. Wilson, Jesus College, Cambridge, Reprint, Vol. I. p. 27, *E. T.* September, 1863.]

Let QPR be the rod, of length $4c$; draw tangents QT, RT, and normals QO, RO, to the outer parabola. We know by Geometry that $QP=PR$, and therefore PT is vertical. Now the rod is kept at rest by four forces, two of which, viz., gravity and the resistance at P, pass through P; therefore the resultant of the pressures at Q, R acts along OP. But OP, bisecting QR, is half the diagonal of the completed parallelogram (OQ, OR); hence the resistances at Q, R, are as the normals OQ, OR; that is, as $\sin ORQ : \sin OQR$, or as $\cos TRP : \cos TQP$. Now draw a tangent (ECF) to the outer parabola at the point (C) where TP meets it; then, putting $AB=k$, $4a$=principal parameter of BRQ, and $\angle ECP=\theta$, the equation of BRQ, referred to CP, CQ, will be

$$y^2=4a\operatorname{cosec}^2\theta\,.\,x \qquad\text{(1)}.$$

At the point P, $x=k$, $y=2c$,

$$\therefore\ c^2\sin^2\theta=ak \qquad\text{(2)}.$$

The tangents QT, RT are represented by

$$k^2y^2=c^2(x+a)^2 \qquad\text{(3)}.$$

Therefore by the usual formula for oblique axes

$$\frac{\cos TFC}{\cos TEC}=\left(\frac{c-k\cos\theta}{c+k\cos\theta}\right)\left(\frac{c^2+k^2+2ck\cos\theta}{c^2+k^2-2ck\cos\theta}\right)^{\frac{1}{2}}.$$

This, therefore, is the ratio required.

[The following general remarks accompany a solution of 1418, Reprint, Vol. I. p. 30, *E. T.* November, 1863.]

Consider any two conics, U, V; any point (ξ, η, ζ) has two polars, ΔU, ΔV, where Δ stands for

$$\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}.$$

These meet in a point which we shall call the *polar opposite* of (ξ, η, ζ). Similarly, the line joining the poles of a right line may be called its *polar opposite*.

Now consider the equations $S=0$, $S=LM$, where L, M are common chords.

The polars are $\Delta S=0$, $\Delta S=L\Delta M+M\Delta L$.

If $\Delta L=0$, these meet on L; that is,

(α) If any point lie on a common chord, its polar opposite lies on the same chord.

If also $\Delta M=0$, they coincide, or

(β) The intersection of a pair of common chords has only one polar. It is easily shewn that this is a line joining intersections of common tangents.

It follows that

(γ) The polar opposite of any point in a straight line with two opposite intersections of common tangents is an intersection of common chords.

Let $\Delta(L+KM)=0$, then the polars meet on $L-KM=0$. That is

(δ) Lines joining two polar opposites to an intersection of common chords, form, with the chords, an harmonic pencil.

Next, let the equations be $LM+N^2$, $LM+R^2$, so that L, M are common tangents. The polars are now

$$\left.\begin{aligned}(L\Delta M+M\Delta L)+2N\Delta N=0\\(L\Delta M+M\Delta L)+2R\Delta R=0\end{aligned}\right\}.$$

If $\Delta N=0$, these intersect on R, or

(ϵ) If a point lie on one chord of contact of a pair of common tangents, its polar opposite lies on the other.

If $\Delta(N+KR)=0$, the polars meet on $R+KN=0$; or

(ζ) If the locus of a point is a line through the intersection of the chords of contact of a pair of common tangents, the locus of its polar opposite is another line through the same intersection.

Thirdly, consider the case of double contact, S, $S+L^2$. Here the polars are ΔS, $\Delta S+2L\Delta L$. These always meet on L, shewing that

(η) If two conics have double contact, the polar opposite of any point whatever lies on the chord of contact.

If $\Delta L=0$, they coincide, or

(θ) A point on the chord of contact has only one polar, which is also the locus of its polar opposites.

(ι) In general, if the locus of a point be a straight line,

$$l\xi+m\eta+n\zeta=0,$$

the locus of its opposite is the conic

$$\begin{vmatrix}\dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz}\\[2ex] \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz}\\[2ex] l, & m, & n\end{vmatrix}=0,$$

which we may call the *polar conic* of the line (lmn). As the discriminant is of the third degree in (lmn), it appears that the envelop of lines whose polar conics break up into two right lines is a curve of the third class.

1409. For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary. [July, 1863. Reprint, Vol. I. p. 33, *E. T.* December, 1863.]

Take the point A for origin, and the rectangular tangential equation used in Question 1387 [cf. supra], but in the more convenient form

$$(\xi - a)^2 = 4b\,(\eta - c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is evidently equivalent to the one there given.

The line BC is represented by

$$\xi - a = 0, \quad = \eta - b - c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

it always passes through the pole of the normal, and is in fact the polar of the point of intersection of chords subtending a right angle at A.

If we assume for the general equation of a point on the curve

$$\xi - a = m\,(\eta - c) + \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

then the equation

$$\{m\,(\eta - c) + \beta\}^2 = 4b\,(\eta - c)$$

must have equal roots for η, which gives $\beta = \dfrac{b}{m}$. We shall call this the point m.

Let the intersection of BD, CE, be the point m_1, and the points D, E, m_2, m_3. The equations of B, C, will therefore be

$$\left(m_2 + \frac{1}{m_2}\right)\{(\xi - a) - m_1\,(\eta - b - c)\} = \left(m_1 + \frac{1}{m_1}\right)\{(\xi - a) - m_2\,(\eta - b - c)\} \ldots\ldots (4),$$

$$\left(m_3 + \frac{1}{m_3}\right)\{(\xi - a) - m_1\,(\eta - b - c)\} = \left(m_1 + \frac{1}{m_1}\right)\{(\xi - a) - m_3\,(\eta - b - c)\} \ldots\ldots (5).$$

These equations may be easily verified.

The angle between AB and the axis of ξ is therefore

$$\tan^{-1}\frac{m_2\left(m_1 + \dfrac{1}{m_1}\right) - m_1\left(m_2 + \dfrac{1}{m_2}\right)}{\left(m_2 + \dfrac{1}{m_2}\right) - \left(m_1 + \dfrac{1}{m_1}\right)} = \tan^{-1}\frac{m_1 + m_2}{m_1 m_2 - 1}.$$

Hence the angle BAC is

$$\tan^{-1}\frac{m_1 + m_2}{1 - m_1 m_2} - \tan^{-1}\frac{m_1 + m_3}{1 - m_1 m_3}.$$

That is, its tangent is equal to that of the angle between the lines joining the points m_2, m_3 to the origin. For the latter angle is clearly

$$\tan^{-1}\frac{m_2 - m_3}{1 + m_2 m_3}.$$

Since, therefore, the angles BAC, DAE have their tangents equal, they are either equal or supplementary. If the conic be a circle, it is easily seen that the line BC is always at an infinite distance.

[A solution of this "very elegant theorem" is given by Prof. Cayley on p. 40 of the same volume.]

1442. [The same circle around the origin being employed in the operations of reciprocation and inversion, shew that the first positive and negative pedals of a given curve coincide, respectively, with the inverse of its reciprocal, and with the reciprocal of its inverse; further, that the reciprocal of the n^{th} pedal is the $(-n)^{\text{th}}$ pedal of the reciprocal, and the $(-n-1)^{\text{th}}$ pedal of the inverse of the primitive; and lastly, that the inverse of the n^{th} pedal is the $(-n)^{\text{th}}$ pedal of the inverse, and hence also the $(-n+1)^{\text{th}}$ pedal of the reciprocal of the primitive. Proposed by Dr Hirst, F.R.S. Reprint, Vol. I. pp. 41—3, *E. T.* January, 1864.]

1. Writing J for the operation of inversion, R for that of reciprocation, and P for that of taking the pedal, we have

$$J^2 = R^2 = 1,$$
$$P = JR;$$
$$JP = J^2R = R,$$
$$RJP = R^2 = 1,$$
$$RJ = P^{-1}.$$

2. These, then, are the laws of combination of the symbols R, J, P. We can now immediately prove the theorems in the question. For

$$R \,.\, P^n = R \,.\, (JR)^n = (RJ)^n \,.\, R = P^{-n}R = P^{-n} \,.\, RJ \,.\, J = P^{-n-1} \,.\, J \quad \ldots \ldots (1),$$
$$J \,.\, P^n = J\,(JR)^n = J^2R \,.\, (JR)^{n-1} = R\,(JR)^{n-1}\,J \,.\, J = (RJ)^n \,.\, J = P^{-n} \,.\, J$$
$$= P^{-n} \,.\, JR \,.\, R = P^{-n+1} \,.\, R \quad \ldots \ldots (2).$$

And we may write down any number of formulæ by this method. For instance, the identities

$$(JR)^n \,.\, J\,(RJ)^m = (JR)^{n+m} \,.\, J = J\,(RJ)^{n+m} \quad \ldots \ldots (3),$$
$$(RJ)^n \,.\, R\,(JR)^m = (RJ)^{n+m} \,.\, R = R\,(JR)^{n+m} \quad \ldots \ldots (4),$$

may be thus interpreted:

"The $(n)^{\text{th}}$ pedal of the inverse of the $(-m)^{\text{th}}$ pedal is the $(n+m)^{\text{th}}$ pedal of the inverse, and the inverse of the $(-n-m)^{\text{th}}$ pedal; and the $(-n)^{\text{th}}$ pedal of the reciprocal of the $(m)^{\text{th}}$ pedal is the $(-n-m)^{\text{th}}$ pedal of the reciprocal, and the reciprocal of the $(n+m)^{\text{th}}$ pedal."

3. Again, any formula may be transformed by interchanging R and J, and reversing the signs of all the indices of P. To derive in this way the second pair of theorems from the first, we shall have to make a further change from n to $-n$.

The formulæ (3) and (4) are immediately convertible.

4. The theory of Derived Surfaces and Curves is simply that of the interpretation of symbols. Let any straight line meet two rectangular axes Ox, Oy

in A, B, and draw OP perpendicular to AB, and PM, PN perpendicular to the axes. Then we have two systems of coordinates; (1) when $\frac{1}{OA}$, $\frac{1}{OB}$ are the coordinates of the point P, (2) when PM, PN are the coordinates of the line AB. The formulæ of transformation, between the first and Cartesian, and between the second and Tangential, coordinates, are

$$(\xi^2+\eta^2)(x^2+y^2)=1,\ \xi y=\eta x \qquad (5).$$

These represent the operation of inversion in the two cases. But it is important to remember that, in *Tangential inversion*, the tangents, not the points, are inverted; that is, to every tangent of the primitive corresponds a line parallel to it, such that the rectangle under their distances from the origin is constant. Now let $U=0$ be an equation in x, y, and constants; and let CU denote the curve which is represented by $U=0$, when we interpret x, y as Cartesian coordinates, TU when we interpret x, y as *Tangential* coordinates, MU according to the *first* system of this article, and NU according to the *second*. Then, for instance, $TU=R.CU$, or, by separation of symbols, $T=RC$. In this way we have the equations

$$M=JC,\ N=RJC=RM=RJRT,$$

which serve to connect any two systems.

5. It appears from (4) that if the equation of any curve be written

$$u_n+u_{n-1}+\ldots+u_2+u_1+u_0=0,$$

then the equation of the inverse is

$$u_n+u_{n-1}(x^2+y^2)+u_{n-2}(x^2+y^2)^2+\ldots+u_0(x^2+y^2)^n=0.$$

This is of degree $2n$ in general, but reduces when the curve is circular, and when the origin is on the curve. If the curve be circular in the degree f, that is, if its equation be of the form

$$v_{n-2f}(x^2+y^2)^f+v_{n-2f+1}(x^2+y^2)^{f-1}+\ldots+u_1+u_0=0,$$

the degree of the inverse is reduced $2f$, and if the origin be a multiple point of the order g, or if

$$u_0=u_1=\ldots=u_{g-1}=0,$$

the degree is reduced by g. Hence generally the degree of the inverse is $2(n-f)-g$. It follows by reciprocation that if n is the class of any curve, and if the lines joining the origin to the circular points at infinity are multiple tangents of the order f, and if the line at infinity is a multiple tangent of the order g, then the degree of the first positive pedal is $2(n-f)-g$.

And again, if a curve has g points at infinity distinct from the two circular points at infinity, and has a multiple point of the order ϕ at the origin, being of degree v, then the *class* of the first negative pedal is $v-\phi+g$. This is easily obtained by inverting the result just proved; it being remarked that the inverse of a curve circular in the degree f has a multiple point of the order $n-2f$ at the origin. The *degree* of the negative pedal is the *class* of the inverse, and consequently is the same as the number of circles which can be drawn through

an arbitrary point (ξ, η) and the origin to touch the primitive curve. To find this number, we must eliminate between

$$U=0 \quad \dots\dots\dots\dots\dots\dots (1),$$

$$x^2+y^2+2Ax+2By=0 \quad \dots\dots\dots\dots\dots\dots (2),$$

$$(x+A)\frac{dU}{dx}=(y+B)\frac{dU}{dy} \quad \dots\dots\dots\dots\dots\dots (3),$$

A and B being converted by the linear relation

$$\xi^2+\eta^2+2A\xi+2B\eta=0 \quad \dots\dots\dots\dots\dots\dots (4).$$

The degree of the eliminant in A and B, which is the degree of the first negative pedal, is *in general* $n(n+2)$, but will of course be reduced by peculiarities in the form of U.

1319. [It is announced at p. 205, Vol. II., 12th ed., Davies's Hutton, that "if a tetrahedron be drawn, formed of four tangent planes to a paraboloid, the sphere described about it will pass through the focus of the paraboloid." Prove or disprove this. Proposed by N'Importe. Reprint, Vol. I. p. 45, *E. T.* February, 1864.]

The statement is not true.

If perpendiculars be drawn from the foci of a conicoid of revolution on any tangent plane, the rectangle of these perpendiculars is equal to the square of the minor axis. If then a conicoid of revolution having foci $\alpha\,\beta\,\gamma\,\delta$, $\alpha_1\,\beta_1\,\gamma_1\,\delta_1$, touch the faces of the fundamental tetrahedron, we must have

$$\alpha\alpha_1=\beta\beta_1=\gamma\gamma_1=\delta\delta_1=b^2.$$

So that if one of the foci lies in the plane

$$l\alpha_1+m\beta_1+n\gamma_1+r\delta_1=0,$$

the locus of the other will be the surface of the third degree

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}+\frac{r}{\delta}=0 \quad \dots\dots\dots\dots\dots\dots (1),$$

which is otherwise interesting (Frost and Wolstenholme's *Solid Geometry*, p. 289). A particular case is when the surface of revolution is a paraboloid, one of whose foci lies on the plane at infinity,

$$A\alpha+B\beta+C\gamma+D\delta=0,$$

and the other on the surface

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}+\frac{D}{\delta}=0 \quad \dots\dots\dots\dots\dots\dots (2),$$

where $A\ B\ C\ D$ are the faces of the tetrahedron.

Now if the theorem of Davies's Hutton were true, we should have found for the locus the equation of the circumscribing sphere.

I write down one or two other instances of the application of this principle. (See Salmon's *Conics*, 4th ed., p. 261, Ex. 13, 15.)

Given five planes connected by the identical relation

$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0,$$

the foci of any conicoid of revolution touching the "frustum" will lie in the surface of the fourth degree,

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}+\frac{e}{\epsilon}=0.$$

Given one focus and the intersection of the focal tangents of a parabola of the third class inscribed in the triangle of reference; the other focus moves on a conic (which is *never* a circle) circumscribing the triangle.

Given four tangents, a focus, and the intersection of the focal tangents, in a curve of the third class; the other two foci move on a curve of the third degree.

The general extension is sufficiently obvious.

1421. [If by the Harmonic centre, relative to a fixed plane, of A, C, points in a line meeting the fixed plane in D, be understood a point B between A and C, such that A, B, C, D form an harmonic system; prove that if through the harmonic centre of either diagonal of any of the three quadrilateral faces of the frustum of a triangular pyramid, and the harmonic centres of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will all pass through one and the same point. Proposed by Professor Sylvester, F.R.S. Reprint, Vol. I. pp. 45, 46, *E. T.* February, 1864.]

The proposer has shewn, in the *Philosophical Magazine* for September, that the theorem is true when we put the arithmetical centre for the harmonic. His proof, by Cartesian coordinates, is exceedingly simple and elegant; but it will be found that the attempt to prove the same case by quadriplanar coordinates involves an enormous amount of algebraic work. However, it is clear that if the requisite operations were performed, the result must be the same as by the other method. Now when we find the arithmetic centre of a line by quadriplanar coordinates, the process is simply to find its harmonic centre with respect to the plane at infinity $A\alpha+B\beta+C\gamma+D\delta=0$. But the proof can make no mention of the meaning of $ABCD$, since the thing proved is general for any tetrahedron, and does not depend at all upon the areas of the faces. *Therefore the proof holds, whatever interpretation we give to the symbols $ABCD$*; that is to say, whatever plane is represented by

$$A\alpha+B\beta+C\gamma+D\delta=0.$$

This principle is evidently identical with the method of Projections in plane geometry. (See Salmon's *Higher Plane Curves*, Art. 246). Quadriplanar equations not connected with the absolute form of the fundamental tetrahedron, will hold good whatever tetrahedron we choose. Now it is analytically possible to choose a tetrahedron with reference to which a given conicoid shall be represented by a given equation. For a conicoid is determined by nine conditions, but each of the four planes involves three independent constants. We may, therefore, in addition, choose the tetrahedron so that the plane at infinity shall be represented by a given equation. Thus any property proved of any one conicoid and a plane, when expressed in quadriplanar coordinates, is true of any other conicoid and plane. The only limitation is that connecting

properties which can be expressed in the coordinates will be retained in transformation. Again, we may analytically transform real lines and planes into imaginary, and *vice versâ*, without loss of continuity. Now, ruled conicoids only differ from others in containing real lines instead of imaginary; therefore, *the distinction between ruled and unruled conicoids is lost in transformation.*

In studying, then, the properties of any figure, the principal point will be the reduction of the figure to what (by an extension of the term) may be called the *canonical form.* For instance, the canonical form of a quadrilateral is a parallelogram; of a conic, a circle; of a conicoid, a sphere; and so on. In particular, we wish to find the canonical form of a tetrahedral frustum, which is the figure formed by the intersection of five planes $ABCDE$. For a quadrilateral $ABCD$ we proceed in this way; we join the vertex $\widehat{AB}$ to the vertex $\widehat{CD}$, and project the joining line to infinity. Hence by analogy in the frustum, we join the vertex $\widehat{ABC}$ to the edge $\widehat{DE}$, and project the joining plane to infinity. The figure is thus reduced to three parallel straight lines cut by two parallel planes. As an instance of the use of these canonical forms, we give the following properties, which may be easily proved: "From a point O, three chords o are drawn to a conicoid, meeting it in six points A. These may be joined again by four pairs of planes α, each pair intersecting in one of four lines β on the polar plane of O. At the points A six tangent planes b are drawn; if any three of these (whose points of contact are not in one plane through O) intersect in X and the other three in Y, then X, Y, O are in a straight line. Again, the planes b will intersect by pairs in three lines γ on the polar plane of O, and these will pass through the three intersections of one of the lines β with the other three. If the chords o cut the polar plane in three points, these will lie in three straight lines through the same intersections. There are thus three coaxal triangles on the polar plane, and their common pole is on the line joining O and two of the intersections XY. The tangent planes b cut the chords o in twelve new points C, four of which lie on each chord. Consider the eight C-points lying on two chords o; they may be divided into two groups, each group having two points on each of the chords. Lines joining points in *either* group intersect on the polar plane of O, but one group has for these two intersections, (1) a vertex of one of the three co-polar triangles, (2) the point where the opposite side cuts the common axis. Points in different groups may be joined by eight lines, intersecting in four points lying on the polar plane of O, and eight points lying on four lines through O, and so on.

"If a straight line be drawn through the vertex of either of the common tangent cones of two conicoids having double contact, to meet either of the planes of common section, and the two conicoids, it will be cut in involution, so that the equal anharmonic ratios of the involution are constant."

1443. [Shew that the locus of the centres of all the conics circumscribing a given quadrilateral is an ellipse if the quadrilateral is re-entrant, and an hyperbola if it is convex. Shew further that two real parabolæ may always be drawn through the angles of any convex quadrilateral. Proposed by Prof. Sylvester, F.R.S. Reprint, Vol. I. pp. 51—54, *E. T.* March, 1864.]

1. First, the locus of the centre is a conic.

Let U, V be the tangential equations of two conics through the four points; then the general equation of a conic through the points is

$$l^2pU + lmF + m^2qV = 0 \text{ ....} (1),$$

where p, q are the discriminants of U, V and F is the conic touched by all the tangents to U and V at the four points of intersection; thus, if

$$U \equiv ax^2 + by^2 + cz^2, \qquad V \equiv a'x^2 + b'y^2 + c'z^2,$$

then $$F \equiv aa'(bc' + b'c)\,x^2 + bb'(ca' + c'a)\,y^2 + cc'(ab' + a'b)z^2.$$

Now let (ξ, η, ζ) be a fixed straight line, and write Δ for

$$\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz};$$

then the pole of (ξ, η, ζ) with respect to (1) is

$$l^2p\Delta U + lm\Delta F + m^2q\Delta V = 0 \text{ ... ...} (2),$$

whose locus is $4pq\Delta U \,.\, \Delta V = (\Delta F)^2$, a conic section.

We obtain the locus of *centres* by simply putting $\xi = \eta = \zeta$.

2. Solution by *trilinear* coordinates.

The equation to the conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \text{ .. ..} (3),$$

subject to the condition

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0. \text{ ....} (4).$$

The polar of a point (ξ, η, ζ) with respect to (3) is

$$\frac{\alpha}{\xi}\left(\frac{m}{\eta} + \frac{n}{\zeta}\right) + \frac{\beta}{\eta}\left(\frac{n}{\zeta} + \frac{l}{\xi}\right) + \frac{\gamma}{\zeta}\left(\frac{l}{\xi} + \frac{m}{\eta}\right) = 0.$$

If this coincide with a fixed line $x\alpha + y\beta + z\gamma = 0$, we must have

$$\frac{\frac{m}{\eta} + \frac{n}{\zeta}}{\xi x} = \frac{\frac{n}{\zeta} + \frac{l}{\xi}}{\eta y} = \frac{\frac{l}{\xi} + \frac{m}{\eta}}{\zeta z} = \frac{\frac{l}{\xi} + \frac{m}{\eta} + \frac{n}{\zeta}}{\frac{1}{2}(\xi x + \eta y + \zeta z)};$$

$$\therefore \quad \frac{\frac{l}{\xi}}{-\xi x + \eta y + \zeta z} = \frac{\frac{m}{\eta}}{\xi x - \eta y + \zeta z} = \frac{\frac{n}{\zeta}}{\xi x + \eta y - \zeta z} \text{ ....} (5).$$

Substitute these values in (4), and we have for the equation to the locus

$$\frac{\xi}{f}(-\xi x + \eta y + \zeta z) + \frac{\eta}{g}(\xi x - \eta y + \zeta z) + \frac{\zeta}{h}(\xi x + \eta y - \zeta z) = 0 \text{ ....} (6).$$

To find the *locus* of *centres*, we may either consider the coordinates trilinear, and put a, b, c for x, y, z; or we may consider them triangular, and put $x = y = z = 1$.

It is clear that by varying the condition (4) we may easily find the locus in other cases. Thus, for instance, "the locus of the centres of all conics passing

through three given points and touching a given straight line ($f\alpha + g\beta + h\gamma = 0$), is the curve of the fourth degree,

$$\sqrt{\{f\xi(-\xi+\eta+\zeta)\}} + \sqrt{\{g\eta(\xi-\eta+\zeta)\}} + \sqrt{\{h\xi(\xi+\eta-\zeta)\}} = 0,$$

the coordinates being triangular." (*Cambridge and Dublin Math. Journal*, Vol. v. p. 148.)

Another solution by trilinear coordinates has been proposed in the *Messenger of Mathematics*, Vol. II. p. 169; we give it here in order to notice one of the theorems which may be deduced from it. Consider the equations in Art. 1 as trilinear; then it may easily be proved that the locus of the pole of a line L, or $\xi x + \eta y + \zeta z = 0$, with respect to all the conics, $lU + mV = 0$, is the Jacobian of U, V, L, that is,

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz} \\ \xi, & \eta, & \zeta \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

But this is precisely the equation which has been elsewhere obtained (see Art. ι, [p. 572, 1418]) as the locus of "*polar opposites*" of points in the line (ξ, η, ζ); that is, the polars of any point in this line with respect to all the conics $lU + mV$ pass through a fixed point in (7). The sides and diagonals of the quadrilateral are evidently cut harmonically by any line and its polar conic; and since the "locus of centres" is the polar conic of the line at infinity, it must coincide with the "nine-point conic," which *bisects* the sides and diagonals, besides passing through the points E, F, G (Fig. 120). That the conic (7) always *does* circumscribe the common self-conjugate triangle of U, V, may be shewn by putting it in the form

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \frac{\xi}{x}, & \frac{\eta}{y}, & \frac{\zeta}{z} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

It appears then that the nine-point conic possesses the following property: *the polars of any point of it, with respect to all the conics circumscribing the quadrilateral, are parallel.* And conversely, *all the diameters conjugate to a fixed straight line pass through a fixed point on the nine-point conic.*

The "curves of the third class," mentioned at the end of the solution of 1418 [p. 573] as the envelop of lines whose polar conics degenerate, is no other than the three vertices of the common self-conjugate triangle, as readily appears from geometrical considerations. By taking, then, the discriminant of the Jacobian (7) with respect to (x, y, z), we obtain a contravariant expression for these vertices, which enables us at once to reduce two quadrics to the canonical form.

3. Construction for the *directions of the asymptotes*. Let $ABCD$ (Fig. 121) be the four given points. Draw any line KL parallel to AB, meeting AD, BC in K, L respectively. Then DL, CK are parallel to the asymptotes of a certain conic through A, B, C, D. This follows immediately from Pascal's theorem. (See Gaskin's *Construction of a Conic Section*, &c., p. 39, Cor. 5.)

4. Construction for the *centre* of the last conic. Describe about $ABCD$ the parallelogram $\alpha\beta\gamma\delta$, having its sides parallel to DL, CK. Let $EFGH$ be the bisections of the sides of $ABCD$.

Then αE, βF, γG, δH will meet in a point X, which is the centre of the conic.

It may be observed that KL, MN, OP, QR are respectively parallel to BA, AD, DC, CB.

5. Construction for the directions of the axes of the two *parabolæ* through the four points. This would clearly be accomplished if we could draw KL so that CK, DL should be parallel. Let then the circle through DAB (Fig. 122) meet UBC in S, and the circle through ABC meet UAD in T. Take $UK^2 = US \,.\, UC$, and $UL^2 = UD \,.\, UT$; then KL is parallel to AB, and DL to CK. By treating V in the same way, we get another direction; but if the quadrilateral be re-entrant, it is easily seen that the construction fails.

This immediately determines the *species* of the conic found in Arts. 1 or 2. For if two parabolæ can be described through the four points, the locus of centres must have two points at infinity, that is, it must be an hyperbola. If no parabola can be so described, the locus has no point at infinity; that is, it is an ellipse.

6. We can now easily find a construction for the locus of centres when the quadrilateral is convex.

For the asymptotes are parallel to the axes of parabolæ found in Art. 5 and the locus must pass through the intersections of AB, CD, of AD, BC, and of AC, BD, each pair of lines being a conic through the four points. We have then the three finite points, and two points at infinity. Construct for the centre by Art. 4 (hence "if on the three sides of any triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of these parallelograms will meet in a point"); thus we can draw the axes and asymptotes; construct by Pascal's theorem the points where the curve meets the major axis, and the thing is done. Or, of course, the length of the axis may be found more simply by performing the geometric operations indicated by the equation

$$CA^2 = CN^2 - \left(PN \,.\, \frac{CA}{CB}\right)^2,$$

P being one of the three given finite points.

7. It appears from Art. 2 that the locus will break up into two straight lines, if E or F (Fig. 120) be at infinity, that is, if two sides of the quadrilateral are parallel. This is also clear from the fact that, when a conic becomes two parallel straight lines, any point midway between them is a centre. The line at infinity is itself part of the locus when it contains two of the points A, B, C, D. If one of these four is at infinity, only one parabola (or rather two coincident parabolæ) can be drawn through them, and the locus of centres is a parabola.

When the quadrangle can be inscribed in two *different* equilateral hyperbolæ, the locus of centres is a circle; and when it can be inscribed in a circle,

the locus of centres is an equilateral hyperbola, whose asymptotes are equally inclined to any pair of opposite sides, and to the two diagonals. If two equilateral hyperbolæ intersect in A, B, C, D, then AB is perpendicular to CD, AC to BD, and AD to BC, each of the points A, B, C, D, being, in fact, the intersection of perpendiculars of the triangles formed by the other three. It is easily seen that the locus of centres is, in this case, the nine-point circle of any of the four triangles ABC, BCD, CDA, DAB. See Note to solution of 1408 [Reprint, Vol. I. p. 28].

8. If we write the equation of the conic in the form $\alpha\beta = \mu\gamma\delta$, it is clear that we shall pass from a possible to an impossible region by changing the sign of *one* or *three* of the quantities $\alpha\beta\gamma\delta$. Attention to this and to Fig. [120] will shew that a given sign of μ the curve must lie wholly in the shaded regions, or wholly in the unshaded regions. We proceed to trace the cyclic succession of these curves, beginning with the pair of straight lines AB, CD, considered as the limit of a hyperbola. It is clear that these may separate into a finite hyperbola in two ways; so as to lie in the shaded regions, or in the unshaded regions. We begin with the former. One branch of the hyperbola lies entirely in (8), where also the centre is; the other branch lies in (6), (7), (10), (11), having its infinite parts in (7). The branch in (8), with the centre, moves off rapidly from E, and when the centre is at an infinite distance, we have a parabola in (6), (7), (10), (11), the infinite parts being in (7). When the parabola closes up into an ellipse, the centre reappears from the infinity of (7), and finally passes into (1). The ellipse again elongates itself, but in the direction of (6), into which the centre passes. In the limit we get another parabola, the centre going off to the infinity of (6). As the parabola merges into a hyperbola, the centre reappears from the infinity of (9), and the limit of the hyperbola is the pair of straight lines FA, FB, the centre being at F. We now pass into the unshaded regions, beginning with a small hyperbola, one branch lying in (2), (1), and (4), and the other in (3), (1), and (5). The centre is in (11), and moves down into (1). The parts of the hyperbola in (1) gradually approximate, giving as a limiting form, the lines AC, BD, when the centre is at G. The branches separate again in the other direction, one lying in (2), (1), and (5), and the other in (3), (1), and (4). The centre moves from (1) into (10), and gradually approaches the point E, where the hyperbola again becomes two straight lines. This is the point from which we started. The points E, F, G lie on the *same* branch of the hyperbola which is the locus of centres, and no part of the locus lies in the regions (2), (3), (4), (5).

The re-entrant quadrangle $ACEF$ may be treated in the same way; this case is simpler, all the conics of the series being hyperbolæ.

9. It may be worth while to notice a property of the nine-point conics of the quadrilateral faces of a tetrahedral frustum. With Prof. Sylvester's own notation, let $Oabc$ be a tetrahedron, the axes of coordinates being Oa, Ob, Oc; and let the plane $\alpha\beta\gamma$ cut off the frustum $\alpha\beta\gamma abc$. Put $4a$ for Oa, and similarly for the others; and consider the quadrilateral $ab\beta\alpha$ in the plane of xy. Its nine-point conic is easily found to be

$$\frac{x}{a\alpha}\{x-2(a+\alpha)\}=\frac{y}{b\beta}\{y-2(b+\beta)\},$$

and we draw through this a cylinder whose generating lines are parallel to the axis of z. There are three such cylinders, and they evidently have common to them the curve section

$$\frac{x}{a\alpha}\{x-2(a+\alpha)\}=\frac{y}{b\beta}\{y-2(b+\beta)\}=\frac{z}{c\gamma}\{z-2(c+\gamma)\}.$$

We are obviously entitled to conclude that, if we take instead the polar conics of the lines in which the faces are cut by any plane, and draw cones to the points where that plane cuts the opposite edges, these three cones will have a common section.

10. The theorem stated incidentally in Art. 6 is a particular case of Brianchon's theorem. It may be put a little more generally as follows:—

Take two points P, Q, and through each of them draw three straight lines. These triads will intersect in nine points, as in the following scheme,

$$\left.\begin{matrix} \overbrace{A \quad B \quad C}^{P} \\ D \quad E \quad F \\ G \quad H \quad K \end{matrix}\right\} Q.$$

Take now three points, one from each of the P-lines, and one from each of the Q-lines, as, for instance, B, F, G. To each pair of these take the opposite diagonal of the quadrilateral, *e.g.*, to BF corresponds CE; then these three lines CE, DK, AH will meet in a point. There are six such systems of lines.

The points P, Q may be considered as a conic inscribed in the hexagon $CKHEDA$, of which CE, DK, AH are the diagonals. The theorem is thus seen to be a particular case of Brianchon's. It will be found to involve also the following theorem of determinants: viz., the determinant whose constituents are the nine determinants,

$$\left\|\begin{matrix} \begin{Vmatrix} C & \dfrac{a}{b} & A \\ c & \dfrac{A}{B} & a \end{Vmatrix} \\ \begin{Vmatrix} A & \dfrac{b}{c} & B \\ a & \dfrac{B}{C} & b \end{Vmatrix} \\ \begin{Vmatrix} B & \dfrac{c}{a} & C \\ b & \dfrac{C}{A} & c \end{Vmatrix} \end{matrix}\right\|$$

vanishes identically.

1416. [Shew that the area of the perspective representation, in a given picture, of a triangle of given area in a fixed plane, varies as the product of the

distances of the angles of the perspective representation from the vanishing line. Proposed by Prof. Sylvester, F.R.S. Reprint, Vol. I. p. 77, *E. T.* June, 1864.]

Let ABC be the perspective representation of the triangle, DE the vanishing line. Let BC meet DE in D, and join AD. If abc is the triangle represented, AD is the picture of a line through a parallel to bc. If therefore B and C are fixed, the point A can only move along AD. But the area ABC varies as the perpendicular from A on BC, which is in a constant ratio to the perpendicular from A on DE, because A lies on a fixed line through the intersection of BC and DE. Since then when two of the perpendiculars on the vanishing line are fixed, the area varies directly as the remaining one; therefore when all vary, the area varies as the product of the three.

1479. Prove that the ordinary inverse of the *Tangential inverse* is the second positive pedal; and that the *Tangential inverse* of the ordinary inverse is the second negative pedal of the primitive.

[February, 1864. Solved, Reprint, Vol. I. p. 78.]

1497. (1) Given three points by equations of the form $lx+my+nz=0$, prove that the area of the triangle contained by them is

$$(l_1m_2n_3)\div(l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3),$$

that of the triangle of reference being unity.

(2) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)(456)\equiv(156)(423)+(164)(523)+(145)(623).$$

[April, 1864. Solved, Reprint, Vol. I. p. 79; Proposer's solution, Reprint, Vol. IV. p. 53; September, 1865 (?).]

1. The area of the triangle formed by three points vanishes only when they are in a straight line, and becomes infinite only when one of them is at infinity. The condition that they may be in a straight line is

$$J\equiv\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=0, \text{ or } (l_1m_2n_3)=0;$$

and the condition that one of them may be at infinity is

$$P\equiv(l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3)=0.$$

Now the expression for the area must be of no dimensions in the coefficients, since we are only concerned with their ratios; and the equation obtained by equating this expression to a constant must be of the first order in each set of coefficients; since, two of the points being fixed, the locus of the other is then a straight line. The expression for the area is therefore some numerical multiple of $\frac{J}{P}$. By putting $x, y, z=0$ for the three points, we find that the area of the fundamental triangle, on the same scale, is unity.

2. To prove the second proposition, take (456) for the fundamental triangle. Then, by applying the above interpretation to the well-known theorem

$$(l_1 m_2 n_3) \equiv l_1(m_2 n_3) + m_1(n_2 l_3) + n_1(l_2 m_3),$$

we find it equivalent to

$$(123)\,(456) \equiv (156)\,(423) + (164)\,(523) + (145)\,(623),$$

the factor P dividing out on both sides.

[Solutions were also sent to *E. T.* 1372, 1373, June, 1863 : Reprint, Vol. I. 1380, p. 16 : 1385, 1386, p. 9 : 1402, p. 19 : 1404, 1405, p. 21 : 1406, p. 22 : 1411, p. 43.]

1505. [If P, Q, 1, 2, 3, 4 be points on a conic, then the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ lie on a conic passing through the points P and Q. Proposed by Prof. Cayley. Reprint, Vol. II. pp. 9, 10, *E. T.* July, 1864.]

1. Let the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ be called S, T, U, V respectively; then

$$\{P\,.\,1234\} = \{Q\,.\,1234\};$$

but $$\{P\,.\,1234\} = \{P\,.\,STUV\},$$

and $$\{Q\,.\,1234\} = \{Q\,.\,TSVU\},$$

also $$[TSVU] = [STUV]^*,$$

therefore $$\{P\,.\,STUV\} = \{Q\,.\,STUV\},$$

which proves that the six points P, Q, S, T, U, V lie on a conic.

2. Let $A=0$, $B=0$, $C=0$, $D=0$, denote respectively the pairs of right lines ($P1$, $Q1$), ($P2$, $Q2$), ($P3$, $Q3$), ($P4$, $Q4$). Then we shall prove presently that there is an identical relation

$$A + B + C + D = 0,$$

constant multipliers being supposed.

3. The Jacobian of any three of the four conics A, B, C, D is obviously the original conic $PQ\,1\,2\,3\,4$, together with the straight line PQ. Now the conic $A+B=0$ is identical with $C+D=0$ (by Art. 1); and it passes through all the intersections of A with B, and of C with D. It must therefore be the very conic $PQSTUV$. And there are clearly two more conics, namely, $A+C$ or $D+B$, and $A+D$ or $B+C$, obtained just in the same way. It may be as well to remark that the three are represented by the equation

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} = 0.$$

Moreover, the original conic may be reproduced by treating $STUV$ in the same way as we treated 1234. We have therefore four conics derived from the two points P, Q in a symmetrical manner. *Each of these conics is the Jacobian of the other three;* the line PQ being of course added. For the three conics $A+B$, $B+C$, $C+A$ have the same Jacobian as A, B, C, that is, the original conic and the line PQ.

* [ ] Auctoris.

The pole of PQ, with regard to the conic $STUV$, is the intersection of 12 and 34. For the chord ST is divided harmonically by PQ and 12, and the chord UV by PQ and 34. Hence *the poles of* PQ, *with regard to any three of the conics, form a self-conjugate triad with regard to the fourth.* For the poles with regard to $A+B$, $B+C$, $C+A$, are the intersections of (12, 34), (14, 23), (13, 42), which form a self-conjugate triad of any conic 1234.

4. By projecting the points P, Q into the circular points at infinity, we may prove M. Laguerre's theorems, proposed in the *Nouvelles Annales* for March, 1864 (p. 141, Question 698).

"Lorsqu'une courbe a quatre foyers sur un cercle, elle en a nécessairement douze autres situés par quatre sur trois autres cercles; tous ces cercles sont orthogonaux entre eux."

It follows that when four circles cut each other orthogonally, each centre is the intersection of perpendiculars of the triangle formed by the other three. Hence one of the circles must be imaginary.

5. We now proceed to prove the statement in (2).

Lemma. When four conics have the same Jacobian, three and three, their equations are connected by an identical linear relation.

Any four conics can be reduced simultaneously to the form

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2+d_2w^2=0,$$
$$\&\text{c.} \qquad \&\text{c.}$$

where $x+y+z+w=0$. This we can see by counting the constants. Let R be the determinant $(a_1b_2c_3d_4)$, and A_1, A_2, &c. its first minors. Then the Jacobians

$$\frac{A_1}{x}-\frac{B_1}{y}+\frac{C_1}{z}-\frac{D_1}{w}=0,$$
$$\&\text{c.} \qquad \&\text{c.}$$

and if these are all identical, we must have $(A_1B_2C_3D_4)=0$, which implies that $(a_1b_2c_3d_4)=0$, or the conics are connected by an identical linear relation.

For a metric interpretation, see Dr Salmon's *Conics*, 4th ed., Art. 94.

1514. [Let P be the points of intersection of the three perpendiculars, and G the centre of gravity of any triangle ABC; also let l, m, n be the middle points of the sides BC, CA, AB; S_l, S_m, S_n, the circles described upon Al, Bm, Cn as diameters, and S_1, S_2, S_3, the circles circumscribing the triangles PBC, PCA, PAB.

It is required to prove,

(α) That the circle which passes through l, m, n passes also through the points of intersection, real or imaginary, of the self-conjugate and circumscribing circles of each of the triangles PBC, PCA, PAB, ABC.

(β) That the six points common to the three pairs of circles S_l, S_1; S_m, S_2; S_n, S_3; lie on another circle Σ.

(γ) That the self-conjugate and circumscribing circles of the triangle, the circle which bisects its sides, the circle upon PG as diameter, the circle Σ, and the director of the maximum ellipse that can be inscribed in the triangle, all pass through the same two points, real or imaginary. Proposed by J. Griffiths, M.A. Reprint, Vol. II. p. 27, *E. T.* August, 1864.]

Let $$U \equiv a^2yz + b^2zx + c^2xy,$$
$$V \equiv (x+y+z)(bc\cos A \,.\, x + ca\cos B \,.\, y + ab\cos C \,.\, z);$$
then $U = kV$ represents respectively (the coordinates being triangular)

(1) when $k=\frac{1}{2}$, the *nine-point* circle of the triangle of reference,

(2) when $k=0$, the *circumscribing* circle;

(3) when $k=1$, the *self-conjugate* circle;

(4) when $k=\frac{1}{3}$, the *director of the maximum ellipse;*

(5) when $k=\frac{2}{3}$, the *circle on* PG as *diameter.*

The last two are the only ones which present any difficulty.

In (4) the tangential equations of the ellipse and the circular points at infinity are respectively

$$yz + zx + xy = 0,$$
$$a^2x^2 + b^2y^2 + c^2z^2 - 2bc\cos A \,.\, yz - 2ca\cos B \,.\, zx - 2ab\cos C \,.\, xy = 0.$$

Forming, by the rule, the *harmonic conic* of these two, we at once write down equation (4). To find equation (5), we make use of the following

THEOREM. If $A=0$, $B=0$, $C=0$, $D=0$ are any four lines in a plane, and if $\Psi AB=0$ denote the condition that A and B may be at right angles, then, the equation to the circle whose diameter is the line joining (AB), (CD) is

$$\begin{vmatrix} \Psi AC, & \Psi AD, & A \\ \Psi BC, & \Psi BD, & B \\ C\ , & D\ , & 0 \end{vmatrix} = 0.$$

In the present case, take the four lines

$$A \equiv x - y;$$
$$B \equiv y - z;$$
$$C \equiv bc\cos A \,.\, x - ca\cos B \,.\, y;$$
$$D \equiv ca\cos B \,.\, y - ab\cos C \,.\, z;$$

then the condition of perpendicularity of $(l_1m_1n_1)$ $(l_2m_2n_2)$ being

$$a^2l_1l_2 + b^2m_1m_2 + c^2n_1n_2 - bc\cos A\,(m_1n_2 + m_2n_1) - \&c. = 0,$$

we have

$$\Psi AC = \Psi BD = abc\,(a\cos A + b\cos B + c\cos C) = 2abc \,.\, a\sin B\sin C,$$
$$\Psi AD = \Psi BC = -abc \,.\, b\sin C\sin A = -\tfrac{1}{2}\Psi AC.$$

Expanding then the determinant, it becomes simply

$$2\,(AC + BD) + AD + BC = 0,$$

and, substituting the values of A, B, C, D, we get equation (5).

1486. If two transversals ABC, DEF cut the sides of any triangle, then AE, BF, CD cut the sides in three points on a straight line X. If moreover the triangle touch a cubic at A, B, C, and cut it in D, E, F, the lines AE, BF, CD meet the curve in three points on a straight line Y, and the lines X, Y, ABC, meet in a point.

[*E. T.* March, 1864. Solved, Reprint, Vol. II. p. 40.]

1519. [ABC is a triangle having the three real points (P, Q, R) of inflexion of a cubic on the sides BC, CA, AB respectively, each of which also passes through two imaginary points of inflexion. The tangents at Q and R meet in D, those at R and P in E, and those at P and Q in F. Shew that AD, BE, CF meet in a point which is fixed for all the cubics having the same nine points of inflexion. Proposed by F. D. Thomson, M.A. Reprint, Vol. II. p. 48, *E. T.* September, 1864.]

Taking ABC for triangle of reference, the equation of the cubic is

$$a^3x^3+b^3y^3+c^3z^3-3dxyz=0,$$

which may also be written in the form

$$\left(\frac{d}{bc}x+by+cz\right)\left(\frac{d}{bc}x+\theta by+\theta^2cz\right)\left(\frac{d}{bc}x+\theta^2by+\theta cz\right)=\frac{d^3-a^3b^3c^3}{b^3c^3}x^3,$$

(where θ is an imaginary cube root of unity) shewing that the tangents at P, Q, R are

$$\frac{d}{bc}x+by+cz=0,$$

$$ax+\frac{d}{ca}y+cz=0,$$

$$ax+by+\frac{d}{ab}z=0.$$

The equation to AD is therefore $by=cz$, so that AD, BE, CF meet in the point $ax=by=cz$.

Now all the points of inflexion are on the axis $xyz=0$; consequently any other cubic having the same points of inflexion can only differ from the above in the coefficient of xyz, of which the point $ax=by=cz$ is independent.

This point is the pole of the line PQR with respect to the triangle ABC.

1517. [If each edge of a tetrahedron is perpendicular to the non-conterminous edge (it being observed that if two pairs of such edges be mutually perpendicular, the third pair will be so too); prove that the nine-point circles of the three triangular faces lie on a sphere; also that the nine-point circle of any triangular face, and the three points of intersection of the perpendiculars of the other three triangular faces, lie on a sphere; and find the equations of all these spheres. Proposed by H. R. Greer, M.A. Reprint, Vol. II. p. 79, November, 1864.]

The triangular equation of the nine-point circle is

$$a^2yz+b^2zx+c^2xy=2(x+y+z)(bc\cos A\,.\,x+ca\cos B\,.\,y+ab\cos C\,.\,z)\ldots(1).$$

Now when two opposite edges of a tetrahedron are perpendicular, a plane may be drawn through either perpendicular to the other, and will therefore contain the perpendiculars from the extremities of the first upon the second. Consequently, if a tetrahedron has *two* pairs of perpendicular edges, the perpendiculars from the vertices to opposite faces will meet in a point, and the foot of any one of them will be the intersection of perpendiculars of the face in which it lies. From this it is obvious that, if $ABCD$ be such a tetrahedron,

$$AB \,.\, AC \cos BAC = AC \,.\, AD \cos CAD = AD \,.\, AB \cos DAB = (A), \text{ suppose.}$$

It follows at once that the nine-point circles of the four faces lie on a sphere; for the tetrahedral equation of this, the "twenty-four-point sphere," is

$$ab^2 \,.\, xy + ac^2 \,.\, xz + \quad = 2(x+y+z+w)\{(A)x + (B)y + (C)z + (D)w\} \ldots. \quad (2).$$

The equation to the sphere which contains the nine-point circle of the face a, and the polar centres of the other three faces, is got by changing the sign of (A) in equation (2). The equation of the self-conjugate sphere is

$$ab^2 \,.\, xy + ac^2 \,.\, xz + \,.. = (x+y+z+w)\{(A)x + (B)y + (C)z + (D)w\};$$

this, therefore, passes through the intersection of the circumscribed and the twenty-four-point spheres.

If through the middle point of each edge of a tetrahedron a line be drawn parallel to the opposite edge, the tetrahedron will be reproduced in an inverted position. In the present case, the two tetrahedra will have the same twenty-four-point sphere, and the sphere self-conjugate to one will circumscribe the other.

[The proposer in his solution refers to an article on this species of tetrahedron by Prof. Wolstenholme (*Quarterly Journal of Mathematics*, Vol. III.).]

1394. [Required a *direct* proof that an ellipse and its osculating circle have a contact of the third order at the ends of its axes; also prove that the deviations of the ellipse from the circle osculating it most closely at the ends of its axes are to each other inversely as the seventh powers of the axes. Proposed by Matthew Collins, B.A. Reprint, Vol. II. pp. 82—85, December, 1864.]

1. We know that if the osculating circle at a point P meet the ellipse again at Q, PQ and the tangent at P are equally inclined to the axes. (Salmon's *Conics*, 4th ed. Art. 244; Taylor's *Geometrical Conics*, p. 85). This shews that the equation of the osculating circle at (ξ, η) may be written in either of the forms

$$\left(\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left(\frac{x\xi}{a^2} + \frac{y\eta}{b^2} - 1\right)\left\{\frac{\xi(x-\xi)}{a^2} - \frac{\eta(y-\eta)}{b^2}\right\} \ldots (1),$$

$$\left(\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left\{\frac{\xi^2}{a^4}(x-\xi)^2 - \frac{\eta^2}{b^4}(y-\eta)^2\right\} \ldots \quad \ldots. (2).$$

When $\xi = 0$, $\eta = \pm b$, (2) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)(y-\eta)^2 = 0 \ldots\ldots\ldots\ldots \ldots\ldots\ldots \quad (3).$$

When $\eta = 0$, $\xi = \pm a$, (2) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \left(\frac{1}{a^2} - \frac{1}{b^2}\right)(x-\xi)^2 = 0 \ldots\ldots\ldots\ldots \quad \ldots (4).$$

This shews that in both these cases the curves meet only where they meet the common tangent, that is, they have *four* consecutive points common; or, what is the same thing, they have contact of the *third* order. This is also seen to follow at once from the property enunciated at the outset.

2. Consider now the geometrical meaning of equation (4). Take any point P on the circle, and draw P_1QQ_1T parallel to the major axis, meeting the ellipse in Q, Q_1, and the tangent at the vertex (A) in T. Then, if C is the centre of the ellipse and (x, y) the coordinates of P, the quantity $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ is equal to $\frac{PQ\,.\,PQ_1}{CA^2}$, and $\xi - x$ is PT. Hence (4) means that

$$\frac{PQ\,.\,PQ_1}{CA^2}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)PT^2.$$

But in the limit $PQ_1=2CA$, so that we may write

$$PQ=\frac{PT^2\,.\,CA}{2}\,.\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If we make a similar construction with small letters near the extremity of the *minor* axis, we shall get from (3)

$$pq=\frac{pt^2\,.\,CB}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right).$$

Hence $$\frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{PT^2}{pt^2}.$$

But PT, pt are as the reciprocals of the radii of curvature, or as

$$\frac{CA^2}{CB}:\frac{CB^2}{CA}.$$

Therefore $$\frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{CA^4}{CB^2}\cdot\frac{CA^2}{CB^4}=\frac{CA^7}{CB^7}.$$

3. We take this opportunity of setting down two other equations of the circle of curvature, which are easily deduced from (1) or (2).

The tangent at any point of an ellipse may be represented by the equation

$$lx+my=\sqrt{(l^2a^2+m^2b^2)}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Let $\sqrt{(l^2a^2+m^2b^2)}=p$, then the common chord of the ellipse and the circle of curvature at the points (l, m) is represented by

$$lx-my=\frac{l^2a^2-m^2b^2}{p}\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Hence from (2) the equation to the osculating circle is

$$p^2(l^2+m^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\{l^2(px-la^2)^2-m^2(py-mb^2)^2\}\ldots..(8).$$

Next, at a point whose eccentric angle is ϕ, the equations to the tangent and the common chord are, respectively,

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1\ldots\ldots\ldots\ldots\ldots\ldots(9);$$

$$\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=\cos 2\phi\ldots\ldots\ldots\ldots\ldots\ldots(10);$$

whence, putting $a^2-b^2=c^2$, we readily obtain the equation of the osculating circle in the form

$$x^2+y^2-2c^2\left(\frac{x}{a}\cos^3\phi-\frac{y}{b}\sin^3\phi\right)=a^2\sin^2\phi+b^2\cos^2\phi-c^2\cos 2\phi\ldots\ldots(11).$$

4. If we put $x=x_1+\alpha$, $y=y_1+\beta$, the equation of the tangent will still be of the form

$$l(x_1+\alpha)+m(y_1+\beta)=\sqrt{(l^2a^2+m^2b^2)}\ldots\ldots\ldots\ldots(12),$$

and the line through (α,β) perpendicular to this is clearly $\frac{x_1}{l}=\frac{y_1}{m}$.

The locus of the foot of this perpendicular is therefore obtained by writing (x_1, y_1) for (l, m) in (12).

Hence, *if the equation of the tangent to any curve can be put in the form*

$$lx+my=F(l,m),$$

then the equation of the pedal with the point (α,β) *for origin is*

$$x(x-\alpha)+y(y-\beta)=F(x-\alpha,y-\beta)\ldots\ldots\ldots\ldots(13).$$

For instance, the locus of the foot of the perpendicular from the centre of the ellipse on the common chord (7) is

$$(x^2+y^2)^2(a^2x^2+b^2y^2)=(a^2x^2-b^2y^2)^2\ldots\ldots\ldots\ldots(14).$$

All that has been here said may be applied to the hyperbola by changing the sign of b^2.

5. The first pedal of the cycloid may be simply obtained by the method of Art. 4. For let the circle begin to roll on the axis of x at the origin, and consider the tangent at a point corresponding to a revolution ϕ of the circle. If its equation be written

$$lx+my=F(l,m),$$

we must have

$$\phi=2\cot^{-1}\left(-\frac{l}{m}\right);$$

now the tangent passes through the point $2x=a\phi$, $y=a$, where a is the diameter of the circle; whence the equation may be written

$$lx+my=a\left(m-l\cot^{-1}\frac{l}{m}\right),$$

and we immediately get the pedal with origin (α,β); viz.,

$$x(x-\alpha)+y(y-\beta)=a\left\{(y-\beta)-(x-\alpha)\cot^{-1}\frac{x-\alpha}{y-\beta}\right\}.$$

Putting $\alpha=0$ and $\beta=0$, we get the pedal from the origin, viz.,

$$x^2+y^2=a\left(y-x\cot^{-1}\frac{x}{y}\right),$$

or, in polar coordinates,

$$r=a(\sin\theta-\theta\cos\theta).$$

1468. Given the centre of a conic, and a conjugate triad; to construct for the directions of the asymptotes. [Proposed, *E. T.* January, 1864: solution, February, 1865.]

[If O be the given centre of the conic; A, B, C the three points of the self-conjugate triad, and OX, OY, OZ the three lines through O parallel to BC, CA, AB respectively; the two double rays (real or imaginary) OM and ON of the involution determined by the three angles AOX, BOY, COZ are the two asymptotes required. For the three pairs of conjugates, OA and OX, OB and OY, OC and OZ, determining the involution, being evidently pairs of conjugate diameters of the conic, divide, therefore, harmonically the angle (real or imaginary) MON determined by the two asymptotes OM and ON.

The same construction (with some slight and obvious modifications) applies also to the following more general problem, of which the above is evidently a particular case: viz., given a point and a line, pole and polar with respect to a conic and a conjugate triad; to construct the two tangents (real or imaginary) from the point to the curve, and the two intersections (real or imaginary) of the line with the curve. For if P and L be the point and line; A, B, C, as before, the three points of the triad; X, Y, Z and X', Y', Z' the six intersections of L with BC, CA, AB, and with PA, PB, PC, respectively; then, as X and X', Y and Y', Z and Z' are evidently pairs of conjugate points with respect to the conic, the two double points M and N of the involution determined by the three segments XX', YY', ZZ', as cutting them all harmonically, are the two intersections required; and as PX and PX', PY and PY', PZ and PZ', are evidently pairs of conjugate lines with respect to the conic, the two double rays PM and PN of the involution determined by the three angles XPX', YPY', ZPZ', as cutting them all harmonically, are the two tangents required.

The two corresponding problems in geometry of three dimensions, viz., Given, of a quadric, the centre and a self-conjugate tetrahedron, to construct the asymptotic cone of the surface; or, more generally: Given, of a quadric, a point and plane, pole and polar to each other, and a self-conjugate tetrahedron, to construct the tangent cone from the point to the surface, and the conic of intersection of the plane with the surface, may be readily solved by application of the above.]

Corollary i. Let any two straight lines parallel to two conjugate diameters be called *conjugate* with respect to a conic; then it is shewn above that the pairs of lines 12, 34; 13, 24; 14, 23, joining the points 1234, are conjugates with respect to the conic which has the point 1 for a centre, and 234 for a conjugate triad. But the symmetry of this statement shews that they are also conjugates with respect to the conic which has any other of the five points for centre, and the remaining three for a conjugate triad. We may draw four such conics; and since the asymptotes are determined in direction by *two* pairs of conjugates, it follows that these four conics are all similar and similarly situated. So, in the more general case, we shall have four conics intersecting in two points on the given straight line.

Corollary ii. Let a straight line and plane, drawn parallel to any diameter and its conjugate diametral plane, be called *conjugates* with respect to a com-

coid. Then if we are given five points 12345, of which 1 is the centre, and 2345 a self-conjugate tetrahedron of a given conicoid, it is evident that since 2 is the pole of the plane 345, 12 is conjugate to 345, and so on. We thus get four pairs of conjugates. Again, since 23 is the polar line of 45, 123 is conjugate to 45, and 145 to 23, and so on. This gives us six more pairs of conjugates. But this amounts to saying that if we join any three of the five points by a plane, and the other two by a line, the line and plane are conjugates. This statement makes no mention of the particular point taken for centre; and we conclude as before, that if five conicoids are drawn, by taking each of five points in succession for centre, and the remaining four for a self-conjugate tetrahedron, these five conicoids will be similar and similarly situated. A line and its conjugate plane cut the plane at infinity in a point and line which are pole and polar with respect to the section which the plane at infinity makes of the conicoid. The problem is therefore equivalent to that of describing a conic, being given the poles of certain lines. Three points and their polars are sufficient to determine a conic; for let A, B, C be the points, and let AB meet the polar of A and B in P, Q respectively. Then the foci of the involution determined by AP, BQ, are evidently points on the conic. In this way we can determine six points on the sides of the triangle ABC, and six more on the sides of the reciprocal triangle; and it would be interesting to prove *a priori* that these twelve points must lie on the same conic, when the triangles are in perspective.

Corollary iii. In the plane case we are given three pairs of conjugates to determine two points at infinity; and we conclude that any transversal is cut in involution by the six lines joining four points. Similarly we conclude from the solid problem that "if five points in space are joined every way by ten lines and ten planes, the system will be cut by any plane in a system of points and lines which are poles and polars with respect to a certain conic." The analogy of this relation of points and lines with involution, may be illustrated analytically. Let $U \equiv ax^2+by^2=0$ be a pair of points; then if we put Δ for

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right),$$

a point and its harmonic conjugate will be represented by the equations

$$\begin{vmatrix} x, & y \\ \xi, & \eta \end{vmatrix}=0, \text{ and } \Delta U=0.$$

And a system of such harmonic conjugates is of course a system in involution. Next let U represent a conic, and Δ stand for

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right),$$

then a point and its polar will be represented by the equations

$$\left\|\begin{matrix} x, & y, & z \\ \xi, & \eta, & \zeta \end{matrix}\right\|=0, \text{ and } \Delta U=0,$$

and the analogy is obvious.

Corollary iv. Lastly, the four conics mentioned in Cor. i. are all similar to the *nine-point conic* of the quadrangle, or locus of centres of all conics through

the four points. This proposition was set in a problem paper, at St John's College, Cambridge, in Dec. 1862; but I do not know to whom it is due.

It follows at once from the equation to the nine-point conic given in Art. 2 of the solution to Question 1443 [Reprint, Vol. I. p. 51, supra, p. 580]; for an equation of the second degree in x, y, 1, in which the coefficient of xy is zero, obviously represents a conic with respect to which the axes are conjugates. Thus the lines 12, 34; 13, 24; 14, 23, are conjugates with respect to the nine-point conic, and therefore its asymptotes are parallel to those of the other four. These, therefore, are ellipses when the quadrangle is re-entrant, and hyperbolas when it is convex.

[Reprint, Vol. III. pp. 35, 36. Prof. Townsend and the proposer are credited with the solution. I felt sure that the Corollaries were Clifford's work, and on asking Mr Miller I find that the part enclosed in brackets above was due to Prof. Townsend, and that Clifford instead of having his solution, a short one, printed, added the above Corollaries to the "proof."]

1652. Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R; RB meets Aa in P; PC meets Bb in Q; QA meets Cc in r; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Shew that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane. [*E. T.* February, 1865. Reprint, Vol. III. p. 66, *E. T.* April, 1865, which contains a solution by Prof. Cayley.]

Let x, y, z be the sides of the triangle ABC, and let $ay=z$, $bz=x$, $cx=y$ be the three lines drawn through them. Start with the line AR or $y=z$, which meets Cc or $cx=y$ on $cx=z$, which meets $ay=z$ on $cx=ay$, which meets $bz=x$ on $cbz=ay$, which meets $cx=y$ on $bz=ax$, which meets $ay=z$ on $by=x$, which meets $bz=x$ on $y=z$; so that we have come round again. The extension of this is most easy; I write down two enunciations:—

Consider a plane polygon of an odd number of sides; let the two sides adjacent to any given side be produced to meet, and through their intersections let an arbitrary line be drawn; then treating these lines in the same way as Aa, Bb, Cc were treated in the case of the triangle, we may go twice round the polygon, and shall always come back to the same point.

Let $ABCD$ be a skew quadrilateral in space, and through the four sides AB, BC, CD, DA let arbitrary planes be drawn; let any line through A meet the plane through CD in a; aB meets the plane DA in b; and so on; after going *three* times round the quadrilateral we shall come back to the same point.

The theorem is not true for a plane polygon of an even number of sides; I have not been able to find an analogue in this case.

1679. [To find the envelope of the straight line joining the feet of the perpendiculars drawn on the sides of a triangle from a point in the circumference of the circumscribed circle. Proposed by H. R. Greer, B.A. Reprint, Vol. III. pp. 81, 2, *E. T.* May, 1865.]

The line in question is known to be the tangent at the vertex of a parabola which touches the sides of the triangle. Now this straight line, being always at right angles to the axis of the parabola, determines on the line at infinity a series of points in involution with the series determined by the parabola itself; we have then a series of conics touching four given lines, and a series of points on one of the lines, homographic with the series of conics; and we want to find the envelope of the remaining tangent, drawn from each point to its corresponding conic. Let then $U=kV$ be the tangential equation of the series of conics and $P=\lambda Q$ of the series of points. We obtain the required envelope by eliminating k; it is $UQ=VP$, a curve of the third class touching the common tangents of U and V, and the line PQ. When, as in the case we are considering, the line PQ coincides with one of the common tangents of U, V, then it is a double tangent to the curve $UQ=VP$, and the points of contact are the double points of the involution; in this case, the circular points at infinity. Since the curve is of the third class, and has one double tangent (that is, all it can) it is of the fourth order; and because the double tangent has imaginary contacts, the curve has three *real* cusps. To determine the position of these cusps, and the general form of the curve, we have to study a most singular figure.

Consider four points, 1, 2, 3, 4, such that each is the intersection of perpendiculars of the triangle formed by the other three. About the triangles 234, 341, 412, 123 describe circles; it is known that these circles are all equal, and that their centres 1′, 2′, 3′, 4′ form another quadrangle, exactly similar and equal to 1 2 3 4, but in an inverted position, their centre of (inverse) similitude being the centre of the nine-point circle. Now suppose that the feet of the perpendiculars from any point in the circle 234 to the sides of the triangle 234 are joined by a line X. Then I say that *if at the points where the line X cuts the six connectors of the quadrangle* 1234, *perpendiculars be drawn to these six connectors respectively, the perpendiculars will concur three by three, in four points*, 1″, 2″, 3″, 4″, *situate one on each of the four circumscribing circles, and forming a quadrangle equal, similar, and similarly situated to* 1′ 2′ 3′ 4′. And *the centre of* (inverse) *similitude of* 1 2 3 4 *and* 1″ 2″ 3″ 4″ *is situated on the line X, and bisects the segment determined on it by any pair of connectors.* Hence we see (1), that the line X is connected with the whole quadrangle, and not with three particular points of it; (2), it is cut by the connectors in an involution, one double point of which is at infinity; and therefore is an *asymptote* of some conic passing through the points 1, 2, 3, 4.

Now, take any connector 1 2, and find a point on it, symmetrical in respect of 1, 2, with the point where it is cut by 34. Then the envelope of X touches all the connectors at the points thus determined.

Since writing the above, I have read a paper on the subject by Steiner, in the 53rd volume of Crelle's *Journal* *. He asserts that the curve is a hypocycloid of three branches, and gives a simple construction for the cusps.

The property of a quadrangle enunciated above, is in fact this:—If four parabolas be drawn, having their axes parallel, each inscribed in one of the four triangles determined by a quadrangle, these four will have a common tangent:

* [An abridgment of Steiner's paper is printed on pp. 97—100 of this Vol. III. of the *Reprint*.]

which is at once seen to be a particular case of the reciprocal of this: The four circles, each circumscribing one of the triangles determined by a quadrilateral, have a common point. And this again is a particular case of that wonderful proposition, the involution of cubics:—All the cubics which pass through eight fixed points pass also through a ninth point.

Finally, reciprocate the whole figure in respect of the self-conjugate circle of any of the triangles 234, &c. We then get the locus of a point where the normal at (1) meets again a rectangular hyperbola circumscribing the quadrangle; it is a cubic having its asymptotes parallel to the sides of 234, and with a double point at (1), the tangents to which are the asymptotes of the polar circle. In fact, this problem is rather easier than its reciprocal.

1680. [(1) Prove that the envelope of the asymptotes of a rectangular hyperbola described about a given triangle is a curve of the third class, touching the sides of the triangle, the three perpendiculars, lines through the feet of the perpendiculars parallel to the opposite sides of the triangle formed by joining them, and also the line at infinity.

(2) Prove that the envelope of the asymptotes of a conic inscribed in a given quadrilateral, is a curve of the third class touching the sides and diagonals of the quadrilateral, the line at infinity, and the line joining the middle points of the diagonals. Proposed by F. D. Thomson, M.A. Reprint, Vol. III. pp. 82, 3, *E. T.* May, 1865.]

(1) It is shewn in the solution of 1679 that the line whose envelope is there considered is an asymptote of *some* rectangular hyperbola circumscribing the quadrangle; whence the two envelopes must be identical. This may also be proved thus: the proposition is that a rectangular hyperbola may circumscribe any triangle which circumscribes a parabola, and have for an asymptote the tangent at the vertex of the parabola. Let β be the axis of the parabola, α the tangent at its vertex, γ the line at infinity; then the respective equations to the hyperbola and parabola are

$$\gamma^2 + 2p\alpha\gamma = 2\mu\alpha\beta, \qquad \beta^2 = 2\lambda\gamma\alpha\ ;$$

whence $$\Theta = -p^2, \quad \Theta' = 2p\lambda, \quad \Delta = -\mu^2, \quad \Delta' = -\lambda^2,$$

and the condition $\Theta'^2 = 4\Theta\Delta'$ is satisfied. In fact, the triangle $(\alpha\beta\gamma)$ is inscribed in the hyperbola, and circumscribes the parabola.

Hence (i) the envelope of the asymptotes of all conics through four given points is a three-cusped quartic touching the six connectors of the given points, and the line at infinity at the points of contact of the parabola through them. (ii) If two tangents to a three-cusped quartic divide harmonically the double tangent, their intersection lies on a conic through the points of contact of the double tangent. This conic touches the quartic in three points. (iii) If a chord of a nodal cubic subtend harmonically the double point, its envelope is a conic touching the tangents at the double point, and the curve itself in three points.

M. Chasles gets the result (i) by his method of characteristics (Theor. XVI.). The envelope of the asymptotes is in general of class $\mu + \nu$, and has a ν-ple tangent at infinity; where μ is the number of conics of a system that can be

drawn through a given point, and ν the number that can be drawn to touch a given line.

(2) Here again M. Chasles's method shews that the envelope is of the third class, and touches *once* the line at infinity. Let U, V be two inscribed conics, and (ξ, η, ζ) the coordinates of the line at infinity; and write also Δ for $(\xi\delta_x + \eta\delta_y + \zeta\delta_z)$; then a conic of the system is $U = kV$, the centre $\Delta U = k\Delta V$, and the envelope required $U\Delta V = V\Delta U$, which is of the third class, touching the sides of the quadrilateral, and the line $\Delta U = 0$, $\Delta V = 0$, which joins the middle points of the diagonals. If for U, V we write AB, CD, the equation is

$$AB(C\,.\,\Delta D + D\,.\,\Delta C) = CD(A\,.\,\Delta B + B\,.\,\Delta A),$$

shewing that the curve touches the lines $(A = 0,\ B = 0)$ and $(C = 0,\ D = 0)$; that is the diagonals of the quadrilateral.

1724. The equations of three conics being given in the following forms, viz.,

$$a_1x^2 + b_1y^2 + c_1z^2 + d_1w^2 = 0,$$
$$a_2x^2 + \&c. = 0,$$
$$a_3x^2 + \&c. = 0,$$

where $$x + y + z + w \equiv 0,$$

shew that a straight line $(\xi x + \eta y + \zeta z + \omega w = 0)$ will be cut in involution by them, if

$$\Sigma\,.\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}.\,(\xi - \eta)(\xi - \zeta)(\xi - \omega)\,.\,(\text{to four terms}) = 0.$$

[Proposed, *E. T.* May, 1865. Solved Reprint, Vol. IV. p. 52.]

1733. [To find the area of a triangle, the equations of whose sides in trilinear coordinates are

$$l_1\alpha + m_1\beta + n_1\gamma = 0, \qquad l_2\alpha + \ldots = 0, \qquad l_3\alpha + \ldots = 0.$$

Proposed by W. A. Whitworth, M.A. Reprint, Vol. IV. pp. 55, 6, *E. T.* September, 1865.]

Call the three lines 1, 2, 3. Then we have to find the area of the triangle included by the points (23), (31), (12); that is, by the points

$$L_1x + M_1y + N_1z = 0, \quad L_2x + \ldots = 0, \quad L_3x + \ldots = 0;$$

where $L_1\ldots$ are the first minors of the determinant $(l_1m_2n_3)$. But by the Solution to 1497 [p. 584], this area is a numerical multiple of $\dfrac{J}{P}$, where

$$J \equiv (L_1M_2N_3) \equiv (l_1m_2n_3)^2,$$
$$P \equiv (aL_1 + bM_1 + cN_1)(aL_2 + \ldots)(aL_3 + \ldots)$$

in the Trilinear system.

In the case of the fundamental triangle we find that $\dfrac{J}{P} = \dfrac{1}{abc}$. Hence the *ratio* has the value given in the foregoing solutions* [and the area $= abc\,.\,\Delta\,.\,J \div P$].

* [Two other modes of solution were printed.]

The general formula for all systems (see my paper on "Analytical Metrics" [p. 90 supra]), is

$$\frac{\{J(ABC)\}^2}{J(BC\infty).J(CA\infty).J(AB\infty)},$$

where $J(ABC)=0$ is the condition that the lines $A, B, C, =0$ may meet in a point, and $J(BC\infty)=0$ is the condition that B and C may be parallel. The *ratio* to the area of the fundamental triangle may easily be found in any particular case by the method used above.

In the same way it may be shewn that the volume of a tetrahedron is

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty).J(CDA\infty).J(DAB\infty).J(ABC\infty)},$$

where $J(ABCD)=0$ is the condition that the planes $A, B, C, D, =0$ may meet in a point, and $J(BCD\infty)=0$ is the condition that B, C, D may be parallel to the same line.

In quadriplanar coordinates, for instance, if $\alpha, \beta, \gamma, \delta$, denote the areas of the faces of the fundamental tetrahedron, the equation to the plane at infinity is

$$\alpha x+\beta y+\gamma z+\delta w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the above expression for the volume, if calculated by means of (1), must be multiplied by the product $\alpha\beta\gamma\delta$ to give the ratio of the volume of the given tetrahedron to that of the fundamental one.

1732. [Prove that the characteristics of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. Proposed by T. A. Hirst, F.R.S. Reprint, Vol. IV. pp. 56, 7, *E. T.* September, 1865.]

In a system of conics satisfying four conditions (Z, Z', Z'', Z''') let μ be the number of conics that pass through an arbitrary point, and ν the number that touch an arbitrary line. Suppose that the polars of a point P, in respect of all the conics of the system, envelope a curve of class x. Then from the point P, x tangents can be drawn to the curve, that is to say, there are x polars of P which pass through P. But a point which lies on its polar in respect of a given conic is a point on the conic. Therefore x conics pass through P. But μ conics (by hypothesis) pass through P; so that $x=\mu$. Thus we get Chasles's Prop. xii., —*the polars of an arbitrary point, in respect of a system of conics* (μ, ν) *envelope a curve of class* μ. It follows that there are μ polars of P which pass through another arbitrary point Q; that is to say, there are μ conics of the system which divide harmonically a given segment PQ. This is Chasles's Prop. xxviii.

Suppose now that the condition Z''' is that the conics shall pass through a given point. Call S the condition that they shall divide harmonically the segment PQ. Then (by the above) the number of conics satisfying the conditions (Z, Z', Z'', S), and passing through the given point, is μ; that is to say, the *first characteristic* (μ') of the system (Z, Z', Z'', S) is equal to the *first characteristic* (μ) of the system (Z, Z', Z'', Z''') where Z''' is the condition of passing through a given point. In the next place let Z''' be the condition of touching a given line. Then the number of conics which satisfy the conditions

(Z, Z', Z'', Z''', S) is the same as the number which satisfy the conditions $(Z, Z', Z'', Z'''$, point); that is to say, the *second characteristic* (ν') of the system (Z, Z', Z'', S) is equal to the *second characteristic* (ν) of the system $(Z, Z', Z''$, point). Thus neither of the characteristics is altered when we substitute for the condition of passing through a given point, the condition S of dividing harmonically a given segment PQ.

By similar reasoning it may be shewn that neither characteristic is altered when we substitute for the condition of touching a given line, the condition of subtending harmonically a given angle.

1750. Given four straight lines whose equations are connected by the syzygy $x+y+z+w=0$; shew that the straight lines

$$lx+my+nz+sw=0, \quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if

$$(l\mu+\lambda m)+(n\sigma+\nu s)=(l\nu+\lambda n)+(s\mu+\sigma m)=(l\sigma+\lambda s)+(m\nu+\mu n).$$

Shew also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three; then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle.

[*E. T.* July, 1865. Solution Reprint, Vol. IV. p. 63.]

1775. If a straight line meet the faces of the tetrahedron $ABCD$ in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

[*E. T.* August, 1865. Solution Reprint, Vol. IV. pp. 66—8.]

1675. If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_2, n_3, be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc\,.\,ca\,.\,ab}=\frac{2m^2}{n_1n_2n_3}(\text{area of } ABC).$$

[*E. T.* March, 1865. Solution Reprint, Vol. IV. pp. 90, 1.]

1823. The conicoids which pass through six fixed points in space, intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilateral. (DEF. A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically.)

[*E. T.* October, 1865. Solution Reprint, Vol. IV. p. 110.]

1795. (1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Shew that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P.

[*E. T.* September, 1865. Solution Reprint, Vol. IV. p. 116.]

1638. Find the condition that the general equation of the third order may represent a cubic whose asymptotes form an equilateral triangle; and shew that this is always the case when the curve passes through three points and their three pairs of antifoci. [*E. T.* January, 1865. Solution Reprint, Vol. V. pp. 44, 5, February, 1866 ?]

Three lines forming an equilateral triangle meet the line at infinity in a point-cubic whose Hessian is the circular points. Now let

$$(a, b, c, d)(x, y)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

be the terms of the highest order in the general equation of the third degree in Cartesian coordinates; then the three lines represented by (1) are parallel to the asymptotes. Now the Hessian of (1) is

$$(ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2;$$

and in order that this may be identical with x^2+y^2, we must have

$$(ad=bc,\ ac-b^2=bd-c^2),$$

which are the conditions required.

To prove the proposition, let A, B, C be the three points, and P, Q the circular points at infinity. Let the equation to the three lines PA, PB, PC be $U=0$, and to the three lines QA, QB, QC, $V=0$. Then the nine intersections of the cubics U, V are the three points, and their three pairs of antifoci. Any other cubic through those intersections may be represented by $U=kV$. Let U', V' be the terms of highest order in U, V; then $U'-kV'$ will be the terms of highest order in $U-kV$. But U', V' must be perfect cubes, representing the circular points; say x^3, y^3. Then $U'-kV'$ is x^3-ky^3. But the Hessian of x^3-ky^3 is $-kxy$. That is, every cubic represented by $U=kV$ meets the line at infinity in a point-cubic whose Hessian is the circular points. Or, which is the same thing, the asymptotes of every such cubic form an equilateral triangle.

There is no difficulty in finding the conditions when the equation is given in a homogeneous form. We substitute for z, from the equation of the line at infinity, in the cubic and in any circle; let the former substitution give (1), and the latter, $Ax^2+2Bxy+Cy^2=0$; then the conditions are

$$\frac{ac-b^2}{A}=\frac{ad-bc}{2B}=\frac{bd-c^2}{C}.$$

1888. [(1) Amongst the conics which have three-pointic contact with a cubic at a given point, there are, in general, three which have a three-pointic contact elsewhere, and a fourth passes through the points of contact of these three with the cubic. The number of such conics is reduced to one, when the cubic has a cusp.

(2) Amongst the conics which have four-pointic contact with a cubic at a given point, there are three which touch the cubic elsewhere. There is but one

such conic when the cubic has a node, and none when it has a cusp. Proposed by E. de Jonquières. Reprint, Vol. v. p. 56, *E. T.* March, 1866.]

1. Let A be the given point on the cubic, and let F be any point of inflexion, or flex. Join AF, and let AF meet the curve again in B. Then a conic may be drawn having three-pointic contact with the cubic at the points A and B. For, consider these three cubic curves: (a) the cubic itself; (b) the line ABF taken three times over; (c) a conic having three-pointic contact at A and touching the cubic at B, together with the tangent at the flex F. The cubic (a) passes through eight out of the nine points of intersection of the cubics (a) and (b); consequently, by the theorem known as the involution of cubics, it passes through the ninth point. That is to say, a conic having three-pointic contact at A, and touching the cubic at B, will necessarily have three-pointic contact at B.

By joining the point A, therefore, to the *nine* flexes F, we shall obtain *nine* points B, and therefore nine conics fulfilling the required conditions; but only three of these points B will be real when the point A is real.

It remains to shew that a conic having three-pointic contact at A passes through the three real points B. Let F_1, F_2, F_3 be the three real flexes, which are known to be in one straight line; and let B_1, B_2, B_3 be the corresponding points B. Draw a conic U having three-pointic contact at A and passing through B_1, B_2. Then consider these three cubic curves: (a) the cubic itself; (b) the straight lines AB_1F_1, AB_2F_2, AB_3F_3; (c) the conic U and the line $F_1F_2F_3$. The cubic (c) passes through eight out of the nine intersections of the cubics (a) and (b); consequently it passes also through the ninth. That is to say, the conic U passes through the point B_3.

A cusped cubic has only one flex; in this case, therefore, the number of conics is reduced to one.

2. Let A be the given point. By COTTERILL's Theorem (which again is a particular case of the involution of cubics), if a conic have four-pointic contact with the cubic at A, its remaining chord of intersection with the cubic will pass through a fixed point M on the curve. Now the tangent at A, taken twice over, may be regarded as a conic having four-pointic contact at A; whence it appears that the point M is the second tangential of A. The number of conics of the system which touch the cubic at some other point is therefore the number of tangents that can be drawn from M to the curve; that is, four in general, two when the cubic has a node, and one when it has a cusp. But in this number there is always included that conic which is made up of the tangent at A taken twice over; and this is not a proper solution.

1996. If four circles $A=0$, $B=0$, $C=0$, $D=0$ are mutually orthotomic, the square of the radius of a circle $lA+mB+nC+sD=0$ is

$$(l^2r_1^2+m^2r_2^2+n^2r_3^2+s^2r_4^2)\div(l+m+n+s)^2,$$

where r_1, r_2, r_3, r_4 are the radii of A, B, C, D.

[*E. T.* July, 1866. Solutions Reprint, Vol. VI. pp. 74, 5.]

1878. A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is $n2^{1-n}$; and

(2) the chance, that the sum of the squares described on them does not exceed $\frac{a^2}{n-1}$, is

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \cdot \frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}} \cdot \frac{1}{n^{\frac{1}{2}}}.$$

[*E. T.* January, 1866. Reprint, Vol. VI. pp. 83—87, *E. T.* November, 1866. A solution by Prof. Wolstenholme is given Vol. XI. pp. 17, 18.]

1. Let us define as follows. A point is taken *at random* on a (finite or infinite) straight line, when the chance that the point lies on a finite portion of the line varies as the length of that portion. And, a line is broken up at random when the points of division are taken at random.

Now, the n pieces will always be capable of forming a polygon except when one of them is greater than the sum of all the rest; that is greater than half the line. The *first* part of the question may therefore be stated thus: $n-1$ points are taken at random on a finite line; to find the chance that some one of the intervals shall be greater than half the line.

2. *First* solution. Bisect the line AB at C. Then the chance that one of the points of division shall lie within BC is $\frac{1}{2}$; therefore the chance that all the $n-1$ points shall lie within BC is 2^{1-n}. But this is the chance that the *first piece* (reckoning from A) shall be greater than AC. Next, I say that the chance of the r^{th} piece being greater than half the line is equal to the chance of the $(r+1)^{\text{th}}$ piece being greater. For let PQ be the portion which is made up of the r^{th} and $(r+1)^{\text{th}}$ pieces. And take $PR=QS=AC$*. Then if the point of division between the r^{th} and $(r+1)^{\text{th}}$ pieces lies within RQ, the r^{th} piece is greater than AC; and if it lies within PS, the $(r+1)^{\text{th}}$ piece is greater than AC. But $RQ=PS$; therefore by definition the chances are equal. Consequently, the chance that any one of the n pieces shall be greater than AC is equal to the chance that any other of the n pieces shall be greater than AC. And all these n events are mutually exclusive; while we have proved that the chance of the first of them is 2^{1-n}. Therefore the chance that some one piece is greater than AC is $n2^{1-n}$.

3. *Second* solution. I am convinced that there is a fallacy in the above, and have therefore tried to get a rigorous proof in this way. Take P a point in AC, and let $AP=x$. Consider a small element dx at P. I want to find the chance that the r^{th} piece, reckoning from A, may begin at P (within the element dx) and be greater than AC. This requires, first, that one of the $n-1$ points of division shall be within dx; the chance of this is $(n-1)\frac{dx}{a}$; next, $r-2$ of the remaining points must be within AP, and the chance of this is

$$\frac{|n-2}{|n-r\;|r-2}\left(\frac{x}{a}\right)^{r-2};$$

lastly, the $n-r+1$ points left must be within RB; whose chance is

$$\left(\frac{1}{2}-\frac{x}{a}\right)^{n-r+1}.$$

* [To draw the figure, take the points in the order A, P, S, C, R, Q, B.]

Therefore the chance required is

$$\frac{\lfloor n-1}{\lfloor n-r \lfloor r-2}\left(\frac{x}{a}\right)^{r-2}\left(\tfrac{1}{2}-\frac{x}{a}\right)^{n-r+1}\frac{dx}{a}.$$

Now, if we integrate this with respect to x from 0 to $\frac{1}{2}a$, we shall get the entire chance that the r^{th} piece may be greater than AC. The integral is easily found to be 2^{1-n}. And as there are thus n equal chances, whose events are all mutually exclusive, the chance that some one of these events will happen is $n2^{1-n}$.

4. *Third* solution. To make this clear, I will state first the previously-known analogous solutions in the cases where $n=3$ and $n=4$. When the line is divided into three pieces, call them x, y, z, and take their lengths for the coordinates of a point P in geometry of three dimensions. Then, since

$$x+y+z=a \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and x, y, z are all positive, the point P must be somewhere on the surface of the equilateral triangle determined on the plane (1) by the coordinate planes. Now, consider those points on the triangle for which $x>\frac{1}{2}a$. These are cut off by the plane $x=\frac{1}{2}a$; and it is easy to see that this plane cuts off from one corner of the triangle a similar triangle of *half the linear dimensions*, and therefore of the fourth the area. Now, there are three corners cut off; their joint area is therefore three-fourths of the area of the triangle; and the chance required is accordingly $\frac{3}{4}$.

When the line is divided into *four* pieces, take the *first three* pieces as the coordinates of a point in space. Then we have $x+y+z<a$, and x, y, z all positive; so the point must lie within the content of the tetrahedron bounded by the plane $x+y+z=a$ and the coordinate planes. Now, if $x+y+z<\frac{1}{2}a$, the *fourth piece* must be *greater* than $\frac{1}{2}a$. The points for which this is the case are cut off by the plane $x+y+z=\frac{1}{2}a$; and it is easily seen as before that this plane cuts off from one corner of the tetrahedron a similar tetrahedron of half the linear dimensions, and therefore of one-eighth the volume. So also the plane $x=\frac{1}{2}a$ cuts off from another corner a similar tetrahedron of half the linear dimensions. Since therefore there are four corners cut off, their joint volume is ($\frac{4}{8}$ or) one half of the volume of the tetrahedron; and the chance required is accordingly $\frac{1}{2}$.

5. Now, consider the analogous cases in geometry of n dimensions. Corresponding to a closed area, and a closed volume, we have something which I shall call a *confine*. Corresponding to a triangle, and to a tetrahedron, there is a confine with $n+1$ corners or vertices, which I shall call a *prime* confine, as being the simplest form of confine. A prime confine has also $n+1$ faces, each of which is, not a plane, but a prime confine of $n-1$ dimensions. Any two vertices may be joined by a straight line, which is an *edge* of the confine. Through each vertex pass n edges. A prime confine may be *regular*, which it is when any three vertices form an equilateral triangle; or *rectangular*, which it is when the edges through some one vertex are all equal and at right angles to one another.

To solve the question for general values of n, we may adopt as a type either of the geometrical solutions given for the cases $n=3$ and $n=4$. First, take the

lengths of the n pieces for the coordinates of a point in geometry of n dimensions. Then, since their sum is a, and they are all positive, the point must lie within a certain regular prime confine of $n-1$ dimensions. The supposition that a certain piece is greater than $\frac{1}{2}a$ cuts off from one corner of the confine a similar confine of half the linear dimensions, and therefore of 2^{1-n} times the content. And as there are n corners, their joint content is $n \cdot 2^{1-n}$ times the content of the confine; the chance required is consequently $n2^{1-n}$. Or, take the lengths of the *first* $n-1$ *pieces* as the coordinates of a point in geometry of $n-1$ dimensions; the point will then lie within a certain *rectangular* confine of $n-1$ dimensions; and the investigation proceeds as before, the n corners being cut off in the same manner.

6. It will be seen that this *third* solution involves in a geometrical form the assumption of which some sort of proof was given in the *first* solution. Let us make this extension of our fundamental definition :—A point is taken *at random* in a (finite or infinite) space of n dimensions, when the chance that the point lies in a finite portion of this space varies as the contents of that portion. The assumption is that when the lengths of the pieces into which a line is broken up are taken as coordinates of a point, then if the line is broken up at random the point is taken at random, and *vice versâ*. The proof of this assumption may be shewn to involve a geometrical proposition equivalent to the integration by parts of the differential in Art. (3).

Making this assumption, we may solve the second part of the question by the method of the *third* solution of the first part. I will first state the previously known analogous solution of the case where $n=3$. The question is in this case,—If a line of length a be broken into three pieces at random, find the chance that the sum of the squares of these pieces shall be less than $\frac{1}{2}a^2$. Take the lengths of the three pieces for coordinates x, y, z of a point P in geometry of three dimensions; then, as before, the point must lie somewhere in the area of the equilateral triangle determined on the plane $x+y+z=a$ by the coordinate planes. But if also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain circle determined on the plane $x+y+z=a$ by the sphere $x^2+y^2+z^2=m^2$. Now, in the case where $m^2=\frac{1}{2}a^2$ this circle is the circle inscribed in the equilateral triangle; so that the question reduces itself to this one :—

To find, in terms of the area of an equilateral triangle, the area of its inscribed circle.

Now let us go a little further, and consider the case in which $n=4$. Here we shall have to take a point P in geometry of four dimensions; the point must lie somewhere in the regular tetrahedron determined on the hyper-plane

$$x+y+z+w=a$$

by the coordinate hyper-planes. If also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain sphere determined on the hyper-plane $x+y+z+w=a$ by the quasi-sphere

$$x^2+y^2+z^2+w^2=m^2.$$

In the particular case where m is the perpendicular from the vertex on the base of a rectangular tetrahedron, each of whose equal edges is of length a, or

$$m^2 = \tfrac{1}{3}a^2,$$

this sphere is the sphere inscribed in the regular tetrahedron. The question is therefore reduced to this one :—

To find, in terms of the volume of a regular tetrahedron, the volume of its inscribed sphere.

Now, a similar reduction holds in the general case; viz., the question can always be reduced to this one :—

To find, in terms of the contents of a regular prime confine of $n-1$ dimensions, the contents of its inscribed quasi-sphere.

This question I proceed to solve.

7. Let $n-1=p$. *The perpendicular from any vertex on the opposite face of a regular prime confine in* p *dimensions* $= \left(\dfrac{p+1}{2p}\right)^{\frac{1}{2}}$. (edge).

For, let O be the vertex in question, OA, OB, ... the p edges through O. Draw through each vertex A a space of $p-1$ dimensions parallel to the face opposite to A. The p spaces thus drawn will intersect in a point P, such that OP is the diagonal of a confine analogous to a parallelogram and to a parallelepiped. Then OP is p times the perpendicular from O on the opposite face of the regular confine; for the perpendicular is the projection of *one* edge at a certain angle, while OP is the projection at the same angle of a broken line consisting of p edges.

We have also

$$OP^2 = OA^2 + OB^2 + OC^2 + \ldots + 2OA \,.\, OB \cos AOB + \ldots$$
$$= \Sigma OA^2 + \Sigma OA \,.\, OB \text{ (since } \cos AOB = \tfrac{1}{2}, \text{ \&c.)},$$
$$= \{p + \tfrac{1}{2}p(p-1)\} \,.\, OA^2 = \tfrac{1}{2}p(p+1) \,.\, OA^2,$$
$$\therefore \text{ (perpendicular)}^2 = \frac{OP^2}{p^2} = \frac{p+1}{2p} \,.\, \text{(edge)}^2.$$

{If the confine were *rectangular*, or all the angles at O right angles, we should have $\cos AOB = 0$, &c., and so

$$\text{(perpendicular)}^2 = \frac{1}{p}\,\text{(edge)}^2 = \frac{a^2}{n-1};$$

which proves that the question *does* always reduce itself to the one now under consideration}.

The content *of a regular prime confine in* p *dimensions whose edge is* a, *is*

$$= \frac{a^p}{\lfloor p} \left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Suppose this formula true for $p-1$ dimensions; that is, let

$$V_{p-1} = \frac{a^{p-1}}{\lfloor p-1} \left(\frac{p}{2^{p-1}}\right)^{\frac{1}{2}}.$$

Now, content of confine

$$=\frac{1}{p}\times\text{perpendicular}\times\text{content of face},$$

or $$V_p=\frac{a}{p}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.V_{p-1}=\frac{a^p}{\lfloor p}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Hence the formula, if true for one value of p, is true for the next; now it can be immediately verified in the case of $p=1$; therefore it is generally true.

The radius of the inscribed quasi-sphere

$$=\frac{a}{\{2p\,(p+1)\}^{\frac{1}{2}}}.$$

We can divide the regular confine into $p+1$ equal confines, each having the centre of the inscribed quasi-sphere for vertex; and the content of one of these

$$=\frac{\rho}{p}\times\text{content of face};$$

but the sum of them all is equal to the content of the whole confine. Hence $(p+1)\,\rho=$perpendicular of confine

$$=a\left(\frac{p+1}{2p}\right)^{\frac{1}{2}},\text{ or, }\rho=\frac{a}{\{2p\,(p+1)\}^{\frac{1}{2}}}.$$

The content of the quasi-sphere

$$=\rho^p.\frac{\{\Gamma(\frac{1}{2})\}}{\Gamma(\frac{1}{2}p+1)}.$$

For it is the value of

$$\iiint\ldots dx\,dy\,dz\ldots$$

the integral being so taken as to give to the variables all values consistent with the condition that $x^2+y^2+z^2+\ldots$ is not greater than ρ^2. (See Todhunter's *Integral Calculus*, Art. 271.)

Let C_p denote this content; then

$$C_p=\rho^p\frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)}=\frac{a^p}{(2p^2+2p)^{\frac{1}{2}p}}.\frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)},$$

therefore $$\frac{C_p}{V_p}=\left(\frac{\pi}{p^2+p}\right)^{\frac{1}{2}p}.\frac{\Gamma(p-1)}{\Gamma(\frac{1}{2}p+1)}.\frac{1}{(p+1)^{\frac{1}{2}}}.$$

Restore $n-1$ for p, and we get the answer to the question, namely,

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)}.\frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}}.\frac{1}{n^{\frac{1}{2}}}.$$

8. The following are applications of the same method.

If a line be broken up at random into n pieces, the chance of an assigned two of them (the p^{th} and q^{th} from one end) being together greater than half the line, is $n2^{1-n}$.

If n pieces be cut off at random, one from each of n equal lines, the chance that the pieces cannot be made into a polygon is $\frac{1}{\lfloor n-1}$.

2253. If four circles have a common radical centre, it is possible to find four planes which intersect, two and two, at angles equal to those at which the circles intersect, but not otherwise.

[*E. T.* October, 1866. Reprint, Vol. VII. p. 22.]

2220. A, B, C, D are four points on a circle, and through every pair, as AB, another circle (AB) is drawn; then the pair of circles (AB), (CD) intersects the pair (AC), (DB) in four new points on a circle U, the pair (AC), (DB) meets (AD), (BC) on a circle V; and the pair (AD), (BC) meets (AB), (CD) on a circle W; also the thrée circles U, V, W have a common radical axis. (This may be extended to spheres; and there are also analogous properties of rectangular hyperbolas).

[*E. T.* August, 1866. Reprint, Vol. VII. p. 37.]

[2135.* Reprint, Vol. VII. p. 45, is merely a repetition of 1378 (p. 566), with a different solution. It is noteworthy that there is no question so numbered in the *E. T.*, for the August (1866) No. gives 2110 in succession to 2009 and 2220 next to 2119.]

1962. Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n.

[*E. T.* May, 1866. Reprint, Vol. VII. p. 47.]

2343. A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D. AD, AC meet BC, BD in E, F; shew that EF is a fixed straight line and meets CD on the polar of A.

[*E. T.* February, 1867. Reprint, Vol. VIII. pp. 64, 5.]

2522. Prove (1) that the perpendiculars of a circular triangle have a common radical axis; and (2) that if the perpendiculars from the pairs of vertices of one circular triangle on the sides of another meet in a point, then *vice versâ*. (*Def.* A, B, C being circles, a circle coaxal with A, B, and orthogonal to C, is called the perpendicular from AB on C.)

[*E. T.* November, 1867. Reprint, Vol. IX. p. 42.]

2383. A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; shew that DE and CF meet in a fixed point G, and that ABG is a straight line.

[*E. T.* April, 1867. Reprint, Vol. x. p. 81.]

2732. An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, shew that the locus of its centre is an ellipse.

[*E. T.* September, 1868. Reprint, Vol. x. p. 96.]

2748. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

[*E. T.* October, 1868. Reprint, Vol. x. pp. 105, 6.]

2301. A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

[*E. T.* December, 1866. Reprint, Vol. xi. p. 31.]

2776. Through A, the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, shew that there is a fixed point in AB at which also they subtend a right angle.

[*E. T.* November, 1868. Reprint, Vol. xi. p. 64.]

2108. Required Analogues in Solid Geometry to the following propositions in Plane Geometry:—

(*a*) The perpendiculars of a triangle meet in a point.

(*b*) The middle points of the diagonals of the quadrilateral are in one straight line.

(*c*) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.

(*d*) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.

(*e*) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.

(*f*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(*g*) The sum of the angles of a triangle = two right angles.

(*h*) In any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

[*E. T.* July, 1864, where it is numbered 1526, cf. 2135, p. 607. Reprint, Vol. xi. pp. 102, 103, May, 1869.]

I have an analogue for each of the four (*d*), (*e*), (*f*), (*g*), and more than one of each of the others obtained by extensions of Mr Greer's methods*.

E.g.:—

(*c*) A straight line cuts the faces of a tetrahedron $ABCD$ in a, b, c, d; the spheres whose diameters are Aa, Bb, Cc, Dd, have a common radical axis. Hence the middle points of these four lines are in one plane.

* [? Reprint, Vol. ii. p. 80; cf p. 588, supra.]

Let a conicoid whose asymptotic cone has three generating lines at right angles be called a rectangular conicoid.

(*d*), (*e*) Every rectangular conicoid circumscribing a tetrahedron whose perpendiculars meet in a point, passes through the point. And every rectangular conicoid to which a tetrahedron is self-conjugate, passes through the centres of the eight touching spheres.

(*f*) $$\sin(ABC) = \sin(ABD + BCD + CAD)$$

$$= \sin(BCD)\,.\,\cos\widehat{AD} + \sin(CAD)\,.\,\cos\widehat{BD} + \sin(ABD)\,.\,\cos\widehat{CD},$$

where A, B, C, D are four lines in space, and

$$\sin^2 ABC = \begin{vmatrix} 1, & \cos\widehat{AB}, & \cos\widehat{AC} \\ \cos\widehat{AB}, & 1, & \cos\widehat{BC} \\ \cos\widehat{AC}, & \cos\widehat{BC}, & 1 \end{vmatrix}.$$

(*g*) In the triangle case this should be written

$$(BC) + (CA) + (AB) = 0.$$

The analogue is then obviously

$$(BCD) - (CDA) + (DAB) - (ABC) = 0,$$

A, B, C, D being any four planes.

(*h*) In any tetrahedron,

$$\frac{AC\,.\,DB}{\sin\widehat{AC}\,.\,\sin\widehat{DB}} = \frac{abcd}{V^2} = \frac{a}{\cos A} = \frac{V^6}{(abcd)^2}\,.\,\cos A\cos B\cos C\cos D,$$

where a, b, c, d are the faces, and

$$\cos^2 A = \begin{vmatrix} 1, & \cos\widehat{BC}, & \cos\widehat{BD} \\ \cos\widehat{BC}, & 1, & \cos\widehat{CD} \\ \cos\widehat{BD}, & \cos\widehat{CD}, & 1 \end{vmatrix},$$

($\widehat{AB}$, &c. denoting angles between planes).

2793. C is the single focus of a semicubical parabola, and from any point O three tangents are drawn to the curve; if CD, CE, CF be perpendicular to them, shew that DE and CF are equally inclined to the direction of the infinite branches.

[*E. T.* December, 1868. Reprint, Vol. XII. p. 22.]

2932. [Given the inscribed and circumscribed circles of a triangle, the envelope of the polar circle is a bicircular quartic. Proposed by the Rev. J. Wolstenholme. Reprint, Vol. XII. p. 52.]

Let B, C, X be the inscribed, circumscribed, and polar circles respectively. The circle X has to be such that a triangle self-conjugate with regard to it can be circumscribed to B and inscribed to C; that is, it is subject to a condition of the first and a condition of the second degree in its coefficients. Two such circles can therefore be drawn through an arbitrary point.

Now, *any series of circles, such that two of them can be drawn through an arbitrary point, is one system of bitangent circles of a bicircular quartic.* For the equation of a circle of the series must contain a variable parameter in the second order; that is, it must be of the form

$$X+2\theta Y+\theta^2 Z=0^*,$$

where $X=0$, $Y=0$, $Z=0$ are circles. But the envelope of this is $XZ=Y^2$, a bicircular quartic.

This important remark is made by Cremona (*Teoria Geometrica*, ii. 21) in the case of a series of curves of any order and of index 2; that is, such that two of them can be drawn through an arbitrary point. The envelope is always what Prof. Cayley (*Edinb. Phil. Trans.*, 1868) calls a *trizomal* curve; viz X, Y, Z being any three curves of the series, its equation may be written

$$\sqrt{\alpha X}+\sqrt{\beta Y}+\sqrt{\gamma Z}=0$$

2923. [In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle. Proposed by S. Roberts, M.A. Reprint, Vol. XII. p. 57. *E. T.* September, 1869.]

A bicircular quartic is its own inverse with regard to any focal circle (Moutard, *Nouvelles Annales*, 1866). The bitangent circles divide themselves into four systems, all the circles of any one system being cut orthogonally by the corresponding focal circle. Through any point of the plane can be drawn two bitangent circles of each system. The two bitangent circles, then, that can be drawn through the centre of the focal circle of their system, are in fact straight lines touching the curve in two pairs of inverse points, which consequently lie on a circle.

The corresponding theory in anallagmatic surfaces is that the centre of each one of the five principal spheres is vertex of a quadric cone doubly tangent to the surface; the curve of contact being the intersection of this cone with a sphere. These five cones noticed by Moutard are independently arrived at by Kummer (*Berlin. Monatsber.*) in the case of the general quartic surface with a nodal conic.

2924. [On a focal chord PSQ of a parabola are taken p, q, on opposite sides of S, such that $Sp\,.\,Sq=SP\,.\,SQ$, and any parabola is described through p, q, and having its axis parallel to that of the former: prove that their chord of intersection will pass through S. Proposed by the Rev. J. Wolstenholme. Reprint, Vol. XII. pp. 62, 3. *E. T.* September, 1869.]

I consider the following more general question:—

Through a point a let a line B be drawn meeting a conic in l, m; then the quantity

$$al\,.\,am \sin BP\,.\,\sin BQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

* [With reference to this solution Mr Wolstenholme remarks (note p. 53), that "it is not necessary that X, Y, Z should be all circles; it is sufficient that one be a circle and the others straight lines. Thus the envelope might, so far as depends on this reasoning, be a circular cubic."]

(where P and Q are the asymptotes) is independent of the position of the line B, and may be called the *distance* of the point a from the conic. What now is the locus of a point equidistant from two given conics?

Let $C_2=0$, $D_2=0$ be the equations of the conics, and let a^2C_2 denote the result of substituting the coordinates of the point a for the variables in C_2; also let i, j be the circular points at infinity. Then I find that the distance (above defined) of the point a from C_2 is

$$\frac{a^2C_2}{(aij)^2 \cdot \sqrt{(i^2C_2 \cdot j^2C_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where, of course, (aij) means the determinant formed with the coordinates of the points a, i, j.

This being so, the equation of the required locus is

$$\frac{C_2}{\sqrt{(i^2C_2 \cdot j^2C_2)}} = \frac{D_2}{\sqrt{(i^2D_2 \cdot j^2D_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

showing that the locus is a conic passing through the intersections of the two given ones.

Now if we are using Cartesian coordinates, and if the two conics are similar and similarly situated, it is easy to see that the terms of the second order have entirely disappeared from the equation (3); which indicates that the line at infinity is part of the locus. The remainder of it is then their finite chord of intersection; which is a true radical axis, in the sense that if any line whatever is met by the radical axis in a, by C_2 in l, m, and by D_2 in l', m', we must have always

$$al \cdot am = al' \cdot am' \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

To apply this to the question we have only to observe that parabolæ with parallel axes are homothetic, or similar and similarly situated curves; and that the equation

$$Sp \cdot Sq = SP \cdot SQ$$

indicates that the focus is situated on their radical axis.

The theorem of the radical axis of two homothetic conics may of course be proved for ellipses by orthogonal projection from the circle, and then extended by the doctrine of continuity to the rest.

2446. PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Shew that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence shew by inversion that, if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

[*E. T.* July, 1867. Reprint, Vol. XII. p. 91.]

2960. The envelope of a series of surfaces of order n, such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in the form

$$\sqrt{(\alpha X)} + \sqrt{(\beta Y)} + \sqrt{(\gamma Z)} = 0,$$

where $X=0$, $Y=0$, $Z=0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n, such that two of them can be drawn through *two* arbitrary points, is a surface of order $2n$, whose equation referred to any four surfaces of the net is of the same form as the equation of a quadric referred to four tangent planes.

[*E. T.* September, 1869. Reprint, Vol. XII. p. 96.]

2942. Let p, q be the foci, and P, Q the asymptotes of a conic; θ the angle it subtends at a point a, and $\{A\}$ the chord it cuts off from a line A. Then

1. If a line B is drawn through the point a meeting the conic in l, m,

$$al \,.\, am \,.\, \sin BP \sin BQ = \frac{ap^2 \;\; aq^2 \,.\, \sin^2\theta}{pq^2}.$$

2. If from a point b on the line A tangents L, M are drawn to the conic,

$$\sin AL \sin AM \,.\, bp \,.\, bq = \frac{\sin^2 AP \sin^2 AQ \,.\, \{A\}^2}{\sin^2 PQ}.$$

(Here al means the distance between the points a, l, and BP means the angle between the lines B, P.)

3. Find analogous propositions for a curve of any order on a plane or on a sphere.

[*E. T.* August, 1869, 4724, July, 1875. Reprint, Vol. XII. pp. 99—101. *E. T.* November, 1869. Solutions given by the Rev. J. Wolstenholme and Mr J. J. Walker, and Prof. Clifford remarks]

The extensions for a plane are

(curve of class n) $.al \,.\, am \,.\, an \;\; .\sin BP \,.\, \sin BQ \;\; .$

$$= \frac{(ap \,.\, aq \,.\, ar \;\;)^{2(n-1)} \sin^2 LM \,.\, \sin^2 LN \ldots}{pq^2 \,.\, pr^2 \,.\, qr^2},$$

(curve of order m) $.\sin AL \,.\, \sin AM. \;.bp \,.\, bq \;\; .$

$$= \frac{(\sin AP \,.\, \sin AQ \ldots)^{2(m-1)}\, lm^2 \,.\, ln^2 \,.\, mn^2 \,.}{\sin^2 PQ \,.\, \sin^2 PR \,.\, \sin^2 QR},$$

where P, Q, R, .. are the asymptotes, and p, q, r, . the real foci. These give me ideas of the "distance" of a point from a line or surface, and they may be extended so as to give the distance of two curves from one another.

3021. The three pairs of foci of a sphero-conic are a, a'; b, b', c, c'; and p is any point on the sphere.

Prove the formulæ

$$\sin aa' \,.\, \sin bb' \,.\, \sin cc' = 8 \,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(\sin aa')^{-\frac{2}{3}} + (\sin bb')^{-\frac{2}{3}} + (\sin cc')^{-\frac{2}{3}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{(\sin pa \,.\, \sin pa')^3}{\sin^2 aa'} = \frac{(\sin pb \,.\, \sin pb')^3}{\sin^2 bb'} = \frac{(\sin pc \,.\, \sin pc')^3}{\sin^2 cc'} \ldots\ldots\ldots\ldots (3).$$

[*E. T.* December, 1869. Reprint, Vol. XIII. p. 50.]

2979. Two triads of points abc, $\alpha\beta\gamma$ being taken on a line, let the two triads be called *harmonic* of one another when

$$a\alpha\,.\,b\beta\,.\,c\gamma+a\beta\,.\,b\gamma\,.\,c\alpha+a\gamma\,.\,b\alpha\,.\,c\beta+a\gamma\,.\,b\beta\,.\,c\alpha+a\beta\,.\,b\alpha\,.\,c\gamma\,.+a\alpha\,.\,b\gamma\,.\,c\beta=0;$$

then (1) the envelope of a line cut harmonically by two cubics is of the third class. (The contravariant $\overline{\alpha 11'^3}$)—(2) this line is also cut harmonically by every pair of cubics through the intersections of the first two. (3) The envelope of a line cut harmonically by a given cubic and the cubic made up by the polar line and conic of a given point is the mixed concomitant $\overline{\alpha 12}\,.\,\overline{\alpha 13}^2$. (4) Two cubics having the same inflexions cut harmonically any line whatever.

[*E. T.* October, 1869. Reprint, Vol. XIII. p. 52.]

3197. If the epicycloid described by a point on the circumference of a circle rolling on an equal fixed circle be loaded with matter proportional to its curvature at every point, the centre of gravity of the whole will be at the centre of the fixed circle.

[*E. T.* August, 1870. Reprint, Vol. XIV. p. 98.]

3282. It is known that the circles circumscribing the triangles formed by four lines meet in a point, and that the points so belonging to the five tetragrams formed by five lines lie in a circle. Prove that the circles so belonging to the six pentagrams formed by six lines meet in a point, and so on; the series of theorems being interminable.

To every $2n+1$ lines there belongs in this way a circle. If from any point p on this circle perpendiculars be let fall on the straight lines, their feet will all lie on a curve of order n, having a $(n-1)$-ple point at p.

[*E. T.* December, 1870. Reprint, Vol. XV. p. 47.]

3885. If A be the single focus of a semi-cubical parabola, there exists a straight line BC, such that if two tangents at right angles cut it in B, C, the angle BAC is also a right angle.

[*E. T.* October, 1872; 2674, June, 1868. Reprint, Vol. XVIII. p. 82.]

3876. [Shew that there are 5184 positions in a cubic curve such that at each of them curves of the 50th order may be drawn having 90-point contact with the cubic. Proposed by J. J. Sylvester, F.R.S. Reprint, Vol. XIX. p. 46. April, 1873?]

The gross number of points where a curve of order n can have $3n$-point contact with a cubic is $9n^2$. The problem is in fact the same as that of the divisions of the periods of an elliptic function by $3n$, and as there are two periods, there are $9n^2$ solutions. (Clebsch, *Anwendung der Abelschen Functionen in der Geometrie*, Crelle, LXII.) But in the case when n is a composite number, all the curves whose order is a division of n, and which have complete contact with the cubic, are included in the result. Thus each inflexional tangent, taken n times over, constitutes a curve of order n having $3n$-point contact with the cubic; and the nine inflexional tangents are thus always

included in the $9n^2$ solutions. To obtain the number of *proper* solutions, then, we must subtract all these improper ones. When $n=30$, the result is

$$9\{30^2-15^2-10^2-6^2+5^2+3^2+2^2-1^2\}=9\times 576=5184.$$

Here it is to be observed that the curves of order 5 are *twice* subtracted, with the curves of order 15 and 10; so that they have to be added in again. The same remark applies to the orders 3 and 2. The curves of order 1 (inflexional tangents) having been thrice subtracted and thrice added, must finally be subtracted again.

4010. Prove that the lines of curvature of a quadric surface are projected from an umbilic on a plane parallel to its tangent plane into a series of confocal Cartesian ovals.

[*E. T.* March, 1873. Reprint, Vol. XIX. pp. 73, 4.]

4034. Prove that the forty umbilics of a cubic surface which passes once, or of a quartic surface which passes twice, through the imaginary circle at infinity, lie by fives upon sixteen straight lines.

[*E. T.* April, 1873; 3308, January, 1871. Reprint, Vol. XIX. p. 77.]

2020. [Reprint, Vol. XIX. p. 84. The question and solution are identical with those of 1379, p. 567 supra: here the Authorship of the Question is ascribed to N'Importe.]

4097. If about a prolate conicoid of revolution there be described an octahedron so that its three diagonals pass through a focus, shew that they must be at right angles to each other.

[*E. T.* June, 1873; 1415, August, 1863. Reprint, Vol. XIX. p. 100.]

2022. [Reprint, Vol. XIX. p. 108. This is 1378, of which see Solution p. 566 supra.]

4069. 1. Curves of order $2n+1$ pass n times through each circular point and through n^2+4n+1 other fixed single points (or their equivalent in multiple points); shew that the envelope of their asymptotes is a tricusped hypocycloid.

2. Curves of order $2n+2$ pass n times through each circular point and through n^2+6n+4 other fixed points, and their real asymptotes are at right angles; shew that the envelope of their asymptotes is a tricusped hypocycloid.

[*E. T.* May, 1873. Reprint, Vol. XX. pp. (31) 50—53.]

Prof. Wolstenholme's remark in his solution of this question, given on p. 31 of this volume of the *Reprint*, that the first part "is not quite true as it stands," has led me to examine the whole with the help of his method; and it turns out, singularly enough, that it is the *second* part that requires correction, not the first. The way in which this comes about is instructive, and the corrected theorem leads us to consider a somewhat interesting series of curves.

1. I will first state the grounds on which I originally concluded that these theorems were true. It is required to find the envelope of the asymptotes of a pencil of curves which if of odd order have *one* real point at infinity besides

the circular points, if of even order *two* which are at right angles or harmonic of the circular points. The intersections with the line infinity at the circular points are due to multiplicity of these points, not to contact with the line infinity.

Now, first, *the line infinity is a tangent to this envelope at each of the circular points and no otherwhere.* For the line infinity can only become an asymptote by the variable one point or one of the variable two points at infinity coming to coincide with one of the circular points. In the second case the variable two points being harmonic of the circular points, if one of them coincide with a circular point, the other must coincide with it. In both cases, then, there are two curves of the pencil which have the line infinity for asymptote; and it is clear that the intersection of the line infinity with the next consecutive asymptote (i.e. its point of contact with the envelope) is the circular point at which it is an asymptote.

Next, from any point at infinity not a circular point, one tangent distinct from the line infinity can be drawn to the envelope. For there is one curve of the pencil that passes through this point.

If, then, the line infinity is an ordinary tangent at each of the circular points, we see that from any point at infinity three tangents may be drawn to the envelope; viz., the line infinity counting twice, and one other. The envelope therefore is of the third class, having the line infinity for double tangent whose points of contact are the circular points; that is to say, a hypocycloid of three branches.

In fact, the tangential equation of the curve may be at once written down. Let $i=0$, $j=0$ be the equations to the circular points, $k=0$ that to some other point; then the equation is $ijk+(i,j)^3=0$. It is, in fact, of the same form as the equation of a cubic curve having a node at the origin to which the axes are tangents. If for k we write $k+\lambda i+\mu j$, it is clear that by proper choice of λ, μ we can get rid of the two middle terms of $(i,j)^3$; the equation then becomes $ijk+\alpha i^3+\beta j^3=0$, which is the same as $p=a\cos 3\theta$, where p is the distance of a tangent from the origin $k=0$, and θ the angle it makes with a fixed line. (Salmon, *Higher Plane Curves*, p. 271, Ex. 5.)

This result is true *if the line infinity is an ordinary tangent at each of the circular points.* Now this holds good in the *first case* of the question; for in this the one variable point at infinity is made to move up to a multiple point, and so only one branch acquires an ordinary contact; in virtue of this, then, the line infinity counts only once as an asymptote for each circular point. It also holds good in the already well-known case of curves of the second order, i.e. in the second case of the question when $n=0$. For in this only the two variable points at infinity coincide at a circular point, making again an ordinary contact.

But in the second case of the question, when n is not zero, something different happens. Here the two variable points at infinity simultaneously approach a circular point which is already multiple on the curve; they approach it on the same branch, and *produce a point of inflexion on that branch.* In respect of each circular point, therefore, the line infinity counts

for *two* asymptotes ; the envelope is raised to the *fifth* class, and *has the line infinity for inflexional tangent at each circular point.*

2. This synthetic discussion shall now be confirmed by analysis. In the first case, the equation of a curve of the pencil is

$$(x+\lambda y)(x^2+y^2)^n+k(\lambda,\ 1\between x,\ y)^2\,.\,(x^2+y^2)^{n-1}+\ldots=0,$$

and its real asymptote is

$$(x+\lambda y)(1+\lambda^2)+k(\lambda,\ 1\between\lambda,\ -1)^2=0,$$

whose envelope is of the third class, touched by $k=0$ (the line infinity) for the two values $\lambda=\pm(-1)^{\frac{1}{2}}$; whence as before.

In the second case, it is convenient to write the equation of the variable curve in the form

$$\left\{x^2+\left(\lambda-\frac{1}{\lambda}\right)xy-y^2\right\}.\,(x^2+y^2)^n+k\left(\lambda-\frac{1}{\lambda},\ 1\between x,\ y\right)^3.\,(x^2+y^2)^{n-1}+\ ..=0.$$

The two real asymptotes are

$$\left(x-\frac{y}{\lambda}\right)(1+\lambda^2)^2+k\left(\lambda-\frac{1}{\lambda},\ 1\between 1,\ \lambda\right)^3=0,$$

$$(x+\lambda y)(1+\lambda^2)^2+\lambda k\left(\lambda-\frac{1}{\lambda},\ 1\between\lambda,\ -1\right)^3=0.$$

These have the same envelope, as one equation is got from the other by writing $-\lambda^{-1}$ for λ. The envelope is of the fifth class, touched by $k=0$ twice for each of the values $\lambda=\pm(-1)^{\frac{1}{2}}$. The line infinity is therefore a double tangent with united contacts (*i. e.* an inflexional tangent, just as a cusp is a double point with united branches) at each of the circular points.

3. It remains to investigate the nature of a curve of the fifth class having the line infinity for inflexional tangent at each of the circular points. This singularity being equivalent to two inflexions and four double tangents, Plücker's equations at once tell us that the curve is of the sixth order and has five cusps and five nodes. Its tangential equation may be at once written down, being of the same form as that of a quintic curve having a quadruple point at the origin, two of whose branches coincide with each of the axes; namely, it is $i^2j^2k+(i,j)^5=0$, where $i=0$, $j=0$ are the circular points, and $k=0$ is some other point. As before, we may suppose k to have been so selected as to get rid of the two middle terms of $(i,j)^5$. Now a particular case of the equation is

$$i^2j^2k+\alpha i^5+\beta j^5=0,\ \text{or}\ p=a\cos 5\theta,$$

which represents the hypocycloid [Fig. 123] described by a point on a rolling circle whose radius is two-fifths of the radius of the fixed circle.

The general equation may be transformed into

$$p=a\cos 5\theta+b\cos 3\theta+c\sin 3\theta,$$

or, if we write

$$p_1=2a\cos 5\theta,\ p_2=2b\cos 3\theta+2c\sin 3\theta,$$

the equation is

$$2p=p_1+p_2.$$

Now p_2 and p_1 are the distances from the origin of parallel tangents to a three-cusped and a sextic five-cusped hypocycloid respectively; whence we learn that *the curve in question is the envelope of a line midway between parallel tangents to two such hypocycloids.*

These hypocycloids have only to be concentric; and their relative size and orientation are the two variable elements in the equation of our curve. The method of description by tangents, however, gives us immediately a description by points, since it is clear that the point of contact of the variable tangent bisects the line joining the points of contact of the two tangents to which it is parallel and intermediate. In this way it is easy to draw roughly a few typical forms.

4. A hypocycloid in which the radii of the rolling and fixed circles are to one another as n to $2n+1$ is a curve of order $2n+2$, class $2n+1$, with $2n+1$ cusps, $(n-1)(2n+1)$ nodes, and has $(n+1)$-pointic contact with the line infinity at each circular point. Its tangential equation is

$$i^n j^n k + \alpha i^{2n+1} + \beta j^{2n+1} = 0 ;$$

or, which is the same thing, $p = a\cos(2n+1)\theta$.

To this simplest class of roulettes, all whose tangential singularities are at infinity, it may be permissible to give the name "stars." Thus an ordinary tricusp is a three-rayed star, the curve in Fig. [123] is a five-rayed star, and so on. We may now state the following proposition:

Every curve of class $2n+1$, which has $(n+1)$-pointic contact with the line infinity at each circular point, is the envelope of a line which is the mean of the parallel tangents of n concentric stars of all odd classes up to $2n+1$.

Namely, its equation is $\qquad i^n j^n k + (i, j)^{2n+1}$,

or $$p = a_3\cos 3\theta + b_3 \sin 3\theta + \, .. + a_{2n+1}\cos(2n+1)\theta,$$

from which the proposition is obvious. The curve has the same number of nodes and cusps as a star of class $2n+1$; only they need not, as in the case of the star, be all real.

I remark, in conclusion, first, that the point-equation of one curve of the fifth class is

$$\text{Discrt. } (a, b, x+iy, x-iy, e, f \between \lambda, \mu)^5 = 0,$$

which is worked out in Dr Salmon's *Higher Algebra**; and secondly, that the Hessian of the tangential equation is easily calculated and shews the cuspidal tangents to be common tangents of a three- and a five-rayed star.

4236. C is the double point of a circular cubic, and a straight line cuts the curve in DEF; join CD, CE, CF, and on the two latter lines take A, B, so that $CA \cdot CE = CB \cdot CF$; then prove that AB and CD are equally inclined to the tangents at C.

[*E. T.* November, 1873; 2817, January, 1869. Reprint, Vol. xx. p. 88.]

* [Pp. 208, &c., Third Edition.]

4199. Three ternary quadrics U, V, W, break up into linear factors 1, 1′; 2, 2′; 3, 3′ respectively. Prove that

$$\square\,(U,\ V,\ W) \equiv 123\,.\,1'2'3' + 1'23\,.\,12'3' + 12'3\,.\,1'23' + 123'\,.\,1'2'3,$$

where $\square\,(U,\ V,\ W)$ is the coefficient of $\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required developments and interpretations.

[*E. T.* October, 1873; 1907, February, 1866. Reprint, Vol. XXI. p. 38.]

4641. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

[*E. T.* April, 1875. Reprint, Vol. XXIII. p. 59.]

4696. Six circles pass through twelve points on a conic in the following order,

$$(a)\ A_1A_2A_3A_4,\qquad (b)\ B_1B_2B_3B_4,\qquad (c)\ C_1C_2C_3C_4,$$
$$(d)\ A_1A_2B_3C_4\qquad (e)\ B_1B_2C_3A_4,\qquad (f)\ C_1C_2A_3B_4;$$

prove that two circles and another point may be taken arbitrarily, and that the circles abc meet the circles def in six new points which lie on the circumference of another circle.

[*E. T.* June, 1875, 2281, November, 1866. Solution of the *first* part given in Reprint, Vol. XXIV. pp. 42, 43. The solver called in question the truth of the second part, but, on seeing the following solution on pp. 76, 77, admitted the correctness of this 'beautiful' theorem.]

Three circles taken together constitute a sextic curve passing three times through each of the circular points at infinity. Now the sextic def passes through all the twelve points of intersection of the sextic abc with the conic, which we may call k; hence, by a well-known theorem, there must be an identical equation of the form

$$\mu\,.\,def = \lambda\,.\,abc + q\,.\,k.$$

Here λ and μ are numerical ratios, and q is a quartic function of the coordinates. The equation may also be written

$$-q\,.\,k = \lambda\,.\,abc - \mu\,.\,def,$$

and in this form it shews that the equation $qk=0$ represents a curve of the sixth order passing three times through each circular point. But, by hypothesis, the conic k does not pass through either circular point. The quartic curve q has therefore two triple points on the line infinity, it must therefore contain that line. The rest of it is a cubic having two double points on the line infinity; it also must therefore contain that line. The final remainder is a conic passing once through each circular point, that is to say, a circle. Calling this circle s, we reduce our equation to the form

$$s\,.\,k\,.\,\infty^2 = \lambda\,.\,abc - \mu\,.\,def,$$

which shews that the remaining six intersections of abc with def lie on a circle s.

It will be observed that the proof holds good if we substitute for the conic in the enunciation a circular cubic or a bicircular quartic. From the latter extension we may obtain a transformed theorem of some interest. Invert the whole figure in regard to a point not in its plane; the bicircular quartic becomes a section of a sphere by an arbitrary quadric surface, and every circle becomes a section of the sphere by a plane. In this form we may substitute for the sphere any quadric surface, and the transformed theorem may then be stated and proved as follows:

If six planes pass through twelve points on a quadriquadric curve in the order above stated, the six lines of intersection ae, af, bf, bd, cd, ce *will meet every quadric surface passing through the curve in six points which lie in one plane.*

It is to be observed that these six lines already meet the quadriquadric curve in the six points A_3, A_4, B_3, B_4, C_3, C_4. Let h_2, k_2 be two quadric surfaces passing through the curve; then the cubic surface *abc* passes through all the twelve intersections of the cubic *def* and the quadrics h_2, k_2. We must therefore have an identical equation of the form

$$\lambda \, . \, abc = \mu \, . \, def + uh_2 + vk_2,$$

where λ and μ are numerical ratios, while u and v are expressions of the first order in the coordinates. Writing this identity in the form

$$vk_2 = \lambda \, . \, abc - \mu \, . \, def - uh_2,$$

we see that the nine lines of intersection of *abc* and *def* must meet the quadric h_2 either on the quadric k_2 (i. e. on the quadriquadric curve) or on the plane v. Now three of these, *ad*, *be*, *cf* meet the curve in two points each, and the rest, *ae*, *af*, *bf*, *bd*, *cd*, *ce* in one point each; consequently these latter must meet the quadric h_2 on the plane v.

The construction of the figure depends first on that of the hexagon $A_4A_3B_4B_3C_4C_3$. In the case of the plane conic the opposite sides of this hexagon are parallel, and the possibility of the construction is assured by Pascal's theorem. When the hexagon has been drawn, it is easy to make a pair of circles pass through the ends of two opposite sides and intersect on the conic. In this way I have drawn the figure as carefully as I can, and it seems to come right. In the case of the quadriquadric curve, each pair of opposite sides is such that a quadric surface can be drawn through them to contain the curve. In the first instance they are given as two chords which are both met by the same third chord; thus the lines A_4A_3, B_3C_4 are both met by A_1A_2. Now the problem, to draw a straight line meeting two given straight lines and a quadriquadric curve twice, admits in general of eight solutions; but in the case where the two given lines are chords of the curve, the four lines joining their points of intersection count for two solutions each, and if there is one other solution, there must be an infinite number; i. e. the two lines and the curve must lie on the same quadric surface. The possibility of the inscription of a hexagon whose opposite sides possess this property may be shewn by a method analogous to that used for the plane conic [pp. 42, 43]. The quadriquadric is a curve of deficiency *one*, and therefore the coordinates of any point on it may be expressed as elliptic functions of a parameter; this may be so taken that the sum of the parameters of four points in one plane shall be con-

gruent to zero (Clebsch "On the application of Abel's functions to Geometry," *Crelle's Journal*). Using the letters A_1, A_2, &c., to represent these parameters, we shall have

$$A_1+A_2+A_3+A_4\equiv 0 \text{ (mod. } \omega,\ \omega', \text{ where } \omega,\ \omega' \text{ are the periods)},$$
$$A_1+A_2+B_3+C_4\equiv 0\,;$$

and therefore, $A_3+A_4\equiv B_3+C_4$ (mod. ω, ω'), as the condition to be satisfied by two opposite sides of the hexagon. Now, if this condition is satisfied by two pairs of opposite sides, it will be satisfied by the third pair; for the congruence $B_4+B_3\equiv C_3+A_4$ follows from the congruences

$$A_4+A_3\equiv B_3+C_4, \qquad C_4+C_3\equiv A_3+B_4.$$

The theorem states that, when such a hexagon has been constructed, lines may be drawn through its vertices which shall meet every quadric surface passing through the curve in six points on one plane. *As the surface varies, this plane passes through a fixed line;* for

$$uh_2+vk_2=u\,(h_2+\rho k_2)+(v-\rho u)\,k_2.$$

Lastly, I observe that not every skew hexagon can have a quadriquadric curve drawn through it so that each pair of opposite sides shall be generators of the same quadric passing through the curve. Let the hexagon $A_4A_3B_4B_3C_4C_3$ be given; through the lines A_4A_3, B_3C_4 and the points B_4C_3, a singly infinite number of quadrics can be drawn, which will intersect in a quadriquadric curve; and one condition is necessary in order that the chords A_3B_4, C_4C_3 may possess the required property, or, which is the same thing, that the three curves which we may then get from the three pairs of opposite sides may be identical. The hexagon therefore possesses a geometrical property which can doubtless be expressed in terms of its diagonal lines or planes; this expression, however, I have not as yet been able to find.

4972. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be any eight tangents of a conic, and let a cubic pass through all the intersections of A's with B's excepting A_1B_1, A_2B_2, A_3B_3, A_4B_4. Then (1) there is a singly infinite number of such octograms inscribed in the cubic and circumscribed to the conic; (2) the groups of eight tangents form an involution of the eighth order; (3) the quadrilaterals $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are totally inscribed in a second fixed cubic; (4) the points A_1B_1, A_2B_2, A_3B_3, A_4B_4 are on a fixed straight line.

[*E. T.* May, 1876. Reprint, Vol. xxv. p. 76.]

4996. If the series

$$1+a\frac{1-x}{1-r}+a^2.\frac{1-x}{1-r}.\frac{1-rx}{1-r^2}+a^3.\frac{1-x}{1-r}.\frac{1-rx}{1-r^2}.\frac{1-r^2x}{1-r^3}+\ldots.$$

be called $\phi\,(a,\,x)$; then prove that

$$\phi\,(1,\,a)\,.\,\phi\,(a,\,x)=\phi(1,\,ax).$$

[*E. T.* June, 1876. Reprint, Vol. xxvi. p. 18.]

5304. Prove that the negative pedal of an ellipse, in regard to the centre, has six cusps and four nodes; find their positions, and the length of the arc external to the ellipse between two real cusps; and account fully for the apparent reduction of the curve to a circle and two parabolas respectively, in special cases.

[*E. T.* June, 1877. Reprint, Vol. XXIX. p. 47.]

4871. Let U, V be any two cubic functions of x; shew that a quantic function $f(x)$ may always be found, such that, by the substitution $y = U : V$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $Mdy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant.

[*E. T.* January, 1876. Reproposed as 6475, October, 1880. Solved, Reprint, Vol. XXXV. (in progress). *E. T.* January, 1881.]

3980. It is known that if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that if five lines be given, the five points so belonging to their five tetragrams lie on a circle, (Miquel's Theorem; see *Diary* for 1861, p. 55 [VIII. supra, pp. 38—54]).

Shew that this series of propositions is interminable; so that if $2n$ lines be given, they determine $2n$ circles which meet in a point; and if $2n+1$ lines be given, they determine in this manner $2n+1$ points which lie on a circle.

[*E. T.* February, 1873; 5423, October, 1877; 6441, September, 1880. Solved, Reprint, Vol. XXXIV. p. 80, *E. T.* November, 1880; cf. 3282, p 613]

5626. The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle; find the envelope of all the binormal circles of one system.

[*E. T.* May, 1878. Proposed November, 1869, as 3000, July, 1873, as 4123. Reprint, Vol. XXXII. p. 17.]

4143. Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings.

[*E. T.* August, 1873. Reprint, Vol. XXXIII. p. 18. Originally proposed as 1459, December, 1863.]

1433. [Prove the following reciprocal cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

a'. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b'. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution. Proposed by W. J. C. Miller, B.A. Reprint, Vol. xxxIII. pp. 50, 51.]

Six points are in involution when the anharmonic ratio of any four is equal to that of their four conjugates.

a. Let then ABC be the triangle [Fig. 124], D the point of concourse; and let a straight line meet the sides and corresponding lines in a, b, c, α, β, γ respectively; then

$$[abc\gamma]=\{A\,.\,DCB\gamma\}=\{A\,.\,DCF\gamma\}=\{B\,.\,DCF\gamma\}=[\beta ac\gamma]=[a\beta\gamma c],^{*}$$

which proves the proposition.

a'. Let now ADF be the triangle, C, B, E the collinear points, and take any point O; then we have

$$\{O\,.\,ABDE\}=\{B\,.\,AODE\}=\{B\,.\,FODC\}=\{O\,.\,FBDC\}=\{O\,.\,CDBF\},$$

which proves the second case.

b. The line at infinity is cut in involution, by prop. a; hence, lines parallel to the given six through any point will form a pencil in involution; turn this pencil through a right angle, and it coincides with the perpendiculars.

b'. Any point at infinity is subtended in involution, by a'; whence the theorem immediately follows.

* [ ] Auctoris.

UNSOLVED QUESTIONS.

1423. Shew that

$$\int_0^{\frac{1}{2}\pi} \cos(a \tan x)\, \epsilon^{\beta \tan x}\, dx = \tfrac{1}{2}\pi\epsilon^{-a}(\cos\beta + \sin\beta).$$

[*E. T.* September, 1863. Reproposed as 2316, January, 1867; 3941, December, 1872; 4794, October, 1875; 5330, July, 1877.]

1448. The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest.

Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

[*E. T.* November, 1863. Reproposed as 3336, February, 1871; 4171, September, 1873, and 6120, November, 1879.]

1507. Consider six planes *ABCDEF*, and join the point *ABC* to the point *DEF*, and so on; we have thus ten finite straight lines, and their middle points lie in a plane.

[*E. T.* May, 1864.]

1585. If three circles are mutually orthotomic, prove that the circles on their common chords as diameters have a common radical axis.

[*E. T.* October, 1864.]

1605. Required the area of the triangle included by three points in space, given by equations of the form

$$lx + my + nz + sw = 0.$$

[*E. T.* November, 1864.]

1691. If the radii of two spheres be ρ_1, ρ_2, and D the distance between their centres; and if a tetrahedron be inscribed in each; prove that the product of the volumes of the tetrahedra into $(D^2 - \rho_1^2 - \rho_2^2)$ may be expressed as an integral function of the squares of the distances between the vertices of the tetrahedra. Hence deduce the condition $(\Theta = 0)$ that four points in a plane may lie in a circle. If they do *not* lie in a circle, what is the meaning of Θ?

[*E. T.* April, 1865.]

1724. The equations of three conics being given in the forms:

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2+d_2w^2=0,$$
$$a_3x^2+b_3y^2+c_3z^2+d_3w^2=0,$$

where $x+y+z+w\equiv 0$, shew that a straight line

$$(\xi x+\eta y+\zeta z+\omega w=0)$$

will be cut in involution by them, if

$$\Sigma . \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} . (\xi-\eta)(\xi-\zeta)(\xi-\omega) \text{ (to four terms)}=0.$$

[*E. T.* May, 1865.]

1748. Let $X, Y, Z, U, V=0$ be the Cartesian equations, and r_1, r_2, r_3, r_4, r_5, the radii, of five spheres, cutting each other orthogonally; then identically

$$\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{U^2}{r_4^2}+\frac{V^2}{r_5^2}=0, \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}=0.$$

[*E. T.* June, 1865.]

1918. It is known that the conic of five pointic contact at any point A of a cubic meets the curve again in a point B, constructed by joining the point A to its second tangential; let this point be called the *conic tangential* of A. Then the conic tangential of B will be the second conic tangential of A, and so on. Shew how, having given the conic tangential of any order, and also the line tangential of any order, we can construct for the original point A by the ruler alone.

[*E. T.* March, 1866. Reproposed as 4299, January, 1874.]

1929. 1. If 1 2 3 4 be four concyclic foci of an anallagmatic (bi-circular) quartic curve, on a plane or on a sphere, and P any point of the curve; the arc at P is equally inclined to the circles $P12$, $P34$.

2. The bitangent circles of an anallagmatic are arranged in four systems, orthotomic respectively of the four focal circles. Two bitangents of the same system can be drawn to cut orthogonally a given circle A. The four points of contact of these lie on a circle B_1 (*polar circle of* A *in that system*). There are polar circles B_2, B_3, B_4 in the other three systems, and the circles A, B_1, B_2, B_3, B_4 have a common radical axis.

3. The two bitangents of the same system through any given point are equally inclined to each of the two confocal anallagmatics which pass through that point.

4. The bitangent spheres of an anallagmatic quartic surface are arranged in five systems, orthotomic respectively of the five focal spheres. An infinite number of bitangents of the same system can be drawn to cut orthogonally a given sphere A; these envelope a cyclide, whose curve of contact with the anallagmatic surface lies on a sphere B, polar sphere of A in that system. The five polar spheres of A have with A a common radical plane.

5. The five tangent cyclides to an anallagmatic from any point have their focal spheres touched by the three confocal anallagmatics through that point.

6. A tangent cyclide from any focus is a *Tore*.

DEF. A *cyclide* is the envelope of a sphere touching three fixed spheres. When the centres of the three fixed spheres are in one straight line, the cyclide becomes a *tore*, or anchor-ring.

[*E. T.* April, 1866. Reproposed as 4340, March, 1874.]

2229. 1. The distances (r, s, t) of a variable point on *one* focal curve of an anallagmatic quartic surface from any three fixed points on another focal curve of the same surface, are connected by a relation of the form $lr+ms+nt=0$ (*i.e.* a relation of the same form as that which connects the distances of a variable point on a circle from three fixed points on the circle).

2. The distances of a variable point on an anallagmatic quartic surface from four fixed foci of the surface are connected by a relation of the same form as that which connects the distances of a variable point on a sphere from four fixed points on the sphere.

3. The four points in which an anallagmatic quartic curve is cut by any circle may be taken as the foci of an anallagmatic quartic curve which passes through any four concyclic foci of the original curve. (This is true on a plane or on a sphere.)

4. The curve in which any anallagmatic quartic surface is cut by any sphere may be taken as the focal curve of an anallagmatic quartic surface which passes through any one focal curve of the original surface.

5. It is required to find the property of anallagmatic quartic curves which corresponds to the property of conics, $SP \,.\, HP=CD^2$.

6. If two anallagmatic quartic curves or surfaces, A and B, are such that a confocal to A can be inscribed or subinscribed to B: then also a confocal to B can be inscribed or subinscribed to A.

DEF. One anallagmatic curve is *inscribed* to another when it touches it in four points on a circle. One surface is inscribed to another when it touches it all along its curve of intersection with a sphere. One surface is *subinscribed* to another when it touches it in four points on a circle.

[*E. T.* September, 1866. Reproposed as 4754, August, 1875.]

2510. If a conic be inverted into a circular cubic with a double point, the foci and directrices of the conic will invert into foci and directing circles of the cubic.

[*E. T.* October, 1867. Reproposed as 4667, May, 1875.]

2858. If the intersections of two circles $A=0$, $B=0$ are concentric with the antifoci of the intersections of $C=0$, $D=0$, then *vice versâ;* and if this property hold for the pairs AB, CD, and also for the pairs AC, DB, it will hold for the pairs AD, CB.

[*E. T.* March, 1869. Reproposed as 4513, October, 1874; 5691, July, 1878.]

3255. Two planes A, B, are said to have an (x, y) correspondence, when to every point on the plane A correspond y points on the plane B, and to every point on B correspond x points on A.

On each plane there is in general a locus of points, two of whose correspondents coincide: this is called the *cross-curve* (Uebergangscurbe, Clebsch in *Math. Annalen*).

On each plane there is also a locus of these united correspondents; this is called the *node-curve*.

1. If a curve touch the cross-curve in either plane, the corresponding curve in the other plane will have a node lying on the node-curve in that plane.

2. The correspondence may be represented as a (1, 1) correspondence of two multiple planes A', B'; A' consisting of y sheets, and B' of x sheets, which are connected together along the cross-curves.

3. In a $(1, y)$ correspondence, in which to two straight lines in the plane A correspond curves of deficiency p in the plane B, the order of the cross-curves in A is $=2(y+p-1)$.

[*E. T.* November, 1870.]

3308. Prove that the 40 umbilici of an anallagmatic surface lie by fives on 16 straight lines.

[*E. T.* January, 1871. Reproposed as 5274, May, 1877: cf. however 4034, p. 614.]

3961. In a polyhedron having n summits and only triangular faces (Δ-faced n-acron, CAYLEY), let every plane which contains three summits, but is not a face, be called a diagonal plane; and let every plane which contains two summits, but is not an edge, be called a diagonal line: then (a) there is a surface of class $n-4$ touching all the diagonal planes, (b) this surface contains all the diagonal lines; (c) the conditions of passing through the diagonal lines and touching the diagonal planes are just sufficient to determine the surface and no more, and (d) when the surface touches the plane at infinity, the volume of the polyhedron is zero.

[*E. T.* January, 1873. Reproposed as 5210, March, 1877. Cf. XVIII. supra, pp. 168—176.]

4819. A spherical curve of class n has in general n^2 foci. Let n foci such that no two are on a line touching the absolute be called a *set*, and denoted by $p, q, r, \ldots$ If x be any point of the sphere, the quantity $\dfrac{(\Pi \sin xp)^{n+1}}{\Pi \sin^2 pq}$ is the same for all sets.

[*E. T.* November, 1875. Reproposed as 6890, November, 1881.]

4843. If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a doubled plane.

[*E. T.* December, 1875. Reproposed as 5825, December, 1878.]

4897. Let U, $V=0$ be the point-equations, and u, $v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u \pm \lambda v = 0$, they will intersect on $U - \lambda^2 V = 0$. This last conic passes through the points of contact of the conics $u \pm \lambda v = 0$ with the common tangents of u and v.

[*E. T.* February, 1876.]

4925. Let U, V, $W=$ be the point-equations, and u, v, $w=0$ the plane-equations of three quadrics inscribed in the same developable, and let $u+v+w$ be identically zero. Then, if a tangent plane to U, a tangent plane to V, and a tangent plane to W, are mutually conjugate in respect of

$$au + bv + cw = 0,$$

they will intersect on

$$\frac{U}{(b-c)^2} + \frac{V}{(c-a)^2} + \frac{W}{(a-b)^2} = 0,$$

which passes through the curves of contact of the developable with $au+bv+cw$ and one other quadric.

[*E. T.* March, 1876.]

4950. Prove that every matrix of the second order may be expressed in the form $aI+bJ$, where I is the matrix unity, and J a matrix such that $J^2 = -1$. Hence find an expression for any power of such a matrix. (See Cayley on Matrices, *Phil. Trans.* 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function.

[*E. T.* April, 1876.]

5457. A triangle ABC has its vertices A, B, jointed to two rods AD, BE, which can turn about the fixed points D, E; express the coordinates of the point C in terms of elliptic functions of a single parameter.

[*E. T.* November, 1877.]

[N.B. 1724, p. 624, has been solved, see p. 597.]

LECTURE I. ON BOUNDARIES IN GENERAL *.

Syllabus.

Every body distinguishes two adjacent regions of space, one inside and one outside.

(*a*) The surface of the body is surface to both of these regions.

(*b*) It takes up no solid room, or has no thickness.

(*c*) When the body is moved continuously, the surface is moved continuously with it.

(*d*) And yet a surface remains in the same place when the body is taken away.

Congruent regions are those which can be filled at different times by a body which does not alter in size or shape.

The remarks (*a*) (*b*) (*c*) (*d*) are true also of a line, boundary between two adjacent surface-regions, and of a point, boundary between two adjacent line-regions.

A line is also the intersection of two surfaces.

A point is the intersection of two lines, a line and a surface, or three surfaces.

A line is the path or locus of a moving point, a surface of a moving line, and a solid of a moving surface.

The number of points in a piece of line is singly infinite; the number in a piece of surface doubly infinite; and the number in a piece of solid space triply infinite.

A point on a line has one variation; on a surface, two; in solid space, three.

Questions.

1. Can two regions be partly adjacent and partly not? (Distinguish between solid, surface, and line-regions.)

2. Explain how a point is the intersection of three surfaces, and give an example.

* [From information furnished to me by Mr F. Pollock, Mr W. J. Ritchie and others, I find that these Syllabuses belong to a series of lectures given to a class of ladies at South Kensington in the spring and summer of 1869. The "proofs," for a complete set of which I am indebted to Mr C. J. Clay, bear dates ranging from April 8 to June 11, 1869. Lecture 1 is evidently that printed in "Seeing and Thinking," pp. 127—156, NATURE Series: *Macmillan's Magazine*, Vol. XL. No. 238, pp. 359—368, Aug. 1879.]

3. State clearly what is meant by the assertion: A point can be moved continuously from one position to another, with the bodies of whose surfaces it is the intersection; and yet remains when they are taken away.

4. Is the motion of a shadow always continuous?

5. An infinite number of circles can be drawn upon a piece of paper. Is this number singly, doubly, or triply infinite?

LECTURE II. ON PLANE SURFACES AND STRAIGHT LINES.

Syllabus.

Space exists independently of the things in it, but allows them to be moved about without altering their size or shape.

A plane surface may be slid about upon itself or another plane surface, and will always fit.

It may also be turned over and applied to itself so as to fit.

A plane is of infinite extent.

The intersection of two planes (a straight line) is also the intersection of an infinite number of planes: or a plane may turn round it and slide along it.

A straight line divides a plane into two congruent regions.

A straight line is fixed by two points, a plane by three.

Two straight lines, or three planes, can meet only in one point.

Two straight lines, meeting, divide a plane into four regions congruent two and two. If all four are congruent, each is called a right angle, and the lines are perpendicular.

Only one straight line can be drawn perpendicular to a given straight line through a given point.

Only one straight line can be drawn parallel to a given straight line through a given point. It is then parallel to the given line at all points on it.

Regions at an infinite distance in opposite directions on the same straight line are adjacent and separated by one point.

All points at an infinite distance on one plane are in a straight line.

All points at an infinite distance in space are in a plane.

A straight line and a plane are respectively a line and surface *of the first order.*

Questions.

1. Two bodies which have plane surfaces may have those surfaces applied to each other in an infinite number of ways. Is this number singly, doubly, or triply infinite?

2. How do we know that the spaces on the two sides of a plane surface are of the same shape?

3. One plane divides another into two congruent surface regions. Explain what this means, and how you see it to be true.

4. How many straight lines can be drawn through a given point of space to meet each of two given straight lines?

5. In what sense is it true that there is only one point at an infinite distance on a straight line?

Rouché et De Comberousse, p. 12, Ex. 1—4.
Wright, p. 12, Ex. 1—4.
Wilson, p. 10, Ex. 1—7.

LECTURE III. ON THE ROTATION OF PLANE FIGURES.

Syllabus.

The properties of plane figures are divided into *projective* properties which are retained in the shadows of the figures, and *non-projective* properties, which are not so retained.

Properties connected with Rotation are non-projective.

A straight line turning about a point in it, by equal amounts of rotation generates congruent angles.

If a plane figure turn about any point in its plane, the directions of all lines in the figure are altered by the same amount.

The direction of a figure is equally altered by the same amount of turning about two different points.

Every change of position of a figure in its plane may be produced by a single rotation.

(The external angles of a polygon are together equal to four right angles. A triangle is determined by three independent elements. A parallelogram is congruent to itself in one way; a rectangle in three.)

The path of a rotating point is a *circle*. The direction of the point's motion is always perpendicular to the radius.

Equal angles at the centre of a circle cut off equal arcs of its circumference.

Questions.

1. Parallel lines (1) do not meet, (2) make congruent intersections with a third line. Shew that any two lines which possess either of these properties possess also the other.

2. Prove that the exterior angle of a regular hexagon is equal to the interior angle of an equilateral triangle.

3. Shew how to construct a triangle of which you know the height and the two base angles.

4. Can *any* two congruent triangles in the same plane be made to coincide by rotating one of them about some point?

5. How does it appear that the tangent at any point of a circle is perpendicular to the radius?

Rouché et De Comberousse, p. 22, Ex. 4, 7; p. 28, Ex. 1—3; p. 36, Ex. 1—6.

Wright, p. 22, Ex. 4, 7; p. 30, Ex. 1—3; p. 39, Ex. 1—7.

Wilson, p. 17, Ex. 1—7; p. 34, Ex. 1—7.

LECTURE IV. ON SIMILAR FIGURES.

Syllabus.

When a figure is enlarged so as to remain still of the same shape, every straight line in it remains a straight line, and every angle remains congruent to itself.

All the parts of the figure are equally enlarged.

When one figure is an enlarged copy of another, the two are said to be *similar*.

The degree of enlargement necessary to make one figure equal to the other is called their *ratio of similitude*.

The ratio of two lines in the one figure is equal to the ratio of the two corresponding lines in the other.

* If four quantities are proportional, and the first is greater or less than any fraction of the second, the third is greater or less than the same fraction of the fourth.

If two quantities are so connected that each, being given, determines the other: and if to the sum of two values of one corresponds the sum of the two corresponding values of the other: then the ratio of any two values of the first quantity is equal to the ratio of the two corresponding values of the second.

Triangles are similar which have (1) their sides proportional, (2) an angle in one equal to an angle in the other, and the sides about them proportionals.

Similar rectilinear figures are made up of similar triangles.

Questions.

1. Prove that all circles are of the same shape.

2. If two parallelograms have the angles of one equal respectively to the angles of the other, and if the ratio of the two diagonals of one is equal to the ratio of the two diagonals of the other, the parallelograms are similar.

3. Shew that two regular polygons of the same number of sides are similar figures.

4. What is Euclid's definition of proportion? Prove that it follows from the fact (*) stated in the Syllabus.

5. There are two similar triangles such that the first and second sides of one of them are four and five feet long respectively; and the first and third sides of the other are twelve and twenty-one feet long respectively. Find the lengths of the remaining sides.

LECTURE V. THE FIRST PRINCIPLES OF CALCULATION.

Syllabus.

Numbers may be changed into other numbers by the operations of addition, subtraction, multiplication, and division.

Operations of addition or subtraction may be changed into others by the operations of multiplication, division, and reversion.

Multiplication is *commutative* $[ab = ba]$, and *distributive* $[a(b+c) = ab + ac]$.

A *quantity* is measured by the ratio which it bears to some fixed quantity of the same kind, called the unit.

Quantities may be changed into others by the four fundamental operations, which are subject to the same laws as in the case of numbers.

Ratios are approximately represented by the ratios of numbers; viz. (1) by decimal fractions, (2) by continued fractions.

The *position* of a point on a straight line may be represented by its distance from a fixed point on the line; or by the quantity of motion necessary to carry it from that fixed point to its position. Positive and negative motion are in opposite directions.

Questions.

1. What theorem is implied in the idea of number? and what in that of the numerical ratio of two quantities?

2. Distinguish between the multiplication of numbers and the multiplication of operations. Prove that multiplication is distributive in the former case.

3. In what cases can subtraction and division be performed with (*a*) numbers, (*b*) quantities, (*c*) motions?

4. Translate into English:

$$5(3+2)=25; \qquad -7+4=-3; \qquad (-4)(-5)=+20.$$
$$a+b-c=a-c+b; \qquad AB+BC+CA=0.$$

5. Write in symbolic language:

If the operation of adding one number and subtracting another be multiplied by the sum of the numbers, the result will be equivalent to the operation of adding the square of the first number and subtracting that of the second.

6. Shew how to express the ratio of the diagonal of a square to its side in the form of a continued fraction.

LECTURE VI. THE THEOREM OF PYTHAGORAS.

Syllabus.

If squares are described on the sides of a right-angled triangle, the two squares on the sides containing the right angle are together equal to the square on the side opposite the right angle.

I. Clairaut's Proof.

When several figures can be fitted together in different ways, they always cover the same area. The square on the hypothenuse can be cut up into pieces out of which the other two squares may be formed.

II. The Proof in the First Book of Euclid.

Parallelograms of equal base and height have equal areas. The area of a triangle is half that of a rectangle of the same base and height.

The squares on the sides of the right-angled triangle are severally equal to the two parts into which the square on the hypothenuse is cut by the perpendicular.

III. The Proof in the Sixth Book of Euclid.

If one side of a rectangle is altered in any ratio, the rectangle is altered in the same ratio.

The perpendicular from the vertex of the right angle on the hypothenuse of a right-angled triangle divides it into two triangles similar to each other and to the original triangle.

The areas of similar figures are as the squares on their corresponding sides.

The theorem of Pythagoras is true of any three similar figures described on the sides of a right-angled triangle, *e.g.*, three semicircles; the Lune of Hippocrates.

IV. The Figure of the Bride's Chair*.

The area of a rectangle is measured by the product of the measure of its sides.

Squares on the sum and difference of two lines:

$$(a+b)^2=a^2+b^2+2ab\ ; \quad (a-b)^2=a^2+b^2-2ab.$$

The theorem follows from the latter formula.

Numbers proportional to the sides of a right-angled triangle may be obtained from the formula

$$(a^2-b^2)^2+(2ab)^2=(a^2+b^2)^2.$$

Questions.

1. Can a length be equal to an area? Can two areas be proportional to two lengths? Can an area have any ratio to a length?

* [I have not been able to trace the origin of this title: it appears to be the case (3) of De Morgan's Article, "Hypothenuse," in the *English Cyclopædia*, where the demonstration is said to be derived from the Hindu treatises on Algebra. See Note, p. 687.]

2. Give examples of the method of proving two areas equal by cutting them up and rearranging the pieces.

3. Shew how with an inch measure to construct a square containing five square inches; and cut it up into square inches.

4. The square whose height is equal to that of an equilateral triangle is three-fourths of the square on the side of the triangle.

5. One side of a right-angled triangle is 24 inches long. Find the triangle so that each of the other sides may be an exact number of inches.

6. Prove geometrically that

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

LECTURE VII. THE PROPERTIES OF ONE CIRCLE.

Syllabus.

I. The construction of a circle which passes through three given points.

Lines bisecting at right angles the sides of a triangle meet in a point.

The perpendiculars of a triangle meet in a point.

No point but the centre is equidistant from three points in the circumference of a circle.

II. *All the angles in the same segment of a circle are equal to one another.*

A rectangle is the only parallelogram that can be inscribed in a circle. The diagonals of the rectangle are then diameters of the circle. Every angle in a semicircle is a right angle.

If the perpendiculars AF, BG, CH of a triangle ABC meet in the point O, then there are nine points, viz.: the points F, G, H, the middle points of the sides BC, CA, AB, and the middle points of the lines OA, OB, OC, which are all upon the circumference of the same circle.

The equality of angles in the same segment may be proved either, as in Euclid, from the fact that the angles of a triangle are together equal to two right angles, or by the properties of rotation.

Four straight lines make four triangles, and their circumscribing circles meet in a point.

III. If through a point P two lines PAB, PCD are drawn to cut a circle in A, B, C, D, the rectangle contained by PA, PB is equal to the rectangle contained by PC, PD.

Questions.

1. The perpendiculars of a triangle ABC meet in a point D. Shew, by examining the figure, that each of the four points A, B, C, D is the intersection of perpendiculars of the triangle formed by the other three.

2. What is the locus of the centres of all circles that pass through two fixed points?

3. And what is the locus of the intersection of the two tangents to each of the circles at those points?

4. Tangents are drawn to a circle at the angular points of an inscribed rectangle. Prove that the quadrilateral thus formed has all its sides equal.

5. AF, BG, CH are the perpendiculars of a triangle ABC, meeting in O. Show that circles can be described about the quadrilaterals $AGOH$, $BHOF$, $CFOG$; and that the angles BFH, CFG are equal to one another.

LECTURE VIII. THE PROPERTIES OF ONE CIRCLE—*continued.*

Syllabus.

The following are different statements of the same projective property of a circle:—

1. If a series of lines are drawn through a fixed point to meet a circle, and if at the two points of intersection of each of these lines with the circle tangents are drawn; the intersections of all these pairs of tangents will lie on a fixed straight line.

2. If a series of points are taken on a fixed straight line, and from each of them two tangents are drawn to the circle; the several chords of contact of these pairs of tangents will all pass through a fixed point.

Two points A and B are called *inverse* points in respect of a circle whose centre is O, when OAB is a straight line, and the rectangle OA, OB is equal to the square of the radius.

A line through B perpendicular to OAB is then called the *polar* of the point A; and a line through A perpendicular to OAB is called the *polar* of the point B.

When a line is the polar of a point, the point is called the *pole* of the line.

If a point L lies on the polar of M, then M lies on the polar of L.

The polar of a point outside the circle is the chord of contact of tangents from it to the circle.

Questions.

1. Two inverse points in respect of a circle are distant from the centre two inches and eight inches respectively; what is the length of the radius?

2. A circle has its radius one foot long, and a certain straight line is sixteen inches distant from the centre; how far is this straight line from its own pole?

3. Explain in what way every diameter is the polar of some point at infinity.

4. Prove that if one point lies on the polar of a second, the second will lie on the polar of the first How does it follow from this that when three points are in a straight line, their three polars meet in a point?

5. Shew that a circle can be drawn through any two pairs of inverse points.

LECTURE IX. ON THE SHADOWS OF A CIRCLE.

Syllabus.

The *shadow* of a solid body may mean either the solid region which it deprives of light, or the portion of any surface which is within that region. And the *shadow* of a curve may mean either the conical surface composed of straight lines passing through the luminous point and meeting the curve, or the new curve in which this cone cuts any surface.

Besides the *real* shadow, or darkened part of space, it is necessary to consider also the *ideal* or geometrical shadow, obtained by producing the rays of light backwards behind the luminous point.

The surface-shadow of a circle is a cone of the second order.

The shadow of a circle cast on a plane is of one of three shapes, called respectively the Ellipse, Parabola, and Hyperbola.

The Ellipse and Hyperbola have connected with them two points called the *foci*, such that in the Ellipse the sum of their distances from every point of the curve is the same, and in the Hyperbola the difference of their distances from every point of the curve is the same. The Parabola has one focus, which is so related to a certain straight line called the *directrix* that the distance of every point on the curve from the focus is equal to its distance from the directrix.

All these curves are of the *second order*, and all the polar properties of the circle belong to them.

Questions.

1. Explain what is meant by the statement that a right cone is a surface of the second order. Describe the surface, and shew that the statement is true.

2. In how many directions can a circle be cut from an oblique cone?

3. Describe the shape of an ellipse, and shew how it can be practically drawn.

4. If the two foci of an ellipse coincide, what does it become?

5. Why are the two branches of a hyperbola regarded as one curve?

LECTURE X. ON THE ORDER OF GEOMETRICAL PROBLEMS.

Syllabus.

When a point has to be found on a straight line, and the problem of finding it is of the second order, so that it may have two solutions; then in certain particular cases these two solutions coincide and become one, and in other cases disappear altogether.

Geometers however are accustomed to say in the first case that there are two coincident solutions, rather than one solution; and in the second case that the two solutions have become *imaginary* or *invisible*, rather than that they have altogether ceased to exist.

This language contains a reference to a class of problems different from the one directly considered, in which lengths are to be measured, not on a line in one of two directions only, but in a plane and in any direction. In these problems the solutions never disappear, and the number of them is always exactly equal to the order of the problem.

By this use of language belonging to a different branch of science, not only is great generality introduced into the enunciations and proofs of theorems, but an explanation is afforded of the very different forms under which the same theorem presents itself.

[Bride's chair, Lect. VI, see below.*]

[The following, Prof. Henrici tells me, are notes of a course of lectures on Synthetic Geometry and Graphical Statics. I have thought them worthy of a place here as the treatment is somewhat novel, at least, to English readers.]

I. Shadows of a circle, of parallel lines.
Perspective range and pencils of lines and planes.
Range perspective to itself with two points interchanged.
1. is called *harmonic*, construction by quadrilateral.

II. Projective ranges and pencils (assuming only Euclid).
2. If two point-ranges are projective in one position they are in all positions (two cases).

Projective correspondence.

Case 1. Projective correspondence is determined by three pairs of corresponding points.

Case 2. Two on same line may have two united points but not more.

3. Two ranges projective to same range are projective to each other.

* [I am indebted to Mr H. C. Levander for a reference to Dyer's English *Folk-Lore*, p 204 Mr Dyer quotes *Antiquarian Repertory*, 1807, Vol. I. p 107, and Harland and Wilkinson's *Lancashire Folk-Lore*, 1867, p. 265, which contain accounts of two Bride's Chairs. The former work describes the chair of the Venerable Bede, at Jarrow Church, Northumberland. "It is preserved in the vestry of the Church, whither all Brides repair immediately the marriage service is over, to seat themselves upon it... The Chair, which is very rude and substantial, is made of oak; is 4 feet 10 inches high; having an upright back, and sides that shape off for the arms." Mr Levander also points out, in a figure, the resemblance of the diagram to the Constellation of Cassiopeia, or 'the Lady in her Chair.']

4. Construction for corresponding point to given one when three pairs of correspondents are given.

5. Line-pencils standing on same range are projective.

6. Line-pencils on projective ranges are projective.

Plane pencils on same range are called projective.

7. Plane pencils on projective ranges are projective.

III. Same propositions proved without assuming Euclid.

4*. Two co-basic prime-forms having three coincident pairs of corresponding elements are identical.

IV. Position-vectors: ratios of the same.

8. Cross-ratio of pencil = that of range cut by it.

(By areas of triangles.)

9. $Ix \,.\, Jy = k$ (similar triangles).

10. Two projective ranges on same line have two united points, visible, coincident or invisible.

11. One pair of symmetrical correspondents makes all pairs symmetrical. Involution. Specialities of harmonic.

V. Unique Correspondence.

Postulate:—Two uniquely corresponding quantities are connected by an equation of the form

$$axy + bx + cy + d = 0.$$

12. Two uniquely corresponding quantities which vanish and become infinite together are proportional.

13. Two uniquely corresponding quantities which vanish and become infinite together alternately are reciprocal.

14. Uniquely corresponding prime-forms are projective.

9*. $Ix \,.\, Jy = k$.

VI. *Secondary Forms.*

15. Locus of intersection of corresponding rays of two projective pencils is of second order.

Envelop of connectors of corresponding points of two projective ranges is of second class.

16. Locus of intersections of corresponding planes of two projective pencils is of second order.

A. Figure described by join of corresponding elements of two projective prime-forms of same kind is form of second degree.

Construction from five points or five tangents.

17. Form of second order may be described by pencils having vertices at any two of its points.

18. From any point on tangent one and one only other tangent can be drawn.

Construction of simultaneous points and tangents: definition of conic.

19. Pascal, Brianchon.

VII. Poles and Polars.

VIII. Metrical properties (centre and diameters).

IX. Systems of conics, foci, etc.

Preceded by re-treatment of conics by (1, 1) correspondence.

X. Surfaces, corresponding to VII. and VIII.

Appendix A. On the representation of Solid Figures, commonly called Descriptive Geometry.

Appendix B. On Graphical Calculations.

Part 2. Graphical Statics,

I. Polygon of forces, plane and space, stresses on frame-work by method of *sections* (?).

II. Tie-Polygon (plane only).

III. Moments and Parallel Forces. Couples.

IV. Centre of Parallel Forces.

For Appendix B. On Graphical Calculation. (Integration of x^n.)

1. Sum of Geometric Series by anti-parallels *

$$\frac{12+23+34+45}{12}=\frac{56-12}{23-12}=\frac{Oe-Oa}{Ob-Oa},$$

which is obvious.

2. Differentiation of $y=ax^n$

$$\frac{y_1-y_2}{x_1-x_2}=a\frac{x_1^n-x_2^n}{x_1-x_2}=(\text{in limit})\ a\,.\,nx^{n-1}.$$

3. Ditto for n fractional.

4. If ordinate of one curve represents area of another, $\frac{\delta y}{\delta x}$ of this curve will represent ordinate of other.

5. Area for $y=ax^n$.

* [Draw $O123456$ horizontal and $Oabcde$ inclined to it, so that $1a$, $2b$, $3c$, $4d$, $5e$ form one set of parallels and $a2$, $b3$, $c4$, $d5$, $e6$ the other set of parallels.]

NOTES.

The paper *XVI. was printed before I had examined the question 2942 (p. 612) and the solutions of it. I have since submitted it to Mr J. J. Walker, who has placed at my service the following remarks:

Greater clearness is attained, and some errors are avoided, in establishing these fundamental formulæ by starting at once with homogeneous co-ordinates $a_1 a_2 a_3$ of the point a. The constant $(\Delta \div R)$

$$a_1 \sin\alpha + a_2 \sin\beta + a_3 \sin\gamma = \tfrac{\iota}{2} \begin{vmatrix} a_1 & a_2 & a_3 \\ -1 & e^{-\iota\gamma} & e^{\iota\beta} \\ -1 & e^{\iota\gamma} & e^{-\iota\beta} \end{vmatrix}$$

so that calling i the point whose co-ordinates are proportional to

$$-1, \quad e^{-\iota\gamma}, \quad e^{\iota\beta},$$

and j the point whose co-ordinates are proportional to

$$-1, \quad e^{\iota\gamma}, \quad e^{-\iota\beta},$$

the constant may be written symbolically

$$\Delta \div R = \tfrac{\iota}{2}(aij).$$

The numerator in Faure's expression for $\overline{ab}^2$: viz.

$$\overline{ab}^2 = \frac{\Sigma (a_2 b_3 - a_3 b_2)^2 - 2\Sigma (a_3 b_1 - a_1 b_3)(a_1 b_2 - a_2 b_1)\cos\alpha}{(a_1 \sin\alpha + a_2 \sin\beta + a_3 \sin\gamma)^2 (b_1 \sin\alpha + \ldots)^2},$$

may also be expressed as the product of the determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{\iota\gamma} & e^{-\iota\beta} \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{-\iota\gamma} & e^{\iota\beta} \end{vmatrix}$$

or (abi), (abj); so that

$$\overline{ab}^2 = 16 \frac{(abi)(abj)}{(aij)^2 (bij)^2} \text{ or } \overline{ab} = 4 \frac{\sqrt{(abi)(abj)}}{(aij)(bij)}.$$

Again the distance of a from B is

$$\frac{B_1 a_1 + B_2 a_2 + B_3 a_3}{\sqrt{B_1^2 + B_2^2 + B_3^2 - 2B_2 B_3 \cos\alpha - \ldots}},$$

and the denominator may be written as the square root of the product

$$(-B_1 + B_2 e^{\iota\gamma} + B_3 e^{-\iota\beta})(-B_1 + B_2 e^{-\iota\gamma} + B_3 e^{\iota\beta});$$

thus

$$\text{distance } aB = \frac{aB}{\sqrt{iB \cdot jB}}.$$

The well-known formula for $\sin AB$ is

$$\frac{\Sigma\,(A_2B_3 - A_3B_2)\sin\alpha}{\sqrt{iA\,.\,jA\,.\,iB\,.\,jB}} \text{ or } \frac{A\,B\,\overline{ij}}{\dots},$$

wherein $\overline{ij}$ stands for the line containing i, j, viz., as has been shewn, the line

$$x_1\sin\alpha + x_2\sin\beta + x_3\sin\gamma = 0.$$

Also

$$\cos AB = \frac{A_1B_1 + A_2B_2 + A_3B_3 - (A_2B_3 + A_3B_2)\cos\alpha - \dots}{\dots \quad \dots \quad \dots} =$$

$$\tfrac{1}{2}\frac{(-A_1 + A_2e^{\iota\gamma} + A_3e^{-\iota\beta})(-B_1 + B_2e^{-\iota\gamma} + B_3e^{\iota\beta}) + (-A_1 + A_2e^{-\iota\gamma} + A_3e^{\iota\beta})(-B_1 + \dots)}{\dots \quad \dots \quad \dots},$$

or

$$2\cos AB = \frac{Ai\,.\,Bj + Aj\,.\,Bi}{\sqrt{Ai\,Aj\,Bi\,Bj}}.$$

I may also state that I have recently received lecture-notes on the subject from a pupil of Prof. Clifford's at University College*, who has also furnished me with the accompanying proof, given also in lecture, of Ivory's Theorem†.

Def. Corresponding points on two confocal ellipsoids are such as coincide when either ellipsoid is deformed by a homogeneous strain, so as to coincide with the other.

Statement. Let corresponding points Pp be taken on two homogeneous confocal ellipsoids Ee. The x component of the attraction of E on p is to that of e on P as the area of the section of E by the plane yz is to the area of the coplanar section of e, that is in the ratio $\beta\gamma : bc$.

This theorem is true for any law of force.

First we show that for any law of force the attraction of a straight line AB of uniform thickness and density on an external point P depends only on the distances PA, PB. (Fig. 125.)

$$\frac{dr}{dx} = \cos PMB.$$

Thus, when $f(r)$ determines the law of force, attraction of element dx resolved parallel to $AB = f(r)\,\rho\,dx\cos PMB$.

Thus whole attraction parallel to AB is

$$\int_u^v f(r)\,\rho\,dr = \rho\,\{F(v) - F(u)\}.$$

Divide up both ellipsoids (Fig. 126) into strips parallel to the axis of x.

Let Qq be corresponding points.

Then clearly $Q'q'$ are also corresponding points.

* Mr G. W. von Tunzelmann, who says that Prof. Clifford informed him that he had found it easy to extend the method to higher plane curves, but that surfaces were much more difficult and he had not made much progress in its application to them. This pupil has also put at my service other Notes which I hope to make use of hereafter.

† See *Lectures and Essays*, Vol. I. p. 4

The attraction of the strip qq' on any point P on the outer ellipsoid E depends on the thickness of the strip, and on Pq, Pq'; and attraction of QQ' on p, corresponding point on inner ellipsoid e, depends on the thickness of the strip, and on Qp, $Q'p$. But $Qp=Pq$ and $Q'p=Pq'$.

Therefore the whole attractions are in the ratios of the sections by the plane yz. Therefore the Theorem is proved.

XXV. is referred to in Maxwell's *Electricity*, Vol. I. p. 171. Nothing further on the subject has been met with in the papers that have been submitted to me.

*XXXVII. On a loose sheet I find some work slightly differing from that in the text, with the result

$$\cos(2x\cos\phi)=f(-x^2)-2x^2f_2(-x^2)\cos 2\phi+2x^4f_4(-x^2)\cos 4\phi-\&c.,$$

$$\therefore \int_0^\pi \cos(2x\cos\phi)\cos 2n\phi d\phi=(-)^n\pi x^{2n}f_{2n}(-x^2).$$

The following notes are given here as additional illustrations to *XLII., p. 369.

[i] If a line meeting two polar lines be displaced equally along both of them through an angle α, the two positions of the line will be called *parallel* and said to have the same direction. The normal distance from *any* point on one of them to the other is α. Two lines parallel to the same line are parallel to one another, and if a line meet two parallel lines it meets them at equal angles. According as the twist converting a line into a parallel line is right-handed or left-handed, the common direction of the two lines will be called a right or left direction; thus every line has two directions, and through an arbitrary point two lines may be drawn having with it right and left parallelism respectively. All lines having the same right direction meet the same two generators of one system of the absolute; all having the same left direction meet two generators of the other system. If a series of parallel lines be drawn through all the points of any line, they will trace out a surface of zero curvature and finite extent, which is in fact a quadric having quadruple contact with the absolute or meeting it in four straight lines. Starting with three lines meeting at right angles, we may determine a triple series of such surfaces, intersecting everywhere at right angles in lines parallel to the original three; thus every point in space will be characterized by co-ordinates measured parallel to three given axes. If any solid body receive simultaneous equal rotations about two polar lines, all the points of the body will move in parallel lines, and any one of these lines may be regarded as an axis of the motion, viz. the body has equal rotations about this line and its polar. Such a motion may be called a *Vector*; it may be represented by a finite straight line having given magnitude and direction, and will be a *right* or *left* vector according as the parallelism is right or left. The ratio of two vectors of the same side (i.e. both right or both left) is a *quaternion* of the same side; viz., one can be converted into the other by a certain rotation about an axis perpendicular to both but of indeterminate position. Every motor is the sum of a right and a left vector; for we have identically

$$A=\frac{1+\omega}{2}A+\frac{1-\omega}{2}A,$$

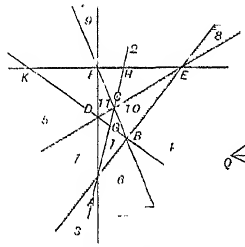


Fig 120.

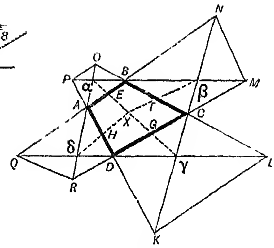


Fig 121

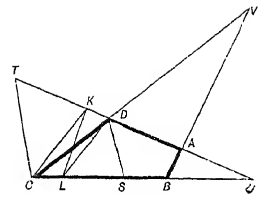
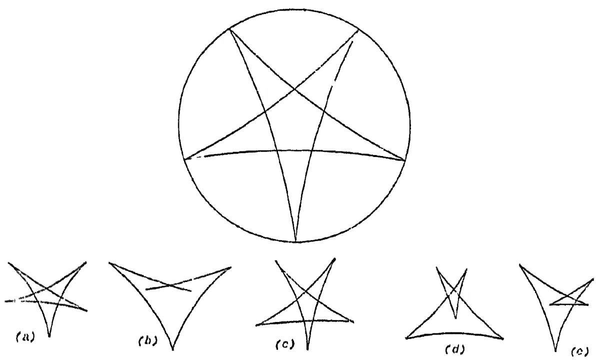


Fig 122.

Fig 123



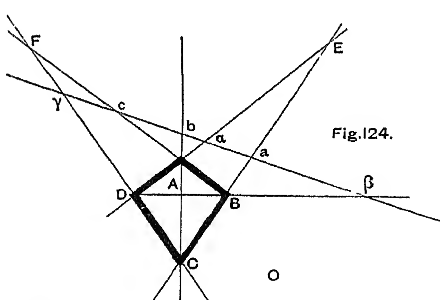


Fig.124.

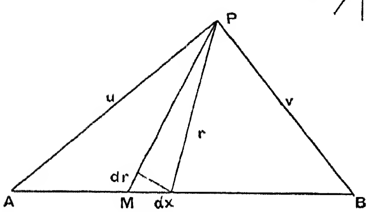


Fig.125.

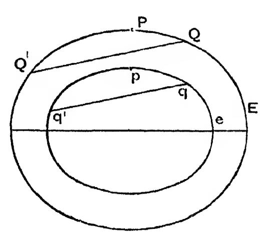


Fig.126

face p 642.

viz. these vector parts are the half sum and half difference of the motor A and its polar motor ωA. If we write $\xi=\frac{1}{2}(1+\omega)$, $\eta=\frac{1}{2}(1-\omega)$, then

$$\xi^n=\xi,\ \eta^n=\eta,\ \xi\eta=0,$$

and we have, if $\alpha\ \beta\ \gamma\ \delta$ are rotors through a fixed point,

$$\frac{\xi\alpha+\eta\beta}{\xi\gamma+\eta\delta}=\xi\frac{\alpha}{\gamma}+\eta\frac{\beta}{\delta},$$

whereby the quotient of any two motors is expressed as a biquaternion. For on substituting for $\xi\ \eta$ their values, the expression becomes $q+\omega r$, where q and r are the quaternions

$$\tfrac{1}{2}\left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}\right),\ \tfrac{1}{2}\left(\frac{\alpha}{\gamma}-\frac{\beta}{\delta}\right).$$

[ii] The co-ordinates of a straight line being $p_{12}\ p_{23}\ p_{31}\ p_{14}\ p_{24}\ p_{34}$, let

$$M=p_{12}\,p_{34}+p_{13}\,p_{42}+p_{14}\,p_{23},$$

and let $\Omega=0$ be the condition that this line touches the absolute or has no length. If we now write

$$\alpha=\alpha_{12}\,\partial_{p_{12}}+\ldots=\Sigma\,\alpha_{ij}\,\partial_{p_{ij}},$$

then $\alpha\beta M=0$ is the condition that the lines α, β (i.e. the lines whose equations are $\alpha M=0$, $\beta M=0$) should meet. $\alpha\Omega=0$ is the equation of the polar line of α.

In general there are two straight lines which meet two lines and their polars. For the five equations $\alpha M=0$, $\beta M=0$, $\alpha\Omega=0$, $\beta\Omega=0$, $M=0$, determine quadricly the ratios of the p. Now these two lines are polars. If γ be one, we have $\alpha\gamma M=0$, $\beta\gamma M=0$, $\alpha\gamma\Omega=0$, $\beta\gamma\Omega=0$, $\gamma^2M=0$.

Moment of two motors $=\dfrac{\alpha\beta M_2}{\sqrt{\alpha^2\Omega_2\,.\,\beta^2\Omega_2}}$.

[iii] $(\xi q+\eta r)(\xi A+\eta A)=$ a pure motor if $S\xi qA=0$, $S\eta rA=0$, or if we write

$$A=\xi\alpha+\eta\beta,$$

then q must directly operate on α and r on β.

Axes of $\xi\alpha+\eta\beta=\frac{1}{2}(T\alpha\pm T\beta)(\xi U\alpha\pm\eta U\beta)$,

$$\xi\alpha+\eta\beta=\frac{\alpha+\beta}{2}+\omega\frac{\alpha-\beta}{2}.$$

This is a rotor if $S.\overline{\alpha+\beta}.\overline{\alpha-\beta}=0$, or if $a^2-\beta^2=0$. *Hence the sum of two right and left vectors of equal length is a pure rotor.*

Or we may say the axes of A are $(T\xi A\pm T\eta A)(\xi U\xi A\pm\eta U\eta A)$.

Observe that $\omega(\xi A+\eta B)=\xi A-\eta B$. The general expression for rotor right parallel to i is $\xi\iota+\eta\beta$, β being of unit length; the axis is $\xi i+\eta\beta$ as it should be, or $\frac{1}{2}(\iota+\beta)+\frac{1}{2}\omega(\iota-\beta)$: therefore if ρ meets this, $\xi(\iota+k\rho)+\eta(\beta+k\rho)$ is a rotor; or

$$T(\iota+k\rho)=T(\beta+k\rho),$$

$$S\iota\rho=S\beta\rho,$$

$$\therefore S(\iota-\beta)\rho=0.$$

Hence ρ *is in a plane equally inclined to* ι and β.

The following fragments have some points of interest, and I print them in their incomplete state as they may suggest lines of working to some readers. The portion on curves [v] is headed *Appendix:* it looks at first sight as if it were intended to be an appendix to Frost's *Curve Tracing*.

[iv] *Geometry in a Quadric Space Q_2.*

Space-sections are represented by quadric-surfaces passing through a fixed conic ω_2. The points of this conic represent a *cone* K_2 lying entirely in the space Q_2. The vertex o of the cone is represented by the plane O of the conic ω_2.

A *Plane* represents a quadric surface containing two lines of K_2, since it meets ω_2 in two points.

The complete section of Q_2 by a space of order n passing a times through o is represented by a surface of order $2n-a$, passing $n-a$ times through the conic ω_2.

A surface of order μ passing b times through ω_2 represents a surface in Q_2 of order $2\mu-2b$, having o for a conical point of order $\mu-2b$.

Q_2 is cut by a *surface* of order r passing a times through o in a curve represented by a curve of order $2r-a$ meeting ω in $2r-2a$ points.

And a curve of order ρ meeting ω in b points represents a curve of order $2\rho-b$ passing $\rho-b$ times through o.

Thus e.g. a quadriquadric surface is represented by a surface of the fourth order passing twice through ω_2; and is therefore unicursal.

A Quadric space can contain no surface of odd order, unless its discriminant vanish. In this case the conic ω_2 is replaced by two lines $\omega\omega'$, and a surface passing a, b times through these represents a surface of order $2\mu-a-b$. Thus the quadric space now contains two pencils of planes represented by the planes through these lines ω, ω'. And a [quadri]quadric surface passing through one of them represents a skew cubic surface, passing once through o, meeting each of the planes of its own system in a line, and each of the others in two points. This cubic surface is ruled, then; and the other lines of the representative quadric represent a pencil of conics upon it, passing through o, but otherwise not meeting each other, but each meeting all the lines. Plane sections of the quadric represent skew cubic curves on the surface, namely its space-sections made by spaces passing through o. But the general space-sections are represented by skew cubics, meeting the conics twice and the lines once. We may generally represent the skew cubic surface upon a plane in the following manner. Its space-sections are represented by conics passing through one point a; sections by space of nth order, curves $2n$, n times through a; curves μ, a times through a represent curves of order $2\mu-a$. Thus the point a itself and the lines through it are the only straight lines on the surface; all other lines in the plane represent a doubly infinite system of conics, any two meeting in one point; each of these is met twice by a space-section, and so on.

Cubic Space C_3.

Space-sections are cubic surfaces any three of which meet in three points; i.e. they have common a quintic curve of deficiency 2 and one fixed point, or else a quartic curve of deficiency 0 and two fixed points.

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[v] *On the general shapes of Algebraic Curves.*

Those who have studied the preceding chapters of this work must still be convinced that, even with the facility which such study has given them, the operation of tracing a curve from its equation is one which requires some time and much close attention for its successful accomplishment. They will readily admit that if, in order to know the chief varieties of form among curves of the first four or five orders, it were necessary to trace one after another the curves represented by all varieties of equations of those degrees, by such processes as have already been explained; this work would be enormous, and beyond the reach of any but a lifelong study. Accordingly, it has never been attempted in the case of any degree exceeding the second. In the enumeration of lines of the third order, either the equation has been reduced by some preliminary discussion (to be presently mentioned) to a very special form, or its consideration has been evaded altogether. Of quartic curves the enumeration, so far as I know, has not yet been systematically attempted; but such general results as have been arrived at were obtained by considerations and methods other than those which the equation itself suggests.

To the explanation of such considerations and methods the present appendix is devoted. The study of them presents many direct advantages to the mathematician, besides the indirect ones of the help they afford him in the tracing of curves and the investigation of their properties. These methods are exceedingly simple and easy of application; they partake more of the nature of a manual craft than of a purely intellectual occupation, and may so be used as a rest from severer studies; and—as we can only imagine things of which we have seen the like—by appealing directly to the senses they extend those powers of concrete realization which the growing complication of modern analysis renders daily more desirable.

They consist of rules for transformation, by the aid of which the figures of a vast variety of curves may be obtained from a few simple types. These rules will be explained under three heads. In the first section is considered Projection, a process by which no alteration is made in the order, the class, nor in any other purely descriptive property of a curve. In the second section those modifications of form are described which leave the *order* of a curve unaltered. In the third, mention is made of those changes which exercise no effect upon the class. In the two latter sections a process will be described which may be called the *composition* of curves; by which a curve of any order or class may be built up out of the simplest elements.

Section 1. *Projection.*

The shadow of a hyperbola may be an ellipse, if it is properly held; and conversely, an ellipse may be so placed as to have a hyperbola for its shadow. More generally, the shadow of a curve may have more, or fewer, infinite branches than the curve itself. Now the fewer infinite branches a curve has, the easier it is to get a general idea of its shape; thus, to return to our first example, the shape of an ellipse is easier to grasp and remember than that of a hyperbola. The object of the method of Projection is to substitute for any given curve that shadow of it which has the least possible number of infinite branches; and thus to classify the enormous variety of shapes as the shadow of a comparatively small number of simple forms.

[vi] *The One-Two Correspondence of Two Planes.*

1. I consider two planes X and Ξ, the points of which are related in such a way that to every point x of X correspond in general two points ξ and ξ' of Ξ, while to every point ξ of Ξ corresponds in general one and only one point x of X.

2. Let the correspondent of a line L in X be a curve Γ_m in Ξ. I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X; the curves C_m and Γ_m being both of the same order m. The curves Γ_m must be such that two of them intersect in two points (variable), and that only one can be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds *one* point in X, m is therefore the number of intersections of L and the correspondent of Λ, which consequently is of order m as asserted.

3. The two representatives on Ξ of a point on X have between themselves a (1, 1) correspondence; let this be of order μ. Then the entire correspondent of C_m is not merely the line Λ but besides a curve Δ_μ which is unicursal and passes through a set of Cremona's principal points. The intersections of C_m, C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the point Δ_μ, Δ'_μ; (3) the 2μ points Λ, Δ'_μ and Λ', Δ_μ—in all $2+2\mu$ points. So then two curves, C_m, C'_m intersect in $\overline{\mu+1}$ variable points. They are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points ab four of them can be drawn; viz., the representatives of the lines $\alpha\beta$, $\alpha\beta'$, $\alpha'\beta$, $\alpha'\beta'$.

4. The curves Γ_m pass through certain fixed points. To single points correspond lines in X, to double ones conics, &c., exactly as in Cremona's theory. These curves are all unicursal. But to common points of the C_m may correspond curves which are not unicursal.

5. The case $m=2$ is of little interest and easily studied. The Γ_2 are conics through two fixed points, with a linear relation; their Jacobian is then a conic K_2 and a line Λ, both passing through the two fixed points. μ is equal to 2, and the Δ_2 are conics through the fixed points and the pole of Λ in respect of K_2. The C_2 are conics through a fixed point correspondent of Λ having double contact with a fixed conic, the correspondent of K_2. To the fixed points on Ξ correspond the tangents from the point to the conic on X.

6. The case $m=3$ gives us two systems: (1) cubics with a node and three points fixed, satisfying besides a linear relation; (2) cubics with seven fixed points. The first case is obtained by quadric transformation from the case $m=2$, whereby we see that the Jacobian is a cubic of the system (not satisfying the linear relation) and the straight lines joining the node to the fixed points. $\mu=3$, and the Δ_3 are cubics through the fixed points and another. The C_3 are cubics having triple contact with a fixed conic and a node at a certain fixed point.

The second system clearly inverts into itself. The Jacobian is a sextic S_6 with seven nodes at the fixed points. The correspondent of this is a curve, N_{12}.

[vii] *On the* (1, n) *Correspondence of Two Planes.*

1. I consider planes X, Ξ so related that to every point x on the first plane correspond in general n points, $\xi^1, \xi^2 \ldots \xi^n$, on the second, while to every point ξ of the second plane corresponds in general only one point x of the first.

2. Let the correspondent of a line L in X be a curve Γ_m in Ξ. I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X; the curves C_m and Γ_m being both of the same order m. The curves Γ_m must be such that two of them intersect in n variable points, and that only one can in general be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds *one* point in X; m is therefore the number of intersections of L and the correspondent of Λ, which consequently is of order m as asserted.

3. Two representatives on Ξ of the same point on X have between them a $(n-1,\ n-1)$ correspondence; let this be of order μ. Then the entire correspondent of C_m is not merely the line Λ, but besides a curve Δ_μ. Two curves Δ_μ, Δ'_μ intersect in $n-1$ variable points. The intersections of C_m, C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the $\overline{n-1}$ points Δ_μ, Δ'_μ; (3) the 2μ points Λ, Δ'_μ and Λ', Δ_μ—in all $n+2\mu$ points. Thus the number of variable intersections of two curves C_m, C'_m is $1+2\dfrac{\mu}{n}$, and consequently (except when $n=2$), μ must be a multiple of n. The curves C_m are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points a, b there may be drawn n^2 of them; viz., those corresponding to the n^2 lines joining the n correspondents of a, $\alpha^1, \alpha^2 \ldots \alpha^n$ to the n correspondents of b, $\beta^1, \beta^2 \ldots \beta^n$.

[viii] *Geometry on a Cubic Surface.*

1. A system of values of the coordinates $X^{(1)}, X^{(2)}, X^{(3)}$ in general determines one point on the surface. There are however six systems of values which determine not points but loci (viz. one-half of a double-sixer). These systems are to be denoted by $a^{(1)}, a^{(2)} \ldots a^{(6)}$, and called the Extension-Absolute (as distinguished from the Measure-Absolute, which is *another* set of six values).

2. Let $K_2^{(1)}$, $K_2^{(2)}$, $K_2^{(3)}$ be quadric functions of the X which are satisfied by the systems $a^{(i)}$, $a^{(j)}$ $a^{(k)}$. In general, a system of ratios of the K_2 determines one point on the surface; but the system $K_2^{(1)}=0$, $K_2^{(2)}=0$, $K_2^{(3)}=0$ has a meaning, viz. the loci $a^{(i)}$, $a^{(j)}$, $a^{(k)}$.

Three systems of values of the K_2 (viz. the other three points a) now represent loci and not points. But besides these there are three other systems which represent the *lines* $a^{(j)}a^{(k)}$, $a^{(k)}a^{(i)}$, $a^{(i)}a^{(j)}$. I call these six value-systems $k^{(1)}$. $k^{(6)}$. The K-coordinates then are precisely equivalent in their properties to the X, except that they vanish all together on the locus $a^{(i)}$, $a^{(j)}$, $a^{(k)}$.

But now the *point* $a^{(i)}$ may be approached in various ways; and it will be found that though the K_2 all vanish at that *point*, yet the ratios in which they vanish depend on the direction in which the *point* is approached. In fact, the vanishing ratios of the K_2 enable us to discriminate between the points of the *locus* $a^{(i)}$. We may then, if we like, consider only their ratios, deprive the system $K_2=0$ of its meaning, and so arrive at a new coordinate-system exactly similar to the one with which we started.

In fact, if $A^{(i)}$ denote $a^{(j)}a^{(k)}$, etc. we can find quadric functions of the K_2 respectively equal to $X^{(1)}A^{(i)}A^{(j)}A^{(k)}$, $X^{(2)}A^{(i)}A^{(j)}A^{(k)}$, $X^{(3)}A^{(i)}A^{(j)}A^{(k)}$.

Here it is clear that $A^{(i)}=0$ makes these three simultaneously vanish, but their ratios are those of the X.

3. I now take three cubic functions $C_3^{(1)}\,C_3^{(2)}\,C_3^{(3)}$ of the X, which are all satisfied doubly by $a^{(i)}$ and singly by all the other points a except $a^{(j)}$. A system of ratios of the C_3 will in general determine *one* point of the surface; but the system $C_3^{(1)}=0$, $C_3^{(2)}=0$, $C_3^{(3)}=0$ has a meaning; viz. the locus $a^{(i)}$ twice, and the other loci a except $a^{(j)}$ once.

The system of values $a^{(j)}$ of the C_3 now represents a locus and not a point. But there are systems of values which represent (1) the lines joining $a^{(i)}$ to the other a except $a^{(j)}$ and (2) the conic passing through all the a but $a^{(j)}$. I call these six values $c^{(1)}...c^{(6)}$. The C_3-coordinates are thus again equivalent to the former systems, with the exception that they all vanish together on the locus $\Sigma a - a^{(j)}$.

The ratios of the C_3 in the immediate neighbourhood of the point $a^{(i)}$ will be found to acquire *two* distinct value-systems for every direction of approach. The locus $a^{(i)}$ is therefore represented by a quadric equation in the C_3. The remaining loci a are represented by linear equations as before.

4. Similar remarks are to be made on quartic functions $Q_4^{(1)}$, $Q_4^{(2)}$, $Q_4^{(3)}$ having nodes at $a^{(i)}$, $a^{(j)}$, $a^{(k)}$ and single points at the other three. They remove entirely the pre-existing absolute, and substitute a new one, $q^{(1)}...q^{(6)}$ consisting of Cremona's well-known principal system. So further the quintics $P_5^{(1)}, P_5^{(2)}, P_5^{(3)}$ which have a node at each a, substitute a new absolute representing the six conics each through five a points. Altogether we have $1+20+30+20+1=72$ equivalent linear systems.

Plane sections of cubic	cubics through 6 points a
skew cubics	straight lines on plane
skew cubics generally	(1) straight lines
	(2) conics through 3 points a
	(3) cubics with node at a_1, passing through a_2, a_3, a_4, a_5
	(4) quartics with nodes at a_1, a_2, a_3, and single points a_4, a_5, a_6
	(5) quintics with a node at each point a.

Any one of Clebsch's 72 nets of skew cubics may be taken for a linear system on the cubic surface. The net cuts twice 6 lines on the cubic; the other half of the double-six forms the absolute. Analytically there are six sets of ratios $X : Y : Z$ which mean not points but these six lines; I call these quasi-points (a, b, c, f, g, h). A *line* $lX + mY + nZ$ which satisfies the condition of passing through one of these say, a, breaks up into the line a and a conic through a', the corresponding line of the double-six. The *line ab* contains these two lines and the line which meets ab', $a'b$. Abstractions made of the absolute, then, the *lines ab*... are the other 15 st. lines of the cubic. The lines a'... are represented by the *conics* through each five of the points.

When the surface has a node, the linear system is that of the plane sections through the node. The points $a\ b\ c\ f\ g\ h$ are then upon one conic.

$$K_2^{(1)} \equiv X^{(2)}X^{(3)} \text{ the point } K_2^{(2)} = 0,\quad K_2^{(3)} \equiv 0 \text{ gives merely } X^{(1)} = 0$$

$$K_2^{(2)} \equiv X^{(3)}X^{(1)} \quad ,, \quad K_2^{(3)} = 0,\quad K_2^{(1)} \equiv 0 \quad ,, \quad X^{(2)} = 0$$

$$K_2^{(3)} \equiv X^{(1)}X^{(2)} \quad ,, \quad K_2^{(1)} = 0,\quad K_2^{(2)} \equiv 0 \quad ,, \quad X^{(3)} = 0$$

[ix] *On the Correspondence between a Doubled Line and a Cubic.*

If I take a fixed point o on a cubic curve C_3, I can draw through it lines which unicursally correspond to the points on a straight line X. I have then for every point x on the line a pair of points c, c' on the cubic. But now I may suppose this line to be doubled, call it XX', and then the two (coincident) points xx' may be held either to correspond as a whole to the pair cc', or x may be held to correspond to c, and x' to c'. I shall now take instead of the doubled *line* XX', a doubled *circle*, so as to have the whole of it within reach. But now from the fixed point o four tangents can be drawn to the cubic; let us assume to begin with that they are all real. Then (calling them $ABCD$) if between A and B the points cc' are real, they must also be real between C and D, but imaginary between B and C and between D and A. Thus the *real* part of the cubic may be represented by two portions of a doubled circle*.

We have in fact substituted for the cubic an anallagmatic quartic on the point of becoming a doubled circle. If two of the tangents are imaginary we

* [There are three figures, (3) is a complete ring, (1) consists of two parts (AB), (CD) of the same ring, and (2) is a part of a ring, semicircular (AD).]

may suppose B and C to coincide, and the curve to be represented by *one* very thin oval; if they are all imaginary, the curve is represented by the entire contour of two indefinitely near circles.

(MS. ends.)

[x] *Syllabus of Lectures**.

General Principle—all the properties of a geometric form depend on its Order. Hence begin by establishing theory of imaginaries, on which that of
History
the order depends. Thus, 1. Historique. 2. Fundamental Hypotheses. Con-
Space
tinuity of Space. Motion without change of size. Infinite extent. Definitions of Line and Plane, perpendicular and parallel. 3. Calculus of Ratios and Position. Primary theorem of number. Deduction of arithmetical rules. Opera-
Quantity
tions. Algebraic calculus of numbers. Primary theorem of continuous quantity. Rules analogous to arithmetical operations. Algebraic calculus of ratios. Calculus of Position in one dimension. 4. Position in two dimensions. Gaussian Formulæ. Discontinuity of the reversor symbol, is made continuous by calculus of position in two dimensions. Idea of functional correspondence, monodromic,
Imaginaries
similarity of the smallest parts. Every equation of nth order has exactly n roots. 5. Position in two dimensions. Cartesian formulæ. Co-ordinates of a point, of a straight line, distance between given points, angle between given lines.
Equations
Equation of straight line, of circle, ellipse, hyperbola, parabola.

6. Projection and Linear Transformation. Passage to homogeneous form. Line at infinity. Projection equivalent to Linear Transformation. Order of curve. Number of points in which it is met by *any* line whatever. General
Invariance
notion of invariant. Invariants of points and lines. Grassmann notation for these. Aronhold notation for invariants in general. Duality, contravariant symbols, contravariant differentiation.

Correspondence
7. Correspondence in one dimension. (1,1) correspondence, anharmonics, harmonics, involution. Harmonics and involution of higher orders.

9. Plücker's equations, the Deficiency. Correspondence on a curve.

10. General theory of Polars. Special application to conics. Conjugirte Kerncurven.

[xi] *Geometric Analysis*.

1. The calculus of Ratios, and of onefold Position.
2. The calculus of twofold Position: (1) the Cartesian; (2) Gauss' plane of numbers.
3. The simpler Cartesian formulæ for geometrical magnitudes.
4. Equation in general. Forms of equation of straight line.
5. Equations of circle and conic sections.

* These are evidently the notes of the lectures printed above (pp. 521–530), and I give them as they show what were the subjects of the missing articles on p. 528.

6. Equation of a curve of any order.
7. Passage to homogeneous co-ordinates. The co-ordinates of a geometrical form in general.
8. The GRASSMANN notation.
9. The imaginary in geometry.
10. Systematic geometry of one dimension. Harmonics and Anharmonics.
11. The Polar Theory of conics.
12. The Bitangent circles of conics [pp. 543—5].
13. General Theory of the circle: powers of circles.
14. Theory of anallagmatic curves [pp. 546—555].
15. Extension of the GRASSMANN notation. General theory of distance.
16. Plucker's Equations. The Deficiency.
17. Polar theory of curves of the nth order.
18. General theorems relating to cubics.
19. The Polar theory of cubics.
20. Passage from the extended GRASSMANN notation to the symbolic form of covariants.

[xii]

The following are the 'heads' of two of Clifford's lectures on Quaternions (pp. 478—515). They will show how little of *written* prepared matter (L. and E. Vol. I. p. 8) he took to his lectures.

Whole numbers; scale of them. Steps of addition and subtraction; sum of any number of steps independent of their order.

Multiplication of numbers; in $2 \times 3 = 6$, 2 is an *operator*. Other interpretation; 2 and 3 are both operators; then 6 is one also.

Multiplication of steps; $2 \times (-3) = -6$; only one interpretation as yet.

Retention and reversal of step; symbols k, r;

$$k2 \times (-3) = -6, \; r2 \times (-3) = +6.$$

Product of operations on steps; $k2 \times r3 = r6$, $r2 \times r3 = k6$. Product independent of order of factors.

Analogy with multiplication of steps; $+$ and $-$ used for k and r; double meaning of $+$ and $-$.

Quantities; all continuous quantities *must* be specified by lines or angles, and angles are conveniently specified by lines. Scale of quantities on straight line.

Ratios of quantities. Ratio as *operator*. Product of ratios. Double meaning of equation $ab = c$.

Ancient ideas about product of quantities. Arabic solution of quadratic equations. Cardan and Tartaglia. Vieta's scale of dimensions. Introduction of ratios by Descartes. His non-recognition of negative quantities. Geometrical view of product.

Addition and subtraction of quantities. Steps on straight line. Ratio of steps. Signed or *scalar* numbers. Hamilton's view of algebra as science of pure time. Reference to geometry and kinematics implied in all the ordinary algebra.

Steps in plane and space. Addition, subtraction. Multiplication by scalars.

Applications. Theory of mass-centre. Equations of uniform and parabolic motion. Mass-centre of number of falling bodies.

Flux of a vector. Flux of product of vector and scalar. Hodograph. Acceleration. Curvature. Tangential and normal acceleration.

Ratio of steps in one plane. Tensor $\times$ versor. Scalar + rectangular versor.

Distributive, associative, commutative laws.

Complex numbers. Exponential. Meaning of $e^{i\theta} = \cos\theta + i \sin\theta$.

Correspondence of points on the planes by complex function. Equation of n^{th} order has n roots. Systems of orthogonal curves.

Ratio of steps in space. Scalar + rectangular versor.

Representation of rectangular versor, or handle. Addition of two. Associative and commutative laws for addition of three or more. Distributive law for operation on sum of vectors.

Product of rectangular versors. Scalar and versor part. Distributive law of product.

Expression in terms of unit rectangular versors at right angles, I, J, K. Laws of multiplication. Product of two versors. Verification. Quaternions.

Product of two quaternions. Associative law. Spherical theorem equivalent to it.

Comparison of multiplication of versors with multiplication of vectors by versors.

Replacement of I, J, K by i, j, k; double meaning of certain expressions.

Geometrical view of scalar and vector products. Applications: velocity-system of a rotation; composition of rotations; moment of momentum; work.

Linear function of a vector. Strain, homogeneous. Representation by conic or quadric when irrotational. Strain-flux due to given displacement. Moment of momentum is pure function of rotation. Motion of a body under no forces.

Slope of a function; condensation, convergence, curl. Rotation in liquid is curl of velocity. Force is slope of potential.

INDEX.

This Index applies to the Clifford Papers only, and the references are made generally to the pages, but the REPRINT Problems are referred to by their numbers, enclosed in brackets, that number being quoted which gives Clifford's own problems for the first time or to which the solution is appended.

K.

L.

M.

N.

O.

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Q.

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